**A**nnals of **F**uzzy **M**athematics and **I**nformatics
Volume 12, No. 6, (December 2016), pp. 757–766
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr



# Max-max operation on intuitionistic fuzzy matrix

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Received 17 March 2016; Revised 6 April 2016; Accepted 17 May 2016

Abstract. In this paper convergences of powers of a transitive intuitionistic fuzzy matrix is considered and some conditions for convergence are explored using the max-min operation on intuitionistic fuzzy matrices. In addition to that max-max operation on intuitionistic fuzzy matrices will be introduced to study the conditions for convergence of intuitionistic fuzzy matrices.

2010 AMS Classification: 03E72, 15B15

Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy matrix, Intuitionistic fuzzy transitive, Intuitionistic fuzzy implication operator.

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## 1. Introduction

The degree of membership was the only basic component of fuzzy sets introduced by Zadeh [40]. Atanassov [3, 4, 5, 6, 7, 8] generalized the concept of fuzzy sets into Intuitionistic Fuzzy Sets (IFS) by giving a degree of membership and non-membership. He showed, the sum of the degree of membership and non-membership should not exceed one. Duan [13] also showed that the powers of an Intuitionistic Fuzzy Matrices (IFMs) have a vital role for studying the transitive closure of the intuitionistic fuzzy relation. Yager [39] defined an Intuitionistic Fuzzy Matrix (IFM) $A$, as $A = [\langle a_{ij_\mu}, a_{ij_\nu} \rangle]$. Where $(a_{ij_\mu})$ and $(a_{ij_\nu})$ denote the membership and non-membership value respectively.

In 1977, Thomoson [38] studied the behavior of powers of fuzzy matrices using max-min operation. Buckley [12], Ran and Liu [34] and Gregory et al. [14] after applying max-min operation on fuzzy matrix found only two results, either the fuzzy matrix convergences to idempotent matrices or oscillates to finite period. More over, Thomoson [38] provided sufficient conditions for convergence of fuzzy matrix. Since then using this max-min operation many results have been obtained by many researchers in fuzzy matrix. Hashimoto [15] studied the convergence of power of a fuzzy transitive matrix. Further, the max-min operation has been extended to IFM by Pal et al. [19]. Bhowmik and pal [9] studied the convergence of the max-min powers of an IFM. Pradhan and pal [31] studied mean powers of convergence of IFMs. Lur et al. [21] studied the powers of convergence of IFMs. Pal [30] studied about intuitionistic fuzzy determinant. The several author's [20, 37, 10, 11, 1, 32, 2, 23, 24, 33] worked on intuitionistic fuzzy matrices and obtained various interesting results. Meenakshi and Gandhimathi [22] studied the intuitionistic fuzzy linear relation equations, Murugadas and Lalitha [25] applied Bi-implication operator to obtain the Sub-inverse and g-inverse of an IFM. Murugadas [26] and Sriram and Murugadas [35] examined IFM theory for obtaining the g- inverse. Pal et al. [19] studied the intuitionistic fuzzy linear transformation. Sriram and Murugadas [36] introduced the implication operator $\rightarrow$ to IFM and studied several properties like sub-inverse, semi-inverse as well as necessary and sufficient condition for the existence of g-inverse using the implication operator. Hashimoto [16, 17, 18] applied implication operators in the fuzzy matrix and studied some interesting properties like traces of fuzzy relation, sub-inverse and reduction of retrieval models. Murugadas and Lalitha [27, 28, 29] used hook implication operator $\leftharpoondown$ for IFS as well as IFM.

In this paper we introduce max-max operation directly to IFMs which is more relevant than max-min operation. For example, consider two IFMs $A$ and $B$ such that,

$$A = \begin{pmatrix} \langle 0.2, 0.7\rangle & \langle 0.4, 0.5\rangle \\ \langle 0.4, 0.5\rangle & \langle 0.1, 0.8\rangle \end{pmatrix} \quad \text{and } B = \begin{pmatrix} \langle 0.1, 0.8\rangle & \langle 0.4, 0.5\rangle \\ \langle 0.3, 0.6\rangle & \langle 0.5, 0.4\rangle \end{pmatrix}.$$

Then

$$\text{max-min } AB = \begin{pmatrix} \langle 0.3, 0.6\rangle & \langle 0.4, 0.5\rangle \\ \langle 0.1, 0.8\rangle & \langle 0.4, 0.5\rangle \end{pmatrix}$$

and

$$\text{max-max } AB = \begin{pmatrix} \langle 0.4, 0.5\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.4, 0.5\rangle & \langle 0.5, 0.4\rangle \end{pmatrix}.$$

Thus max-max $AB \geq$ max-min $AB$. In any retrieval model, if we get maximum membership values, that gives more accurate result. As max-max operation gives maximum membership values than max-min operation, the max-max operation is more relevant than max-min operation.

## 2. Preliminaries

Atanassov introduced operations

$$\langle x, x'\rangle \vee \langle y, y'\rangle = \langle max\{x, y\}, min\{x', y'\}\rangle$$

and

$$\langle x, x'\rangle \wedge \langle y, y'\rangle = \langle min\{x, y\}, max\{x', y'\}\rangle.$$

For any two comparable elements $\langle x, x'\rangle$, $\langle y, y'\rangle \in$ IFS, the operation $\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle$ is defined as

$$\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle = \begin{cases} \langle x, x'\rangle & \text{if } \langle x, x'\rangle > \langle y, y'\rangle, \\ \langle 0, 1\rangle & \text{if } \langle x, x'\rangle \leq \langle y, y'\rangle. \end{cases}$$

For $n \times n$ intuitionistic fuzzy matrices $R = [\langle r_{ij}, r'_{ij}\rangle]$ and $P = [\langle p_{ij}, p'_{ij}\rangle]$ with their elements having values in the unit interval $[0, 1]$, the following notations are well known:

$R \wedge P = (\langle r_{ij} \wedge p_{ij}, r'_{ij} \vee p'_{ij} \rangle)$,

$R \vee P = (\langle r_{ij} \vee p_{ij}, r'_{ij} \wedge p'_{ij} \rangle$.

Here $R \vee P$, $R \wedge P$ are equivalent to $R + P$, $R \odot P$ the component wise addition and component wise multiplication of $R, P$ respectively.

$R \times P = [(r_{i1}, r'_{i1} \wedge p_{1j}, p'_{1j}) \vee (r_{i2}, r'_{i2} \wedge p_{2j}, p'_{2j}) \vee \ldots \vee (r_{in}, r'_{in} \wedge p_{nj}, p'_{nj})]$,

$R \overset{c}{\leftharpoondown} P = [\langle r_{ij}, r'_{ij} \rangle \overset{c}{\leftharpoondown} \langle p_{ij}, p'_{ij} \rangle]$,

here $\overset{c}{\leftharpoondown}$ represents component wise comparison of $R, P$ using $\leftharpoondown$ .

$R^0 = I = [\delta_{ij}, \delta'_{ij}]$ ($\langle \delta_{ij}, \delta'_{ij} \rangle$ where $\langle \delta_{ij}, \delta'_{ij} \rangle = \langle 1, 0 \rangle$ if $i = j$ and $\langle \delta_{ij}, \delta'_{ij} \rangle = \langle 0, 1 \rangle$ if $i \neq j$.),

$R^{k+1} = R^k \times R$, $k = 0, 1, 2, ...,$

$R \leq P$ $(P \geq R)$ if and only if $\langle r_{ij}, r'_{ij} \rangle \leq \langle p_{ij}, p'_{ij} \rangle$ for all $i, j$.

If $R \geq I_n$, then $R$ is reflexive IFM where $I_n$ the $n \times n$ identity IFM. $R = (\langle r_{ij}, r'_{ij} \rangle)$ is weakly reflexive IFM if and only if $\langle r_{ii}, r'_{ii} \rangle \geq \langle r_{ij}, r'_{ij} \rangle$ for all $i, j = 1, 2, ...n$.

Throughout we deal with intuitionistic fuzzy matrices. A matrix R is transitive if $R^2 \leq R$. This matrix represents a intuitionistic fuzzy transitive relation. The above definition of transitivity is equivalent to what is called max-min transitivity. That is, matrix $R = [\langle r_{ij}, r'_{ij} \rangle]$ is transitive if and only if $min(\langle r_{ik}, r'_{ik} \rangle, \langle r_{kj}, r'_{kj} \rangle) \leq \langle r_{ij}, r'_{ij} \rangle$, for all k. This definition is most basic and seems to be convenient when intuitionistic fuzzy matrices are generalized to certain matrices over other algebras.

Thomoson [38] already considered convergences of powers of a fuzzy matrix. Hashimoto [15] examined convergence of powers of a transitive matrix i.e, a matrix R, such that $R \geq R^2$. If R is transitive, then we have $R \geq R^2 \geq R^3 \geq \ldots$. In the sequence of powers of a matrix R, if $R^k = R^{k+1}$ for some positive integers k, then we say R is convergent.

## 3. Main results

Some interesting properties of transitive intuitionistic fuzzy matrices will be shown and some conditions for convergence under the max-min operation will be given. These results are useful when we consider various systems with intuitionistic fuzzy transitivity. Further, we define max-max operation on IFM and exhibit some interesting results. In the following, let $R = [\langle r_{ij}, r'_{ij} \rangle]$, $P = [\langle p_{ij}, p'_{ij} \rangle]$, be IFM of order $n \times n$, and the entries in R and P are comparable.

**Definition 3.1.** For IFMs R and P define, the max-max product of R and P as

$$R \circ P = (\langle \bigvee_{k=1}^{n} (r_{ik} \vee p_{kj}), \bigwedge_{k=1}^{n} (r'_{ik} \wedge p'_{kj}) \rangle).$$

Let $R \circ P$ denote the max-max product of the IFMs R and P.

Clearly $R \circ P$ is also an IFM, $\circ$ is associative and $\circ$ is distributive over addition $(+)$. Also the set of all IFM under $+$ and $\circ$ form a semi-ring.

**Theorem 3.2.** *If $R$ is an $n \times n$ transitive matrix, then*

$$(R \overset{c}{\leftharpoondown} (R \times P))^n = (R \overset{c}{\leftharpoondown} (R \times P))^{n+1} \text{ for any } n \times n \text{ IFM } P.$$

*Proof.* Let $S = (\langle s_{ij}, s'_{ij} \rangle) = R \overset{c}{\leftharpoondown} (R \times P)$, that is,

$$\langle s_{ij}, s'_{ij}\rangle = \langle r_{ij}, r'_{ij}\rangle \xleftarrow{c} \langle \bigvee_{k=1}^{n}(r_{ik}\wedge p_{kj}), \bigwedge_{k=1}^{n}(r'_{ik}\vee p'_{kj})\rangle.$$

(1) Assume that there exist indices $l_1, l_2, \ldots, l_{n-1}$ such that

$$\langle s_{il_1}, s'_{il_1}\rangle \wedge \langle s_{l_1l_2}, s'_{l_1l_2}\rangle \wedge \ldots \wedge \langle s_{l_{n-1}j}, s'_{l_{n-1}j}\rangle = \langle g, g'\rangle > \langle 0, 1\rangle.$$

Let $l_0 = i$ and $l_n = j$. Then $l_a = l_b$ for some a and $b(a < b)$. We define $\langle h, h'\rangle$ by

$$\begin{aligned}\langle h, h'\rangle &= \langle r_{l_{m-1}l_m}, r'_{l_{m-1}l_m}\rangle\\ &= \langle r_{l_al_{a+1}}, r'_{l_al_{a+1}}\rangle \wedge \langle r_{l_{a+1}l_{a+2}}, r'_{l_{a+1}l_{a+2}}\rangle \wedge \ldots \wedge \langle r_{l_{b-1}l_b}, r'_{l_{b-1}l_b}\rangle,\end{aligned}$$

where $a < m \leq b$.
Then $\langle h, h'\rangle = \langle r_{l_{m-1}l_m}, r'_{l_{m-1}l_m}\rangle > \langle \bigvee_{k=1}^{n}(r_{l_mk}\wedge p_{kl_m}), \bigwedge_{k=1}^{n}(r'_{l_mk}\vee p'_{kl_m})\rangle$.
If $\langle r_{l_ml_m}, r'_{l_ml_m}\rangle \leq \langle \bigvee_{k=1}^{n}(r_{l_mk}\wedge p_{kl_m}), \bigwedge_{k=1}^{n}(r'_{l_mk}\vee p'_{kl_m})\rangle$, then

$$\begin{aligned}\langle h, h'\rangle &\leq \langle r_{l_ml_m}, r'_{l_ml_m}\rangle \leq \langle r_{l_mk_1}\wedge p_{k_1l_m}, r'_{l_mk_1}\vee p'_{k_1l_m}\rangle\\ &= \langle r_{l_al_{a+1}}, r'_{l_al_{a+1}}\rangle \wedge \langle r_{l_{a+1}l_{a+2}}, r'_{l_{a+1}l_{a+2}}\rangle \wedge \ldots \wedge \langle r_{l_{b-1}l_b}, r'_{l_{b-1}l_b}\rangle,\end{aligned}$$

for some $k_1$. Since $\langle r_{l_{m-1}l_m}, r'_{l_{m-1}l_m}\rangle = \langle h, h'\rangle$, we have

$$\langle r_{l_{m-1}k_1}, r'_{l_{m-1}k_1}\rangle \geq \langle r_{l_{m-1}l_m}, r'_{l_{m-1}l_m}\rangle \wedge \langle r_{l_mk_1}, r'_{l_mk_1}\rangle = \langle h, h'\rangle.$$

Thus,

$$\begin{aligned}\langle \bigvee_{k=1}^{n}(r_{l_{m-1}k}\wedge p_{kl_m}), \bigwedge_{k=1}^{n}(r'_{l_{m-1}k}\vee p'_{kl_m})\rangle &\geq \langle r_{l_{m-1}k_1}, r'_{l_{m-1}k_1}\rangle \wedge \langle p_{k_1l_m}, p'_{k_1l_m}\rangle\\ &\geq \langle h, h'\rangle,\end{aligned}$$

which is contradiction. So,

$$\langle r_{l_ml_m}, r'_{l_ml_m}\rangle > \langle \bigvee_{k=1}^{n}(r_{l_mk}\wedge p_{kl_m}), \bigwedge_{k=1}^{n}(r'_{l_mk}\vee p'_{kl_m})\rangle.$$

Hence $\langle s_{l_ml_m}, s'_{l_ml_m}\rangle \geq \langle h, h'\rangle \geq \langle g, g'\rangle$. Therefore $\langle s_{ij}^{(n+1)}, s_{ij}'^{(n+1)}\rangle \geq \langle g, g'\rangle$.

(2) Assume that there exist indices $l_1, l_2, \ldots, l_n$ such that

$$\langle s_{il_1}, s'_{il_1}\rangle \wedge \langle s_{l_1l_2}, s'_{l_1l_2}\rangle \wedge \ldots \wedge \langle s_{l_nj}, s'_{l_nj}\rangle = \langle g, g'\rangle > \langle 0, 1\rangle.$$

Let $l_0 = i$ and $l_{n+1} = j$.

(a) Assume $l_a = l_b = l_c$, where $a < b < c$. Then we have

$$\langle s_{l_ml_m}, s'_{l_ml_m}\rangle \geq \langle g, g'\rangle,\ a < m \leq b \text{ for some } l_m.$$

Thus,

$$\begin{aligned}&\langle s_{il_m}^{(m)}, s_{il_m}'^{(m)}\rangle \wedge \langle s_{l_ml_m}^{(c-b-c)}, s_{l_ml_m}'^{(c-b-c)}\rangle \wedge \langle s_{l_ml_b}^{(b-m)}, s_{l_ml_b}'^{(b-m)}\rangle \wedge \langle s_{l_cj}^{(n+1-c)}, s_{l_cj}'^{(n+1-c)}\rangle\\ &\geq \langle g, g'\rangle.\end{aligned}$$

So $\langle s_{ij}^{n}, s_{ij}'^{n}\rangle \geq \langle g, g'\rangle$.

(b) Assume $l_a = l_b$ and $l_c = l_d$.

(i) if $a < b < c < d$, then

$$\langle s_{l_m l_m}, s'_{l_m l_m}\rangle \geq \langle g, g'\rangle,\ a < m \leq b, \text{ for some } l_m.$$

Thus,

$$\langle s^m_{il_m}, s'^m_{il_m}\rangle \wedge \langle s^{(d-c-1)}_{l_m l_m}, s'^{(d-c-1)}_{l_m l_m}\rangle \wedge \langle s^{(c-m)}_{l_m l_c}, s'^{(c-m)}_{l_m l_c}\rangle \wedge \langle s^{(n+1-d)}_{l_d j}, s'^{(n+1-d)}_{l_d j}\rangle$$
$$\geq \langle g, g'\rangle \geq \langle g, g'\rangle.$$

So $\langle s^n_{ij}, s'^n_{ij}\rangle \geq \langle g, g'\rangle$.

(ii) If $a < c < b < d$, then

$$\left\langle s_{l_m l_m}, s'_{l_m l_m}\right\rangle \geq \langle h, h'\rangle \geq \langle g, g'\rangle,\ a < m \leq b, \text{ for some } l_m,$$

where

$$\langle h, h'\rangle = \langle r_{l_{m-1} l_m}, r'_{l_{m-1} l_m}\rangle = \langle r_{l_a l_{a+1}}, r'_{l_a l_{a+1}}\rangle \wedge \ldots \wedge \langle r_{l_{b-1} l_b}, r'_{l_{b-1} l_b}\rangle.$$

Since it is clear that $\langle s^n_{ij}, s'^n_{ij}\rangle \geq \langle g, g'\rangle$ for $m \leq c$, suppose that $m > c$. If

$$\langle r_{l_a l_m}, r'_{l_a l_m}\rangle \leq \langle \bigvee_{k=1}^{n} (r_{l_a k} \wedge p_{k l_m}), \bigwedge_{k=1}^{n} (r'_{l_a k} \vee p'_{k l_m})\rangle,$$

then

$$\langle g, g'\rangle \leq \langle h, h'\rangle \leq \langle r_{l_a l_m}, r'_{l_a l_m}\rangle \leq \langle r_{l_n k_1}, r'_{l_n k_1}\rangle \wedge \langle p_{k_1 l_m}, p'_{k_1 l_m}\rangle,$$

for some $k_1$. Thus,

$$\langle r_{l_{m-1} k_1}, r'_{l_{m-1} k_1}\rangle \geq \langle r_{l_{m-1} l_m}, r'_{l_{m-1} l_m}\rangle \wedge \langle r_{l_m l_a}, r'_{l_m l_a}\rangle \langle r_{l_a k_1}, r'_{l_a k_1}\rangle = \langle h, h'\rangle.$$

We have,

$$\langle \bigvee_{k=1}^{n} (r_{l_{m-1} k} \wedge p_{k l_m}), \bigwedge_{k=1}^{n} (r'_{l_{m-1} k} \vee p'_{k l_m})\rangle \geq \langle r_{l_{m-1} k_1}, r'_{l_{m-1} k_1}\rangle \wedge \langle p_{k_1 l_m}, p'_{k_1 l_m}\rangle$$
$$\geq \langle h, h'\rangle,$$

which contradicts the fact that $\langle h, h'\rangle = \langle s_{l_{m-1} l_m}, s'_{l_{m-1} l_m}\rangle > 0$.
So $\langle s_{l_m l_m}, s'_{l_m l_m}\rangle \geq \langle g, g'\rangle$. Hence

$$\langle s^{(a)}_{il_a}, s'^{(a)}_{il_a}\rangle \wedge \langle s_{l_a l_m}, s'_{l_a l_m}\rangle \wedge \langle s^{(m-a-2)}_{l_m l_m}, s'^{(m-a-2)}_{l_m l_m}\rangle \wedge \langle s^{(n+1-m)}_{l_m j}, s'^{(n+1-m)}_{l_m j}\rangle$$
$$\geq \langle g, g'\rangle.$$

(iii) If $a < c < d < b$, then

$$\langle s_{l_m l_m}, s'_{l_m l_m}\rangle \geq \langle g, g'\rangle,\ a < m \leq b, \text{ for some } l_m.$$

It is clear that $\langle s^{(n)}_{ij}, s'^{(n)}_{ij}\rangle \geq \langle g, g'\rangle$ for $m \leq c$ or $d \leq m$. Suppose that $c \leq m \leq d$. By the same argument as in (ii), we have $\langle s_{l_a l_m}, s'_{l_a l_m}\rangle \geq \langle g, g'\rangle$. Then

$$\langle s^{(a)}_{il_a}, s'^{(a)}_{il_a}\rangle \wedge \langle s_{l_a l_m}, s'_{l_a l_m}\rangle \wedge \langle s^{(m-a-2)}_{l_m l_m}, s'^{(m-a-2)}_{l_m l_m}\rangle \wedge \langle s^{(n+1-m)}_{l_m j}, s'^{(n+1-m)}_{l_m j}\rangle$$
$$\geq \langle g, g'\rangle.$$

□

**Example 3.3.** Let

$$R = \begin{pmatrix} \langle 0.7, 0.2\rangle & \langle 0.4, 0.3\rangle & \langle 0.5, 0.4\rangle \\ \langle 0, 0.6\rangle & \langle 0.2, 0.5\rangle & \langle 0.3, 0.2\rangle \\ \langle 0, 0.6\rangle & \langle 0, 0.5\rangle & \langle 0, 0.6\rangle \end{pmatrix} \text{ and } P = \begin{pmatrix} \langle 0.4, 0.3\rangle & \langle 0.3, 0.4\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.5, 0.2\rangle & \langle 0.3, 0.6\rangle & \langle 0, 0.6\rangle \\ \langle 0, 0.3\rangle & \langle 0.2, 0.5\rangle & \langle 0.2, 0.7\rangle \end{pmatrix},$$

$$R \times P = \begin{pmatrix} \langle 0.4, 0.3\rangle & \langle 0.3, 0.4\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.2, 0.5\rangle & \langle 0.2, 0.5\rangle & \langle 0.2, 0.7\rangle \\ \langle 0, 0.5\rangle & \langle 0, 0.5\rangle & \langle 0, 0.6\rangle \end{pmatrix},$$

$$R^2 = \begin{pmatrix} \langle 0.7, 0.2\rangle & \langle 0.4, 0.3\rangle & \langle 0.5, 0.4\rangle \\ \langle 0, 0.6\rangle & \langle 0.2, 0.5\rangle & \langle 0.2, 0.5\rangle \\ \langle 0, 0.6\rangle & \langle 0, 0.5\rangle & \langle 0, 0.6\rangle \end{pmatrix} \leq R \quad (\text{ R is transitive}),$$

$$S = R \overset{c}{\leftharpoondown} (R \times P) = \begin{pmatrix} \langle 0.7, 0.2\rangle & \langle 0.4, 0.3\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.3, 0.2\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{pmatrix},$$

$$S^2 = S \times S = \begin{pmatrix} \langle 0.7, 0.2\rangle & \langle 0.4, 0.3\rangle & \langle 0.3, 0.2\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{pmatrix}.$$

Then

$$S^3 = S^2 \times S = \begin{pmatrix} \langle 0.7, 0.2\rangle & \langle 0.4, 0.3\rangle & \langle 0.3, 0.2\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{pmatrix} = S^2.$$

Thus we have $S^3 = S^2$.

From Theorem 3.2, we get the following two results.

**Corollary 3.4.** *If $R$ is an $n \times n$ transitive matrix, then*

$$(R \overset{c}{\leftharpoondown} (P \times R))^n = (R \overset{c}{\leftharpoondown} (P \times R))^{n+1},$$

*for any $n \times n$ matrix $P$.*

**Corollary 3.5.** *If $R$ is an $n \times n$ transitive matrix, then $R^n = R^{n+1}$.*

We now consider conditions under which an $n \times n$ transitive matrix R fulfills the relationship $R^{n-1} = R^n$, where $n \geq 2$.

**Theorem 3.6.** *Let $R$ be an $n \times n$ transitive matrix. If*

$$R \wedge I \leq P \leq R,$$

*and the max-max product $R \circ R^T \leq (\langle r_{jj}, r'_{jj}\rangle)$ for some $j$, then $P^{n-1} = P^n$.*

*Proof.* (1) First we know that $P^{n-1} \leq P^n$. Suppose that

$$\langle p_{ij}^{(n-1)}, p_{ij}^{\prime(n-1)}\rangle = \langle c, c'\rangle \geq \langle 0, 1\rangle.$$

Then there exist indices $k_1, k_2, ..., k_{n-2}$ such that

$$\langle p_{ik_1}, p'_{ik_1}\rangle \wedge \langle p_{k_1k_2}, p'_{k_1k_2}\rangle \wedge \cdots \wedge \langle p_{k_{n-2}j}, p'_{k_{n-2}j}\rangle = \langle c, c'\rangle.$$

Thus

$$\langle r_{ik_1}, r'_{ik_1}\rangle \wedge \langle r_{k_1k_2}, r'_{k_1k_2}\rangle \wedge \cdots \wedge \langle r_{k_{n-2}j}, r'_{k_{n-2}j}\rangle \geq \langle c, c'\rangle.$$

Let $k_0 = i$ and $k_{n-1} = j$.

(a) If $k_a = k_b$ for some $a$ and $b(a < b)$, then $\langle p^{(b-a)}_{k_ak_a}, p'^{(b-a)}_{k_ak_a}\rangle \geq \langle c, c'\rangle$ . Thus

$$\langle r^{(b-a)}_{k_ak_a}, r'^{(b-a)}_{k_ak_a}\rangle \geq \langle c, c'\rangle, \quad \langle r_{k_ak_a}, r'_{k_ak_a}\rangle \geq \langle c, c'\rangle, \quad \langle p_{k_ak_a}, p'_{k_ak_a}\rangle \geq \langle c, c'\rangle.$$

So

$$\begin{aligned}&\langle p_{ik_1}, p'_{ik_1}\rangle \wedge \langle p_{k_1k_2}, p'_{k_1k_2}\rangle \wedge \cdots \wedge \langle p_{k_{a-1}k_a}, p'_{k_{a-1}k_a}\rangle \wedge \langle p_{k_ak_a}, p'_{k_ak_a}\rangle\\ &\wedge \langle p_{k_ak_{a+1}}, p'_{k_ak_{a+1}}\rangle \wedge \cdots \wedge \langle p_{k_{n-2}j}, p'_{k_{n-2}j}\rangle\\ &\geq \langle c, c'\rangle.\end{aligned}$$

Hence $\langle p^{(n)}_{ij}, p'^{(n)}_{ij}\rangle \geq \langle c, c'\rangle$.

(b) Suppose that $k_a \neq k_b$ for all $a \neq b$. By hypothesis,

$$\langle \bigvee_{k=1}^{n} (r_{lk_m} \vee r_{k_ml}), \bigwedge_{k=1}^{n} (r'_{lk_m} \wedge r'_{k_ml})\rangle \leq \langle r_{k_mk_m}, r'_{k_mk_m}\rangle \text{ for some } m.$$

Then

$$\langle r_{k_mk_m}, r'_{k_mk_m}\rangle \geq \langle c, c'\rangle, \quad \langle p_{k_mk_m}, p'_{k_mk_m}\rangle \geq \langle c, c'\rangle.$$

Thus

$$\begin{aligned}&\langle p_{ik_j}, p'_{ik_j}\rangle \wedge \langle p_{k_1k_2}, p'_{k_1k_2}\rangle \wedge \cdots \wedge \langle p_{k_{m-1}k_m}, p'_{k_{m-1}k_m}\rangle \wedge \langle p_{k_mk_m}, p'_{k_mk_a}\rangle\\ &\wedge \langle p_{k_mk_{m+1}}, p'_{k_mk_{m+1}}\rangle \wedge \cdots \wedge \langle p_{k_{n-2}j}, p'_{k_{n-2}j}\rangle\\ &\geq \langle c, c'\rangle.\end{aligned}$$

So $\langle p^{(n)}_{ij}, p'^{(n)}_{ij}\rangle \geq \langle c, c'\rangle$.

(2) Next we show that $P^n \leq P^{n-1}$.

Let $\langle p^{(n)}_{ij}, p'^{(n)}_{ij}\rangle = \langle c, c'\rangle \geq \langle 0, 1\rangle$. Then there exists indices $k_1, k_2, ..., k_{n-1}$ such that

$$\langle p_{ik_1}, p'_{ik_1}\rangle \wedge \langle p_{k_1k_2}, p'_{k_1k_2}\rangle \wedge \cdots \wedge \langle p_{k_{n-1}j}, p'_{k_{n-1}j}\rangle = \langle c, c'\rangle.$$

Let $k_0 = i$ and $k_n = j$. Then $k_a = K_b$ for some $a$ and $b(a < b)$. Thus

$$\langle p^{(b-a)}_{k_ak_a}, p'^{(b-a)}_{k_ak_a}\rangle \geq \langle c, c'\rangle.$$

So

$$\langle r^{(b-a)}_{k_ak_a}, r'^{(b-a)}_{k_ak_a}\rangle \geq \langle c, c'\rangle, \quad \langle r_{k_ak_a}, r'_{k_ak_a}\rangle \geq \langle c, c'\rangle, \quad \langle p_{k_ak_a}, p'_{k_ak_a}\rangle \geq \langle c, c'\rangle.$$

Hence

$$\begin{aligned}&\langle p_{ik_1}, p'_{ik_1}\rangle \wedge \langle p_{k_1k_2}, p'_{k_1k_2}\rangle \wedge \cdots \wedge \langle p_{k_{a-1}k_a}, p'_{k_{a-1}k_a}\rangle \wedge \langle p^{(b-a-1)}_{k_ak_a}, p'_{k_ak_a}\rangle\\ &\wedge \langle p_{k_bk_{b+1}}, p'_{k_bk_{b+1}}\rangle \wedge \cdots \wedge \langle p_{k_{n-1}j}, p'_{k_{n-1}j}\rangle\\ &\geq \langle c, c'\rangle.\end{aligned}$$

Therefore $\langle p^{(n-1)}_{ij}, p'^{(n-1)}_{ij}\rangle \geq \langle c, c'\rangle$. □

**Example 3.7.** Let

$$R = \begin{pmatrix} \langle 0, 0.3\rangle & \langle 0.6, 0.1\rangle & \langle 0.7, 0\rangle \\ \langle 0, 0.5\rangle & \langle 0.6, 0.1\rangle & \langle 0.5, 0.2\rangle \\ \langle 0, 0.4\rangle & \langle 0.3, 0.4\rangle & \langle 0.5, 0.2\rangle \end{pmatrix},$$

$$R^2 = \begin{pmatrix} \langle 0, 0.3\rangle & \langle 0.6, 0.1\rangle & \langle 0.5, 0.2\rangle \\ \langle 0, 0.4\rangle & \langle 0.6, 0.1\rangle & \langle 0.5, 0.2\rangle \\ \langle 0, 0.4\rangle & \langle 0.4, 0.3\rangle & \langle 0.5, 0.2\rangle \end{pmatrix} \leq R(\text{ R is transitive}),$$

$$R^3 = \begin{pmatrix} \langle 0, 0.3\rangle & \langle 0.6, 0.1\rangle & \langle 0.5, 0.2\rangle \\ \langle 0, 0.4\rangle & \langle 0.6, 0.1\rangle & \langle 0.5, 0.2\rangle \\ \langle 0, 0.4\rangle & \langle 0.4, 0.3\rangle & \langle 0.5, 0.2\rangle \end{pmatrix} = R^2,$$

$$P = \begin{pmatrix} \langle 0, 0.4\rangle & \langle 0.6, 0.1\rangle & \langle 0.3, 0.3\rangle \\ \langle 0, 0.5\rangle & \langle 0.6, 0.1\rangle & \langle 0.4, 0.2\rangle \\ \langle 0, 0.4\rangle & \langle 0.2, 0.3\rangle & \langle 0.4, 0.2\rangle \end{pmatrix},$$

$$P^2 = \begin{pmatrix} \langle 0, 0.4\rangle & \langle 0.6, 0.1\rangle & \langle 0.4, 0.2\rangle \\ \langle 0, 0.4\rangle & \langle 0.6, 0.1\rangle & \langle 0.4, 0.2\rangle \\ \langle 0, 0.4\rangle & \langle 0.2, 0.3\rangle & \langle 0.4, 0.2\rangle \end{pmatrix},$$

$$P^3 = \begin{pmatrix} \langle 0, 0.4\rangle & \langle 0.6, 0.1\rangle & \langle 0.4, 0.2\rangle \\ \langle 0, 0.4\rangle & \langle 0.6, 0.1\rangle & \langle 0.4, 0.2\rangle \\ \langle 0, 0.4\rangle & \langle 0.2, 0.3\rangle & \langle 0.4, 0.2\rangle \end{pmatrix} = P^2.$$

**Theorem 3.8.** *If $R$ is an $n \times n$ transitive matrix, $R \wedge I \leq P \leq R$ and $P \circ P^T \leq (\langle p_{jj}, p'_{jj}\rangle)$ for some $j$, then $P^{n-1} = P^n$.*

As a special case of theorem 3.6 or theorem 3.8 we obtain the following corollary when $R$ is a transitive IFM.

**Corollary 3.9.** *If $R$ is an $n \times n$ transitive intuitionistic fuzzy matrix and $R \circ R^T \leq (\langle r_{jj}, r'_{jj}\rangle)$ for some $j$, then $R^{n-1} = R^n$.*

## 4. Conclusions

In this paper max-max operation on intuitionistic fuzzy matrices has been introduced. The conditions for convergence of intuitionistic fuzzy matrices are examined under the max-max operation.

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