Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 12, No. 5, (November 2016), pp. 719–738
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

http://www.kyungmoon.com

Applications of r-generalized regular fuzzy closed sets

A. Vadivel, E. Elavarasan

Received 7 January 2016; Revised 14 April 2016; Accepted 23 May 2016

Abstract. In this paper, we introduce the concepts of r-fuzzy regular closure (interior) operators and r-generalized regular fuzzy closed sets in a fuzzy topological spaces in view of the definition of Šostak. We investigate some properties of them. Moreover, we investigate the relationship between generalized regular fuzzy continuous maps, generalized regular fuzzy irresolute maps and generalized regular fuzzy contra continuous maps. Also, some separation axioms of r-generalized regular fuzzy closed sets are introduced and studied.

2010 AMS Classification: 54A40

Keywords: r-fuzzy regular closure (interior) operators, r-generalized regular fuzzy closed (open) sets, Generalized regular fuzzy continuous (irresolute) maps, Generalized regular fuzzy contra continuous (irresolute) maps, r-GRF-regular spaces, r-GRF-normal spaces.

Corresponding Author: A. Vadivel (avmaths@gmail.com)

1. Introduction

Kubiak [14] and Šostak [19] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [4], in the sense that not only the objects are fuzzified, but also the axiomatics. In [20, 21], Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al., [6] have redefined the same concept under the name gradation of openness. A general approach to the study of topological type structures on fuzzy power sets was developed in [7, 8, 9, 14, 15]. Balasubramanian and Sundaram [2] gave the concept of generalized fuzzy closed sets in Chang's fuzzy topology as an extension of generalized closed sets of Levine [16] in topological spaces.

Jin Han Park and Jin Keun Park [17] introduced weaker form of generalized fuzzy closed set and generalized fuzzy continuous mappings i.e, regular generalized fuzzy closed set and generalizations of fuzzy continuous functions. Baby Bhattacharya and Jayasree Chakraborty [3] introduced another generalizatioin of fuzzy closed set i.e, generalized regular fuzzy closed set which is the stronger form of the previous two generalizations.

In this paper, we define r-fuzzy regular closure (interior) operators and r-generalized regular fuzzy closed sets in fuzzy topological spaces (fts's) of Šostak [19]. In section 2 r-fuzzy regular closure and r-fuzzy regular interior operators are introduced in Šostak's fts. Section 3 is devoted to r-generalized regular fuzzy closed (open) sets and their properties. In section 4, we introduce generalized regular fuzzy continuous functions and generalized regular fuzzy irresolute functions and investigate inter-relation between them. In section 5 we introduce generalized regular fuzzy contra continuity in Šostak's fts's. Lastly, some separation axioms of r-generalized regular fuzzy closed sets are introduced and studied in section 6.

2. Preliminaries

Throughout this paper, let X be a nonempty set, $I = [0,\ 1]$ and $I_0 = (0,\ 1]$. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. For $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by

$$x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

Let $Pt(X)$ be the family of all fuzzy points in X. A fuzzy point $x_t \in \alpha$ iff $t < \alpha(x)$. A fuzzy set α is quasi-coincident with β denoted by $\alpha q \beta$, if there exists $x \in X$ such that $\alpha(x) + \beta(x) > 1$. If α is not quasi-coincident with β, we denote $\alpha \overline{q} \beta$. If $A \subset X$, we define the characteristic function χ_A on X by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

All other notations and definitions are standard, for all in the fuzzy set theory.

Lemma 2.1 ([10])**.** *Let X be a nonempty set and $\alpha,\ \beta \in I^X$. Then*

(1) *$\alpha q \beta$ iff there exists $x_t \in \alpha$ such that $x_t q \beta$,*
(2) *If $\alpha q \beta$, then $\alpha \wedge \beta \neq \overline{0}$,*
(3) *$\alpha \overline{q} \beta$ iff $\alpha \leq \overline{1} - \beta$,*
(4) *$\alpha \leq \beta$ iff $x_t \in \alpha$ implies $x_t \in \beta$ iff $x_t q \alpha$ implies $x_t q \beta$ implies $x_t \overline{q} \alpha$,*
(5) *$x_t \overline{q} \bigvee_{i \in \Gamma} \beta_i$ iff there exists $i_0 \in \Gamma$ such that $x_t \overline{q} \beta_{i_0}$.*

Definition 2.2 ([19])**.** A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,
(O2) $\tau(\bigvee_{i \in \Gamma} \beta_i) \geq \bigwedge_{i \in \Gamma} \tau(\beta_i)$, for any $\{\beta_i\}_{i \in \Gamma} \subset I^X$,
(O3) $\tau(\beta_1 \wedge \beta_2) \geq \tau(\beta_1) \wedge \tau(\beta_2)$, for any $\beta_1,\ \beta_2 \in I^X$.

The pair $(X,\ \tau)$ is called a fuzzy topological space (for short, fts).

A fuzzy set α is called an r-fuzzy open (r-fo, for short) set, if $\tau(\alpha) \geq r$. A fuzzy set α is called an r-fuzzy closed (r-fc, for short) set, if $\tau(\overline{1} - \alpha) \geq r$.

Theorem 2.3 ([5])**.** *Let (X, τ) be a fts. Then for each $\alpha \in I^X$ and $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows: $C_\tau(\alpha,\ r) = \bigwedge\{\beta \in I^X : \alpha \leq \beta,\ \tau(\overline{1} - \beta) \geq r\}$. For $\alpha,\ \beta \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following statements:*

(C1) $C_\tau(\overline{0}, r) = \overline{0}$,
(C2) $\alpha \leq C_\tau(\alpha,\ r)$,
(C3) $C_\tau(\alpha,\ r) \vee C_\tau(\beta,\ r) = C_\tau(\alpha \vee \beta,\ r)$,
(C4) $C_\tau(\alpha,\ r) \leq C_\tau(\alpha,\ s)$ *if* $r \leq s$,
(C5) $C_\tau(C_\tau(\alpha,\ r),\ r) = C_\tau(\alpha,\ r)$.

Theorem 2.4 ([5])**.** *Let* (X, τ) *be a fts. Then for each* $\alpha \in I^X$ *and* $r \in I_0$*, we define an operator* $I_\tau : I^X \times I_0 \to I^X$ *as follows:* $I_\tau(\alpha,\ r) = \bigvee\{\beta \in I^X : \beta \leq \alpha,\ \tau(\beta) \geq r\}$. *For* $\alpha,\ \beta \in I^X$ *and* $r, s \in I_0$*, the operator* I_τ *satisfies the following statements:*

(I1) $I_\tau(\overline{1}, r) = \overline{1}$,
(I2) $I_\tau(\alpha,\ r) \leq \alpha$,
(I3) $I_\tau(\alpha,\ r) \wedge I_\tau(\beta,\ r) = I_\tau(\alpha \wedge \beta,\ r)$,
(I4) $I_\tau(\alpha,\ r) \leq I_\tau(\alpha,\ s)$, *if* $s \leq r$,
(I5) $I_\tau(I_\tau(\alpha,\ r),\ r) = I_\tau(\alpha,\ r)$,
(I6) $I_\tau(\overline{1} - \alpha, r) = \overline{1} - C_\tau(\alpha, r)$ *and* $C_\tau(\overline{1} - \alpha, r) = \overline{1} - I_\tau(\alpha,\ r)$.

Definition 2.5 ([18])**.** Let $(X,\ \tau)$ be a fts, $\alpha \in I^X$ and $r \in I_0$.

(i) A fuzzy set α is called r-fuzzy regular open (for short, r-fro), if $\alpha = I_\tau(C_\tau(\alpha, r), r)$.

(ii) A fuzzy set α is called r-fuzzy regular closed (for short, r-frc), if $\alpha = C_\tau(I_\tau(\alpha, r), r)$.

Definition 2.6. Let $(X,\ \tau)$ be a fts, $\alpha,\ \beta \in I^X$ and $r \in I_0$.

(i) A fuzzy set α is called r-generalized fuzzy closed [12] (for short, r-gfc), if $C_\tau(\alpha, r) \leq \beta$, whenever $\alpha \leq \beta$ and $\tau(\beta) \geq r$.

(ii) A fuzzy set α is called r-generalized fuzzy open [12] (for short, r-gfo), if $\overline{1} - \alpha$ is r-gfc.

(iii) A fuzzy set α is called r-regular generalized fuzzy closed (for short, r-rgfc), if $C_\tau(\alpha, r) \leq \beta$, whenever $\alpha \leq \beta$ and β is r-fro.

(iv) A fuzzy set α is called r-regular generalized fuzzy open (for short, r-rgfo), if $\overline{1} - \alpha$ is r-rgfc.

Definition 2.7 ([12])**.** Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f : (X,\ \tau) \to (Y,\ \eta)$ be a function.

(i) f is called fuzzy continuous (F-continuous), if $\eta(\beta) \leq \tau(f^{-1}(\beta))$, for each $\beta \in I^Y$.

(ii) f is called fuzzy open (F-open), if $\tau(\alpha) \leq \eta(f(\alpha))$, for each $\alpha \in I^X$.

(iii) f is called fuzzy closed (F-closed), if $\tau(\overline{1} - \alpha) \leq \eta(\overline{1} - f(\alpha))$, for each $\alpha \in I^X$.

Definition 2.8 ([12])**.** Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f : (X,\ \tau) \to (Y,\ \eta)$ be a function.

(i) f is called generalized fuzzy continuous (for short, gf-continuous), if $f^{-1}(\beta)$ is r-gfc, for each $\beta \in I^Y,\ r \in I_0$ with $\eta(\overline{1} - \beta) \geq r$.

(ii) f is called generalized fuzzy open (for short, gf-open), if $f(\alpha)$ is r-gfo, for each $\alpha \in I^X,\ r \in I_0$ with $\tau(\alpha) \geq r$.

(iii) f is called generalized fuzzy closed (for short, gf-closed), if $f(\alpha)$ is r-gfc for each $\alpha \in I^X,\ r \in I_0$ with $\tau(\overline{1} - \alpha) \geq r$.

(iv) f is called generalized fuzzy irresolute (for short, gf-irresolute), if $f^{-1}(\beta)$ is r-gfc for each r-gfc set $\beta \in I^Y$.

Definition 2.9 ([1])**.** Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f:(X,\ \tau)\to(Y,\ \eta)$ be a function.

(i) f is called fuzzy contra continuous (FC-continuous), if for each $\beta\in I^Y$, we have $\tau(\overline{1}-f^{-1}(\beta))\geq\eta(\beta)$.

(ii) f is called fuzzy contra open (FC-open), if for each $\alpha\in I^X$, we have $\eta(\overline{1}-f(\alpha))\geq\tau(\alpha)$.

(iii) f is called fuzzy contra closed (FC-closed), if for each $\alpha\in I^X$, we have $\eta(f(\alpha))\geq\tau(\overline{1}-\alpha)$.

Definition 2.10 ([1])**.** Let $f:(X,\ \tau)\to(Y,\ \eta)$ be a function and $r\in I_0$. Then f is called:

(i) generalized fuzzy contra continuous(GFC-continuous)(resp. generalized fuzzy contra irresolute (GFC-irresolute)), if $f^{-1}(\beta)$ is r-gfc, for each $\beta\in I^Y,\ \eta(\beta)\geq r$ (resp. $\beta\in I^Y$ is r-gfo),

(ii) generalized fuzzy contra open (GFC-open) (resp. generalized fuzzy contra irresolute open (GFC-irresolute open)), if $f(\alpha)$ is r-gfc, for each $\alpha\in I^X,\ \tau(\alpha)\geq r$ (resp. $\alpha\in I^X$ is r-gfo),

(iii) generalized fuzzy contra closed (GFC-closed) (resp. generalized fuzzy contra irresolute closed (GFC-irresolute closed)), if $f(\alpha)$ is r-gfo, for each $\alpha\in I^X,\ \tau(\overline{1}-\alpha)\geq r$ (resp. $\alpha\in I^X$ is r-gfc).

Definition 2.11 ([11])**.** A fts $(X,\ \tau)$ is said to be:

(i) r-FR_0, if $x_t\overline{q}C_\tau(y_s,r)$ implies $y_s\overline{q}C_\tau(x_t,r)$, for any distinct fuzzy points $x_t,y_s\in Pt(X)$,

(ii) r-FR_1, if $x_t\overline{q}C_\tau(y_s,r)$ implies that there exist $\beta_i\in I^X$ with $\tau(\beta_i)\geq r$, for $i\in\{1,2\}$ such that $x_t\in\beta_1,\ y_s\in\beta_2$ and $\beta_1\overline{q}\beta_2$, for any distinct fuzzy points $x_t,y_s\in Pt(X)$,

(iii) r-FR_2, if (or r-fuzzy regular) $x_t\overline{q}\alpha$ with $\tau(\overline{1}-\alpha)\geq r$ implies that there exist $\beta_i\in I^X$ with $\tau(\beta_i)\geq r$, for $i\in\{1,2\}$ such that $x_t\in\beta_1,\ \alpha\leq\beta_2$ and $\beta_1\overline{q}\beta_2$,

(iv) r-FR_3, iff(or r-fuzzy normal) $\alpha_1\overline{q}\alpha_2$ with $\tau(\overline{1}-\alpha_i)\geq r$, for $i\in\{1,2\}$ implies that there exist $\beta_i\in I^X$ with $\tau(\beta_i)\geq r$, for $i\in\{1,2\}$ such that $\alpha_i\in\beta_i$ and $\beta_1\overline{q}\beta_2$.

Definition 2.12 ([11])**.** A fts $(X,\ \tau)$ is said to be:

(i) r-FT_1, if $\tau(\overline{1}-x_t)\geq r$, for each $x_t\in Pt(X)$,

(ii) r-FT_2, if $x_t\overline{q}y_s$ implies that there exist $\beta_i\in I^X$ with $\tau(\beta_i)\geq r$, for $i\in\{1,2\}$ such that $x_t\in\beta_1,\ y_s\in\beta_2$ and $\beta_1\overline{q}\beta_2$,

(iii) r-$FT_{2\frac{1}{2}}$ if $x_t\overline{q}y_s$ implies that there exist $\beta_i\in I^X$ with $\tau(\beta_i)\geq r$, for $i\in\{1,2\}$ such that $x_t\in\beta_1,\ y_s\in\beta_2$ and $C_\tau(\beta_1,r)\overline{q}C_\tau(\beta_2,r)$,

(iv) r-FT_3, if it is r-FR_2 and r-FT_1,

(v) r-FT_4, if it is r-FR_3 and r-FT_1.

Definition 2.13 ([13])**.** A fts $(X,\ \tau)$ is said to be:

(i) r-GFR_2, if $x_t\overline{q}\alpha$ for each r-gfc $\alpha\in I^X$ implies that there exist $\beta_i\in I^X$ with $\tau(\beta_i)\geq r$ for $i\in\{1,2\}$ such that $x_t\in\beta_1,\ \alpha\leq\beta_2$ and $\beta_1\overline{q}\beta_2$,

(ii) r-GFR_3, if $\alpha_1\overline{q}\alpha_2$ for each r-gfc sets $\alpha_i\in I^X$ and $i\in\{1,2\}$ implies that there exist $\beta_i\in I^X$ with $\tau(\beta_i)\geq r$ such that $\alpha_i\leq\beta_i$ and $\beta_1\overline{q}\beta_2$.

3. Fuzzy regular closure operators

Definition 3.1. Let $(X,\ \tau)$ be a fts. Let $\alpha, \beta \in I^X$ and $r \in I_0$.

(i) The r-fuzzy regular closure of α, denoted by $RC_\tau(\alpha, r)$, and is defined by
$$RC_\tau(\alpha, r) = \bigwedge\{\beta \in I^X | \beta \geq \alpha, \beta \text{ is } r\text{-frc }\}.$$

(ii) The r-fuzzy regular interiror of α, denoted by $RI_\tau(\alpha, r)$, and is defined by
$$RI_\tau(\alpha, r) = \bigvee\{\beta \in I^X | \beta \leq \alpha, \beta \text{ is } r\text{-fro }\}.$$

Proposition 3.2. *A function $RC : I^X \times I_0 \to I^X$ is called a fuzzy regular closure operator if it satisfies the following conditions: for $\alpha,\ \beta \in I^X$ and $r, s \in I_0$,*

(C1) $RC(\overline{0}, r) = \overline{0}$,
(C2) $\alpha \leq RC(\alpha,\ r)$,
(C3) $RC(\alpha,\ r) \vee RC(\beta,\ r) = RC(\alpha \vee \beta,\ r)$,
(C4) $RC(\alpha,\ r) \leq RC(\alpha,\ s)$ *if* $r \leq s$,
(C5) $RC(RC(\alpha,\ r),\ r) = RC(\alpha,\ r)$.

The fuzzy regular closure operator RC generates a fuzzy topology $\tau_{RC}(\alpha) : I^X \to I$ given by

(C6) $\tau_{RC}(\alpha) = \bigvee\{r \in I |\ RC(\overline{1} - \alpha, r) = \overline{1} - \alpha\}$.

Proof. The proof of (C1)-(C5) is straightforward and the proof of (C6) follows from the Proposition 1.3 in [5]. □

In a similar pattern, a fuzzy regular interior operator was defined.

Proposition 3.3. *A mapping $RI : I^X \times I_0 \to I^X$ is called a fuzzy regular interior operator if, for $\alpha,\ \beta \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:*

(I1) $RI(\overline{1}, r) = \overline{1}$,
(I2) $\alpha \geq RI(\alpha,\ r)$,
(I3) $RI(\alpha,\ r) \wedge RI(\beta,\ r) = RI(\alpha \wedge \beta,\ r)$,
(I4) $RI(\alpha,\ r) \geq RI(\alpha,\ s)$ *if* $r \leq s$,
(I5) $RI(RI(\alpha,\ r),\ r) = RI(\alpha,\ r)$,
(I6) $RI(\overline{1} - \alpha,\ r) = \overline{1} - RC(\alpha,\ r)$ *and* $RC(\overline{1} - \alpha, r) = \overline{1} - RI(\alpha,\ r)$.

The fuzzy regular interior operator RI generates a fuzzy topology $\tau_{RI}(\alpha) : I^X \to I$ given by

(I7) $\tau_{RI}(\alpha) = \bigvee\{r \in I |\ RI(\alpha, r) = \alpha\}$.

Proof. Follows from the Proposition 3.2. □

Definition 3.4. Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a function and $r \in I_0$. Then f is called fuzzy regular continuous (for short, FR-continuous) if $f^{-1}(\alpha)$ is r-frc set in X for each $\alpha \in I^Y$ with $\sigma(\overline{1} - \alpha) \geq r$.

4. R-generalized regular fuzzy closed sets

Definition 4.1. Let $(X,\ \tau)$ be a fts. Let $\alpha,\ \beta \in I^X$ and $r \in I_0$.

(i) A fuzzy set α is called r-generalized regular fuzzy closed (for short, r-grfc) set, if $RC_\tau(\alpha, r) \leq \beta$, whenever $\alpha \leq \beta$ and $\tau(\beta) \geq r$.

(ii) A fuzzy set α is called r-generalized regular fuzzy open (for short, r-grfo) set, if $\overline{1} - \alpha$ is r-grfc.

Theorem 4.2. *Let $(X,\ \tau)$ be a fts.*

(1) *If α_1 and α_2 are r-grfc sets, then $\alpha_1 \vee \alpha_2$ is a r-grfc set.*
(2) *If α is r-grfc set and $\alpha \leq \beta \leq RC_\tau(\alpha, r)$, then β is a r-grfc set.*
(3) *If α is r-frc set, then α is a r-grfc set.*

Proof. (1) Let α_1 and α_2 be r-grfc sets, and $\alpha_1 \vee \alpha_2 \leq \beta$ such that $\tau(\beta) \geq r$. For $i \in \{1, 2\}$, $\alpha_i \leq \beta$ such that $\tau(\beta) \geq r$, we have $RC_\tau(\alpha_i, r) \leq \beta$. It implies $RC_\tau(\alpha_1 \vee \alpha_2, r) = RC_\tau(\alpha_1, r) \vee RC_\tau(\alpha_2, r) \leq \beta$. Hence $\alpha_1 \vee \alpha_2$ is a r-grfc.

(2) For $\beta \leq \gamma$ such that $\tau(\gamma) \geq r$, since α is r-grfc set and $\alpha \leq \beta$, $\alpha \leq \gamma$ implies $RC_\tau(\alpha, r) \leq \gamma$. Also, $\beta \leq RC_\tau(\alpha, r)$ implies $RC_\tau(\beta, r) \leq RC_\tau(RC_\tau(\alpha, r), r) = RC_\tau(\alpha, r) \leq \gamma$. Hence β is r-grfc.

(3) It is easily proved from $RC_\tau(\alpha, r) = \alpha$. □

Remark 4.3. The intersection of any two r-grfc set need not be r-grfc from the following example.

Example 4.4. Let $X = \{a,\ b,\ c\}$ be a set and $\alpha,\ \beta,\ \gamma \in I^X$ are defined as follows: $\alpha(a) = 0.8,\ \alpha(b) = 0.4,\ \alpha(c) = 0.7$; $\beta(a) = 0.6,\ \beta(b) = 0.5,\ \beta(c) = 0.8$; $\gamma(a) = 0.6,\ \gamma(b) = 0.4,\ \gamma(c) = 0.7$. We define a fuzzy topology $\tau : I^X \to I$ as follows.

$$\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \alpha = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = 1/3$, α and β are r-grfc set. But $\alpha \wedge \beta = \gamma$ is not r-grfc because $\gamma \leq \gamma$, $\tau(\gamma) \geq r$, $RC_\tau(\gamma, r)(= \overline{1}) \nleq \gamma$. Clearly β is r-grfc set but not r-frc set, so the converse of Theorem 4.2(3) is not true.

Remark 4.5. Every r-grfc set is r-gfc (resp.r-rgfc) set. But not conversely.

Example 4.6. Let $X = \{a,\ b,\ c\}$ be a set and $\gamma \in I^X$ be defined as $\gamma(a) = 0.5,\ \gamma(b) = 0.7,\ \gamma(c) = 0.9$. We define a fuzzy topology $\tau : I^X \to I$ as follows.

$$\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \alpha = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = 1/3$, $\overline{1} - \gamma$ is r-gfc set, where $\overline{1} - \gamma \leq \gamma$, $\tau(\gamma) \geq r$, $C_\tau(\overline{1} - \gamma, r)(= \overline{1} - \gamma) \leq \gamma$. But $\overline{1} - \gamma$ is not r-grfc because $\overline{1} - \gamma \leq \gamma$, $\tau(\gamma) \geq r$, $RC_\tau(\overline{1} - \gamma, r)(= \overline{1}) \nleq \gamma$.

Example 4.7. In Example 4.4, γ is r-rgfc set but not r-grfc set.

The following theorem is easily proved from Theorems 2.4 and 4.2.

Theorem 4.8. *Let $(X,\ \tau)$ be a fts. Let $\alpha,\ \beta \in I^X$ and $r \in I_0$.*

(1) *α is r-grfo iff $\beta \leq RI_\tau(\alpha, r)$, whenever $\beta \leq \alpha$ and $\tau(\overline{1} - \beta) \geq r$.*
(2) *The intersection of any two r-grfo sets is a r-grfo set.*
(3) *If $RI_\tau(\alpha, r) \leq \beta \leq \alpha$ and α is r-grfo, then β is r-grfo.*
(4) *If α is r-fro set and $r \in I_0$, then α is a r-grfo set.*

Theorem 4.9. *Let $(X,\ \tau)$ be a fts. For each $r \in I_0$, $\alpha \in I^X$, we define an operator $GRC_\tau : I^X \times I_0 \to I^X$ as follows:*

$$GRC_\tau(\alpha, r) = \bigwedge\{\beta \in I^X \mid \alpha \leq \beta,\ \beta \text{ is } r\text{-grfc }\}.$$

For $\alpha,\ \beta \in I^X$ and $r \in I_0$, it holds the following properties.

(1) $GRC_\tau(\overline{0}, r) = \overline{0}$.
(2) $\alpha \leq GRC_\tau(\alpha, r)$.
(3) $GRC_\tau(\alpha, r) \vee GRC_\tau(\beta, r) = GRC_\tau(\alpha \vee \beta, r)$.
(4) $GRC_\tau(GRC_\tau(\alpha, r), r) = GRC_\tau(\alpha, r)$.
(5) *If α is r-grfc, then* $GRC_\tau(\alpha, r) = \alpha$.
(6) $GRC_\tau(\alpha, r) \leq RC_\tau(\alpha, r)$.
(7) $RC_\tau(GRC_\tau(\alpha, r), r) = GRC_\tau(RC_\tau(\alpha, r), r) = RC_\tau(\alpha, r)$.

Proof. (1), (2) and (5) are easily proved from the definition of GRC_τ.

(3) Since $\alpha,\ \beta \leq \alpha \vee \beta$, we have

$$GRC_\tau(\alpha, r) \vee GRC_\tau(\beta, r) = GRC_\tau(\alpha \vee \beta, r).$$

Suppose $GRC_\tau(\alpha, r) \vee GRC_\tau(\beta, r) \not\geq GRC_\tau(\alpha \vee \beta, r)$. There exists $x \in X$ and $t \in (0, 1)$ such that

$$GRC_\tau(\alpha, r)(x) \vee GRC_\tau(\beta, r)(x) < t < GRC_\tau(\alpha \vee \beta, r)(x). \tag{4.1}$$

Since $GRC_\tau(\alpha, r)(x) < t$ and $GRC_\tau(\beta, r)(x) < t$, there exist r-grfc sets $\alpha_1,\ \beta_1$ with $\alpha \leq \alpha_1$ and $\beta \leq \beta_1$ such that $\alpha_1(x) < t,\ \beta_1(x) < t$. Since $\alpha \vee \beta \leq \alpha_1 \vee \beta_1$ and $\alpha_1 \vee \beta_1$ is r-grfc, from Theorem 4.2(1), we have $GRC_\tau(\alpha \vee \beta, r)(x) \leq (\alpha_1 \vee \beta_1)(x) < t$. It is a contrdiction for (4.1).

(4) From (2), we only show $GRC_\tau(\alpha, r) \geq GRC_\tau(GRC_\tau(\alpha, r), r)$. Suppose $GRC_\tau(\alpha, r) \not\geq GRC_\tau(GRC_\tau(\alpha, r), r)$. There exists $x \in X$ and $t \in (0, 1)$ such that

$$GRC_\tau(\alpha, r)(x) < t < GRC_\tau(GRC_\tau(\alpha, r), r)(x). \tag{4.2}$$

Since $GRC_\tau(\alpha, r)(x) < t$, there exist r-grfc set α_1 with $\alpha \leq \alpha_1$ such that

$$GRC_\tau(\alpha, r)(x) \leq \alpha_1(x) < t.$$

Since $\alpha \leq \alpha_1$, we have $GRC_\tau(\alpha, r) \leq \alpha_1$. Again $GRC_\tau(GRC_\tau(\alpha, r), r) \leq \alpha_1$. Then $GRC_\tau(GRC_\tau(\alpha, r), r)(x) \leq \alpha_1(x) < t$. It is contradiction for (4.2). Thus $GRC_\tau(\alpha, r) \geq GRC_\tau(GRC_\tau(\alpha, r), r)$.

(6) Since $RC_\tau(\alpha, r)$ is r-grfc, it is easily proved.

(7) Trivially, $GRC_\tau(RC_\tau(\alpha, r), r) = RC_\tau(\alpha, r)$. We only show that

$$RC_\tau(GRC_\tau(\alpha, r), r) = RC_\tau(\alpha, r).$$

Since $\alpha \leq GRC_\tau(\alpha, r)$, $RC_\tau(GRC_\tau(\alpha, r), r) \geq RC_\tau(\alpha, r)$. Suppose

$$RC_\tau(GRC_\tau(\alpha, r), r) \not\leq RC_\tau(\alpha, r).$$

Then there exists $x \in X$ and $t \in (0, 1)$ such that

$$RC_\tau(GRC_\tau(\alpha, r), r)(x) > t > RC_\tau(\alpha, r)(x).$$

Since $RC_\tau(\alpha, r)(x) < t$, by the definition of RC_τ, there exists $\gamma \in I^X$ with $\alpha \leq \gamma$ and $\tau(\overline{1} - \gamma) \geq r$ such that

$$RC_\tau(GRC_\tau(\alpha, r), r)(x) > t > \gamma(x) \geq RC_\tau(\alpha, r)(x).$$

On the other hand, since $\gamma = RC_\tau(\gamma, r)$ is r-grfc, $\alpha \le \gamma$ implies

$$GRC_\tau(\alpha, r) \le GRC_\tau(\gamma, r) = GRC_\tau(RC_\tau(\gamma, r), r) = RC_\tau(\gamma, r) = \gamma.$$

Thus $RC_\tau(GRC_\tau(\alpha, r), r) \le \gamma$. It is a contradiction.
So $RC_\tau(GRC_\tau(\alpha, r), r) \le RC_\tau(\alpha, r)$. □

Theorem 4.10. *Let $(X,\ \tau)$ be a fts. For each $r \in I_0$, $\alpha \in I^X$, we define an operator $GRI_\tau : I^X \times I_0 \to I^X$ as follows:*

$$GRI_\tau(\alpha, r) = \bigvee\{\beta \in I^X \mid\ \beta \le \alpha,\ \beta \text{ is } r\text{-grfo }\}.$$

Then $GRI_\tau(\overline{1} - \alpha, r) = \overline{1} - GRC_\tau(\alpha, r)$.

Proof. For each $\alpha \in I^X$ and $r \in I_0$, we have

$$\begin{aligned} GRI_\tau(\overline{1} - \alpha, r) &= \bigvee\{\beta \in I^X \mid\ \beta \le \overline{1} - \alpha,\ \beta \text{ is } r\text{-grfo }\}. \\ &= \overline{1} - \bigwedge\{\overline{1} - \beta \in I^X \mid\ \overline{1} - \beta \ge \alpha,\ \overline{1} - \beta \text{ is } r\text{-grfc }\}. \\ &= \overline{1} - GRC_\tau(\alpha, r). \end{aligned}$$

□

Example 4.11. In Example 4.4, For $r = 1/3$, α and β are r-grfc sets, that is $GRC_\tau(\gamma, r) = \gamma$, but γ is not r-grfc set. Then, the converse of Theorem 4.9(5) is not true.

From the discussion, we get the following relations.

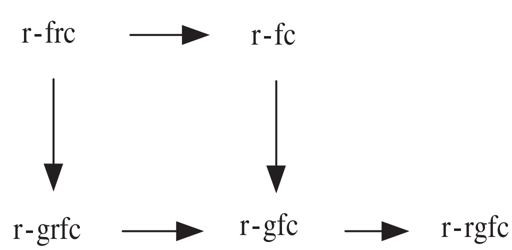

Diagram - I

5. Generalized regular fuzzy continuous mappings

Definition 5.1. Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f : (X,\ \tau) \to (Y,\ \eta)$ be a function.

(i) f is called generalized regular fuzzy continuous (for short, grf-continuous), if $f^{-1}(\beta)$ is r-grfc, for each $\beta \in I^Y,\ r \in I_0$ with $\eta(\overline{1} - \beta) \ge r$.

(ii) f is called generalized regular fuzzy open (for short, grf-open), if $f(\alpha)$ is r-grfo, for each $\alpha \in I^X,\ r \in I_0$ with $\tau(\alpha) \ge r$.

(iii) f is called generalized regular fuzzy closed (for short, grf-closed), if $f(\alpha)$ is r-grfc, for each $\alpha \in I^X,\ r \in I_0$ with $\tau(\overline{1} - \alpha) \ge r$.

Definition 5.2. Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. A function $f : (X,\ \tau) \to (Y,\ \eta)$ is called generalized regular fuzzy irresolute (for short, grf-irresolute), if $f^{-1}(\beta)$ is r-grfc for each r-grfc set $\beta \in I^Y$.

Remark 5.3. (1) Every grf-continuous function is gf-continuous (resp. rgf-continuous).

(2) A function $f : (X,\ \tau) \to (Y,\ \eta)$ is grf-continuous iff $f^{-1}(\beta)$ is r-grfo, for each $\beta \in I^Y,\ r \in I_0$ with $\eta(\beta) \geq r$.

(3) A function $f : (X,\ \tau) \to (Y,\ \eta)$ is grf-irresolute iff $f^{-1}(\beta)$ is r-grfo for each r-grfo set $\beta \in I^Y$.

Example 5.4. Let $X = Y = \{a,\ b,\ c\}$ be a sets and $\gamma \in I^X$ be defined as $\gamma(a) = 0.5,\ \gamma(b) = 0.7,\ \gamma(c) = 0.9$. We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \alpha = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = 1/3$, $\tau(\overline{1} - (\overline{1} - \gamma)) \geq r$, $\overline{1} - \gamma$ is r-gfc set, where $\overline{1} - \gamma \leq \gamma$, $\tau(\gamma) \geq r$, $C_\tau(\overline{1} - \gamma, r)(= \overline{1} - \gamma) \leq \gamma$. But $\overline{1} - \gamma$ is not r-grfc because $\overline{1} - \gamma \leq \gamma$, $\tau(\gamma) \geq r$, $RC_\tau(\overline{1} - \gamma, r)(= \overline{1}) \not\leq \gamma$. Thus an identity function $f : (X, \tau) \to (Y, \tau)$ is gf-continuous but not grf-continuous.

Example 5.5. Let $X = \{a,\ b,\ c\}$ and $Y = \{p, q, r\}$ be a sets and $\alpha \in I^X,\ \beta \in I^Y$ be defined as $\alpha(a) = 0.6,\ \alpha(b) = 0.4,\ \alpha(c) = 0.7;\ \beta(p) = 0.4,\ \beta(q) = 0.6,\ \beta(r) = 0.3$. We define a fuzzy topology $\tau, \eta : I^X \to I$ as follows.

$$\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \alpha = \alpha, \\ 0 & \text{otherwise,} \end{cases} \qquad \eta(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

For $\eta(\overline{1} - \alpha) \geq r$ with $r = 1/3$, α is r-rgfc set on (X, τ), because $\alpha \leq \overline{1}$ and $\overline{1}$ is r-fro, $C_\tau(\alpha, r)(= \overline{1}) \leq \overline{1}$. But α is not r-grfc set because $\alpha \leq \alpha$, $\tau(\alpha) \geq r$, $RC_\tau(\alpha, r)(= \overline{1}) \not\leq \alpha$. Thus an identity function $f : (X, \tau) \to (Y, \eta)$ is rgf-continuous but not grf-continuous.

Theorem 5.6. *Let $f : (X,\ \tau) \to (Y,\ \eta)$ be grf-continuous. Then following statements hold.*

(1) $f(GRC_\tau(\alpha, r)) \leq RC_\eta(f(\alpha), r)$, *for each $\alpha \in I^X$ and $r \in I_0$.*
(2) $GRC_\tau(f^{-1}(\beta), r) \leq f^{-1}(RC_\eta(\beta, r))$, *for each $\beta \in I^Y$ and $r \in I_0$.*
(3) $f^{-1}(RI_\eta(\beta, r)) \leq GRI_\tau(f^{-1}(\beta), r)$, *for each $\beta \in I^Y$ and $r \in I_0$.*

Proof. (1) Since $RC_\eta(RC_\eta(f(\alpha), r), r) = RC_\eta(f(\alpha), r)$ and $\tau_{RC_\eta} = \eta$ from Theorems 2.3 and 2.4, we have $\eta(\overline{1} - RC_\eta(f(\alpha), r)) \geq \eta(\overline{1} - C_\tau(f(\alpha), r)) \geq r$. Since f is grf-continuous, $f^{-1}(RC_\eta(f(\alpha), r))$ is r-grfc. Since

$$\alpha \leq f^{-1}(f(\alpha), r)) \leq f^{-1}(RC_\eta(f(\alpha), r)),$$

we have

$$GRC_\tau(\alpha, r) \leq f^{-1}(RC_\eta(f(\alpha), r)).$$

Hence $f(GRC_\tau(\alpha, r)) \leq f(f^{-1}(RC_\eta(f(\alpha), r))) \leq RC_\eta(f(\alpha), r)$.

(2) For all $\beta \in I^Y$, $r \in I_0$, let $\alpha = f^{-1}(\beta)$. Then by (1),

$$f(GRC_\tau(f^{-1}(\beta), r)) \leq RC_\eta(f(f^{-1}(\beta)), r) \leq RC_\eta(\beta, r).$$

Thus $GRC_\tau(f^{-1}(\beta), r) \leq f^{-1}(RC_\eta(\beta, r))$.

(3) It is easily proved from (2), and Proposition 3.3 and Theorem 4.10. □

Theorem 5.7. *Let $f : (X, \tau) \to (Y, \eta)$ be grf-closed. Then we have $GRC_\eta(f(\alpha), r) \leq f(RC_\tau(\alpha, r))$, for each $\alpha \in I^X$ and $r \in I_0$.*

Proof. For all $\alpha \in I^X$, $r \in I_0$, since $\tau(\overline{1} - RC_\tau(\alpha, r)) \geq r$, $f(RC_\tau(\alpha, r))$ is r-grfc. Then $GRC_\eta(f(\alpha), r) \leq f(RC_\tau(\alpha, r))$. □

The following Theorem is similarly proved as Theorems 5.6 and 5.7.

Theorem 5.8. *Let $f : (X, \tau) \to (Y, \eta)$ be grf-open. Then the following statements hold.*

(1) *$f(RI_\tau(\alpha, r)) \leq GRI_\eta(f(\alpha), r)$, for each $\alpha \in I^X$ and $r \in I_0$.*
(2) *$RI_\tau(f^{-1}(\beta), r) \leq f^{-1}(GRI_\eta(\beta, r))$, for each $\beta \in I^Y$ and $r \in I_0$.*

Theorem 5.9. *Let (X, τ) and (Y, η) be fts's. Let $f : (X, \tau) \to (Y, \eta)$ be grf-irresolute. Then the following statements hold.*

(1) *$f : (X, \tau) \to (Y, \eta)$ is grf-continuous.*
(2) *$f(GRC_\tau(\alpha, r)) \leq GRC_\eta(f(\alpha), r)$, for each $\alpha \in I^X$.*
(3) *$GRC_\tau(f^{-1}(\beta), r) \leq f^{-1}(GRC_\eta(\beta, r))$, for each $\beta \in I^Y$.*

Proof. (1) Let β be r-frc set. Since every r-frc set is r-fc, $\eta(\overline{1} - \beta) \geq r$. By Theorem 4.2(3), β is r-grfc. Since f is grf-irresolute, $f^{-1}(\beta)$ is r-grfc.

(2) Suppose there exist $\alpha \in I^X$ and $r \in I_0$ such that

$$f(GRC_\tau(\alpha, r)) \nleq GRC_\eta(f(\alpha), r).$$

Then there exist $y \in Y$ and $t \in I_0$ such that

$$f(GRC_\tau(\alpha, r))(y) > t > GRC_\eta(f(\alpha), r)(y).$$

If $f^{-1}(\{y\}) = \overline{0}$, it is a contradiction since $f(GRC_\tau(\alpha, r)) = \overline{0}$. If $f^{-1}(\{y\}) \neq \overline{0}$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(GRC_\tau(\alpha, r))(y) \geq GRC_\tau(\alpha, r)(x) > t > GRC_\eta(f(\alpha), r)(f(x)).$$

Since $GRC_\eta(f(\alpha), r)(f(x)) < t$, by the Definition of GRC_η, there exists r-grfc $\beta \in I^Y$ and $f(\alpha) \leq \beta$ such that

$$GRC_\eta(f(\alpha), r)(f(x)) \leq \beta(f(x)) < t.$$

Since f is grf-irresolute, for r-grfc $\beta \in I^Y$, $f^{-1}(\beta)$ is r-grfc and $\alpha \leq f^{-1}(\beta)$. Thus

$$GRC_\tau(\alpha, r)(x) \leq f^{-1}(\beta)(x) = \beta(f(x)) < t.$$

It is a contradiction. Hence $f(GRC_\tau(\alpha, r)) \leq GRC_\eta(f(\alpha), r)$, for all $\alpha \in I^X$ and $r \in I_0$.

(3) It is similarly proved as Theorem 5.6(2). □

Example 5.10. (1) The converse of Theorem 5.6(1) is not true. In Example 5.5, The identity function $f : (X, \tau) \to (Y, \eta)$ is not grf-continuous. By a similar method as in Example 4.11, we can obtain $GRC_\tau(\alpha, r) = \alpha$, for each $\alpha \in I^X$ and $r \in I_0$. Furthermore, $RC_\eta : I^X \times I_0 \to I^X$ as follows.

$$RC_\eta(\alpha, r) = \begin{cases} \overline{0} & \text{if } \alpha = \overline{0}, \\ \frac{1}{2} & \text{if } \alpha = \beta, \\ \overline{1} & \text{otherwise.} \end{cases}$$

Hence $f(GRC_\tau(\alpha, r)) \leq RC_\eta(f(\alpha), r)$, for each $\alpha \in I^X$ and $r \in I_0$.

(2) The converse of Theorem 5.9(2) is not true. Since α is a r-grfc in (Y, η) but not r-grfc set on (X, τ). Thus an identity function $f : (X, \tau) \to (Y, \eta)$ is not a grf-irresolute map. By a similar method as in Example 4.11, $GRC_\tau(\alpha, r) = GRC_\eta(\alpha, r)$, for each $\alpha \in I^X$ and $r \in I_0$.

Definition 5.11. A fts $(X,\ \tau)$ is called $FRT_{1/2}$, if for each r-grfc $\beta \in I^X$ and $r \in I_0$ is r-frc.

Theorem 5.12. *A fts $(X,\ \tau)$ is called $FRT_{1/2}$ iff $GRC_\tau(\alpha, r) = RC_\tau(\alpha, r)$ for each $\alpha \in I^X$ and $r \in I_0$.*

Proof. Let $(X,\ \tau)$ be $FRT_{1/2}$. Then by definition of GRC_τ and RC_τ, we have $GRC_\tau(\alpha, r) = RC_\tau(\alpha, r)$ for each $\alpha \in I^X$ and $r \in I_0$.

Conversely, suppose $(X,\ \tau)$ is not $FRT_{1/2}$. Then there exist r-grfc $\beta \in I^X$ and $r \in I_0$ such that $\tau(\overline{1} - \beta) < r$. Thus $GRC_\tau(\beta, r) = \beta$ but $RC_\tau(\beta, r) \neq \beta$. So $GRC_\tau(\beta, r) \neq RC_\tau(\beta, r)$. □

The following Theorems are easily proved.

Theorem 5.13. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. Let $f : (X,\ \tau) \to (Y,\ \eta)$ be a function.*

(1) If $(X,\ \tau)$ is $FRT_{1/2}$, then f is grf-continuous iff f is FR-continuous.

(2) If $(Y,\ \eta)$ is $FRT_{1/2}$, then f is grf-continuous iff f is grf-irresolute.

(3) If $(X,\ \tau)$ and $(Y,\ \eta)$ is $FRT_{1/2}$, then f is FR-continuous iff f is grf-continuous iff f is grf-irresolute.

Theorem 5.14. *Let $f : (X,\ \tau) \to (Y,\ \eta)$ and $g : (Y,\ \eta) \to (Z,\ \sigma)$ be grf-continuous and $(Y,\ \eta)$ is $FRT_{1/2}$. Then $g \circ f : (X,\ \tau) \to (Z,\ \sigma)$ is grf-continuous.*

Theorem 5.15. *Let $f : (X,\ \tau) \to (Y,\ \eta)$ be grf-irresolute and $g : (Y,\ \eta) \to (Z,\ \sigma)$ be grf-continuous. Then $g \circ f : (X,\ \tau) \to (Z,\ \sigma)$ is grf-continuous.*

Remark 5.16. The composition map of grf-continuous maps need not be grf-continuous.

Example 5.17. Let $X = \{a,\ b,\ c\}$, $Y = \{p, q, r\}$, and $Z = \{x, y, z\}$ be a sets and $\alpha \in I^X, \beta \in I^Y, \gamma \in I^Z$ be defined as $\alpha(a) = 0.5,\ \alpha(b) = 0.7,\ \alpha(c) = 0.4$; $\beta(p) = 0.5,\ \beta(q) = 0.5,\ \beta(r) = 0.5$; $\gamma(x) = 0.5,\ \gamma(y) = 0.3,\ \gamma(z) = 0.6$. We define a fuzzy topology $\tau, \eta, \sigma : I^X \to I$ as follows.

$$\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \alpha \\ 0 & \text{otherwise,} \end{cases} \quad \eta(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \beta \\ 0 & \text{otherwise,} \end{cases} \quad \sigma(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Then the identity function $f : (X, \tau) \to (Y, \eta)$ and $g : (Y, \eta) \to (Z, \sigma)$ is grf-continuous. But $g \circ f : (X, \tau) \to (Z, \sigma)$ is not grf-continuous.

Example 5.18. The converse of Theorem 5.9(1) is not true. Let $X = \{a,\ b,\ c\}$, $Y = \{p, q, r\}$ be a sets and $\alpha \in I^X,\ \beta, \gamma \in I^Y$ be defined as $\alpha(a) = 0.4,\ \alpha(b) = 0.5,\ \alpha(c) = 0.7;\ \beta(p) = 0.4,\ \beta(q) = 0.5,\ \beta(r) = 0.6;\ \gamma(p) = 0.4,\ \gamma(q) = 0.5,\ \gamma(r) = 0.7$. We define a fuzzy topology $\tau, \eta : I^X \to I$ as follows.

$$\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \alpha \\ 0 & \text{otherwise,} \end{cases} \qquad \eta(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

For $r = 1/3$, $\eta(\overline{1} - (\overline{1} - \beta)) \geq r$, $\overline{1} - \beta$ is grfc set in (X, τ). Then the identity function $f : (X, \tau) \to (Y, \eta)$ is grf-continuous but not grf-irresolute, since the fuzzy set γ is r-grfc set in (Y, η) but not r-grfc set in (X, τ).

From the discussion we get the following relations:

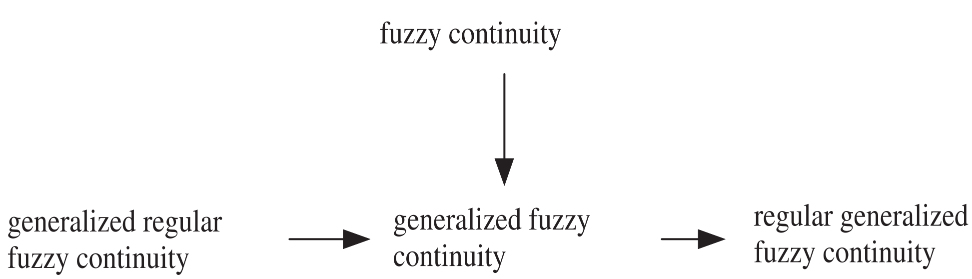

Diagram - II

6. Generalized regular fuzzy contra continuous functions

Definition 6.1. Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a function and $r \in I_0$. Then f is called fuzzy regular contra continuous (for short, FRC-continuous), if $f^{-1}(\alpha)$ is r-frc set in X, for each $\alpha \in I^Y$ with $\sigma(\alpha) \geq r$.

Definition 6.2. Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a function and $r \in I_0$. Then f is called:

(i) generalized regular fuzzy contra continuous (resp. generalized regular fuzzy contra irresolute), if $f^{-1}(\beta)$ is r-grfc, for each $\beta \in I^Y,\ \sigma(\beta) \geq r$ (resp. $\beta \in I^Y$ is r-grfo),

(ii) generalized regular fuzzy contra open (resp. generalized regular fuzzy contra irresolute open), if $f(\alpha)$ is r-grfc, for each $\alpha \in I^X,\ \tau(\alpha) \geq r$ (resp. $\alpha \in I^X$ is r-grfo),

(iii) generalized regular fuzzy contra closed (resp. generalized regular fuzzy contra irresolute closed), if $f(\alpha)$ is r-grfo, for each $\alpha \in I^X$, $\tau(\overline{1}-\alpha) \geq r$ (resp. $\alpha \in I^X$ is r-grfc).

$GRFC$-continuous, $GRFC$-irresolute, $GRFC$-open, $GRFC$-irresolute open, $GRFC$-closed, $GRFC$-irresolute closed are abbreviated to generalized regular fuzzy contra continuous, generalized regular fuzzy contra irresolute, generalized regular fuzzy contra open, generalized regular fuzzy contra irresolute open, generalized regular fuzzy contra closed, generalized regular fuzzy contra irresolute closed.

From the above definition and Example 2.10 in [1] it is clear that the following implication is true but the reverse implication is not true.

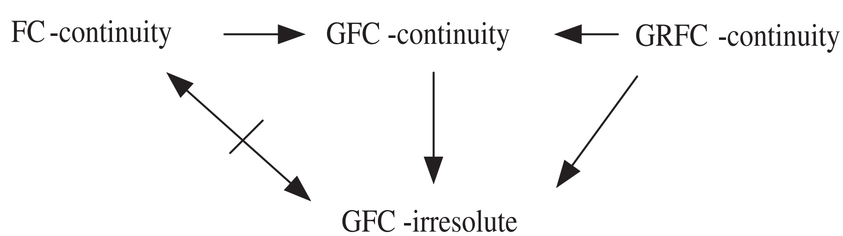

Diagram -III

Example 6.3. Let $X = \{a,\ b,\ c\}$, $Y = \{p, q, r\}$ be a sets and $\alpha \in I^X, \beta \in I^Y$ be defined as $\alpha(a) = 0.6,\ \alpha(b) = 0.7,\ \alpha(c) = 0.5$; $\beta(p) = 0.4,\ \beta(q) = 0.3,\ \beta(r) = 0.5$. We define a fuzzy topology $\tau, \sigma : I^X \to I$ as follows.

$$\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \alpha \\ 0 & \text{otherwise,} \end{cases} \qquad \sigma(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is GFC-continuous but not $GRFC$-continuous.

Example 6.4. Let $X = \{a,\ b,\ c\}$, $Y = \{p, q, r\}$ be a sets and $\alpha \in I^X, \beta \in I^Y$ be defined as $\alpha(a) = 0.5,\ \alpha(b) = 0.7,\ \alpha(c) = 0.9$; $\beta(p) = 0.5,\ \beta(q) = 0.3,\ \beta(r) = 0.1$. We define a fuzzy topology $\tau, \sigma : I^X \to I$ as follows.

$$\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \alpha \\ 0 & \text{otherwise,} \end{cases} \qquad \sigma(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is GFC-irresolute but not $GRFC$-continuous.

Theorem 6.5. *A mapping $f : (X,\ \tau) \to (Y,\ \sigma)$ is GRFC-continuous (resp. GRFC-irresolute) iff $f^{-1}(\beta)$ is r-grfo for each $\beta \in I^Y$, $\sigma(\overline{1}-\beta) \geq r$ (resp. $\beta \in I^Y$ is r-grfc) and $r \in I_0$.*

Theorem 6.6. *Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be bijective GRFC-continuous mapping and $r \in I_0$. Then following statements hold.*

(1) $f(GRC_\tau(\alpha, r)) \geq RI_\sigma(f(\alpha), r)$, *for each* $\alpha \in I^X$.
(2) $GRC_\tau(f^{-1}(\beta), r) \geq f^{-1}(RI_\sigma(\beta, r))$, *for each* $\beta \in I^Y$.
(3) $f^{-1}(RC_\sigma(\beta, r)) \geq GRI_\tau(f^{-1}(\beta), r)$, *for each* $\beta \in I^Y$.

Proof. (1) For each $\alpha \in I^X$, $\sigma(RI_\sigma(f(\alpha), r)) \geq r$. Since f is GRFC-continuous, $f^{-1}(RI_\sigma(f(\alpha), r))$ is r-grfc set of X. Since $f(\alpha) \geq RI_\sigma(f(\alpha), r)$, $f^{-1}(f(\alpha)) \geq f^{-1}(RI_\sigma(f(\alpha), r))$. Since $\alpha = f^{-1}(f(\alpha))$,

$$GRC_\tau(\alpha, r) \geq f^{-1}(RI_\sigma(f(\alpha), r)) \text{ (by } f \text{ is injective).}$$

Since f is surjective, $f(GRC_\tau(\alpha, r)) \geq RI_\sigma(f(\alpha), r)$.

(2) For all $\beta \in I^Y$ and $r \in I_0$, put $\alpha = f^{-1}(\beta)$. From (1), we have

$$f(GRC_\tau(f^{-1}(\beta), r)) \geq RI_\sigma(f(f^{-1}(\beta)), r).$$

Since f is surjective, $\beta = f(f^{-1}(\beta))$. Thus $f(GRC_\tau(f^{-1}(\beta), r)) \geq RI_\sigma(\beta, r)$. Since f is injective,

$$GRC_\tau(f^{-1}(\beta), r) = f^{-1}(f(GRC_\tau(f^{-1}(\beta), r))) \geq f^{-1}(RI_\sigma(\beta, r)).$$

(3) It is easily proved from $RC_\tau(\overline{1} - \alpha, r) = \overline{1} - RI_\tau(\alpha, r)$ and $GRC_\tau(\overline{1} - \alpha, r) = \overline{1} - GRI_\tau(\alpha, r)$. □

Theorem 6.7. *Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be GRFC-open mapping and $r \in I_0$. Then following statements hold.*

(1) $f(RI_\tau(\alpha, r)) \leq GRC_\sigma(f(\alpha), r)$, *for each* $\alpha \in I^X$.
(2) $RI_\tau(f^{-1}(\beta), r) \leq f^{-1}(GRC_\sigma(\beta, r))$, *for each* $\beta \in I^Y$.
(3) $GRC_\sigma(f(\alpha), r) \leq f(GRC_\tau(\alpha, r))$, *for each* $\alpha \in I^X$, $\tau(\alpha) \geq r$.

Proof. (1) For each $\alpha \in I^X$ and $r \in I_0$, since $RI_\tau(\alpha, r) \leq \alpha$, $f(RI_\tau(\alpha, r)) \leq f(\alpha)$. Since f is GRFC-open, $f(RI_\tau(\alpha, r))$ is r-grfc. Then $f(RI_\tau(\alpha, r)) \leq GRC_\sigma(f(\alpha), r)$.

(2) For each $\beta \in I^Y$ and $r \in I_0$, put $\alpha = f^{-1}(\beta)$. From (1),

$$f(RI_\tau(f^{-1}(\beta), r) \leq GRC_\sigma(f(f^{-1}(\beta)), r) \leq GRC_\sigma(\beta, r).$$

Then $RI_\tau(f^{-1}(\beta), r) \leq f^{-1}(GRC_\sigma(\beta, r))$.

(3) Since f is GRFC-open,

$$GRC_\sigma(f(\alpha), r)) = f(\alpha) \leq f(GRC_\tau(\alpha, r))$$

for each $\alpha \in I^X$, $\tau(\alpha) \geq r$ and $r \in I_0$. □

Theorem 6.8. *Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be GRFC-closed mapping, $\alpha \in I^X$ and $r \in I_0$. Then following statements hold.*

(1) $f(RC_\tau(\alpha, r)) \geq GRI_\sigma(f(\alpha), r)$.
(2) $GRI_\sigma(f(\alpha), r) \geq f(GRI_\tau(\alpha, r))$, *for each* $\alpha \in I^X$, $\tau(\overline{1} - \alpha) \geq r$.

Proof. (1) For each $\alpha \in I^X$ and $r \in I_0$, since $\alpha \leq RC_\tau(\alpha, r)$, $f(\alpha) \leq f(RC_\tau(\alpha, r))$. Since $\tau(\overline{1} - RC_\tau(\alpha, r)) \geq r$, $f(RC_\tau(\alpha, r))$ is r-grfo set of Y. Then $f(RC_\tau(\alpha, r)) \geq GRI_\sigma(f(\alpha), r)$.

(2) Since f is GRFC-closed,

$$GRI_\sigma(f(\alpha), r) = f(\alpha) \geq f(GRI_\tau(\alpha, r))$$

for each $\alpha \in I^X$, $\tau(\overline{1} - \alpha) \geq r$ and $r \in I_0$. □

Theorem 6.9. *Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a bijective mapping. f is GRFC-closed (resp. GRFC-irresolute closed) iff f is is GRFC-open (resp. GRFC-irresolute open).*

Remark 6.10. The composition of two $GRFC$-continuous (resp. $GRFC$-open and $GRFC$-closed) mappings need not be $GRFC$-continuous (resp. $GRFC$-open and $GRFC$-closed) as shown by the following example.

Example 6.11. Let $X = \{a,\ b,\ c\}$, $Y = \{p, q, r\}$, and $Z = \{x, y, z\}$ be a sets and $\alpha \in I^X, \beta \in I^Y, \gamma \in I^Z$ be defined as $\alpha(a) = 0.5,\ \alpha(b) = 0.7,\ \alpha(c) = 0.4$; $\beta(p) = 0.5,\ \beta(q) = 0.5,\ \beta(r) = 0.5$; $\gamma(x) = 0.5,\ \gamma(y) = 0.7,\ \gamma(z) = 0.4$. We define a fuzzy topology $\tau, \sigma, \eta : I^X \to I$ as follows:

$$\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \alpha \\ 0 & \text{otherwise,} \end{cases} \quad \sigma(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \beta \\ 0 & \text{otherwise,} \end{cases} \quad \eta(\alpha) = \begin{cases} 1 & \text{if } \alpha = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \alpha = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Then the identity function $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ is $GRFC$-continuous. But $g \circ f : (X, \tau) \to (Z, \eta)$ is not $GRFC$-continuous.

Theorem 6.12. *Let $f : (X,\ \tau) \to (Y,\ \sigma)$ and $g : (Y,\ \sigma) \to (Z,\ \eta)$ be mappings. Then $g \circ f$ is:*

(1) GRFC-continuous, if g is F-continuous (resp. GRFC-continuous) and f is GRFC-continuous (resp. grf-irresolute),

(2) GRFC-open, if f is F-open (resp. GRFC-open) and g is GRFC-open (resp. grf-irresolute closed),

(3) GRFC-closed, if f is F-closed (resp. GRFC-closed) and g is GRFC-closed (resp. grf-irresolute open),

(4) GRFC-irresolute, if g is grf-irresolute (resp. GRFC-irresolute) and f is GRFC-irresolute (resp. grf-irresolute),

(5) GRFC-irresolute open, if f is grf-irresolute open (resp. GRFC-irresolute open) and g is GRFC-irresolute open (resp. grf-irresolute closed),

(6) GRFC-irresolute closed, if f is grf-irresolute closed (resp. GRFC-irresolute closed) and g is GRFC-irresolute closed (resp. grf-irresolute open).

Theorem 6.13. *Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a mapping and $r \in I_0$. The following statements hold.*

(1) If $(X,\ \tau)$ is $FRT_{1/2}$, then the concepts of FRC-continuity and GRFC-continuity are equivalent.

(2) If $(Y,\ \sigma)$ is $FRT_{1/2}$, then the concepts of GRFC-continuity and GRFC-irresolute are equivalent.

(3) If $(X,\ \tau)$ and $(Y,\ \sigma)$ is $FRT_{1/2}$, then the concepts of FRC-continuity, GRFC-continuity and GRFC-irresolute are equivalent.

Theorem 6.14. *Let $f : (X,\ \tau) \to (Y,\ \sigma)$ and $g : (Y,\ \sigma) \to (Z,\ \eta)$ be GRFC-continuous and $(Y,\ \sigma)$ is $FRT_{1/2}$. Then $g \circ f : (X,\ \tau) \to (Z,\ \eta)$ is GRF-continuous.*

7. Applications of r-generalized regular fuzzy closed sets

Definition 7.1. A fts $(X,\ \tau)$ is said to be:

(i) r-GRF-regular, if for each r-grfc $\alpha \in I^X$, $x_t\overline{q}\alpha$ implies that there exist $\beta_i \in I^X$ with $\tau(\beta_i) \geq r$, for $i \in \{1,2\}$ such that $x_t \in \beta_1$, $\alpha \leq \beta_2$ and $\beta_1\overline{q}\beta_2$,

(ii) r-GRF-normal, if for each r-grfc sets $\alpha_i \in I^X$, for $i \in \{1,2\}$, $\alpha_1\overline{q}\alpha_2$ implies that there exist $\beta_i \in I^X$ with $\tau(\beta_i) \geq r$ such that $\alpha_i \leq \beta_i$ and $\beta_1\overline{q}\beta_2$.

Theorem 7.2. *Let (X,τ) be a fts and $r \in I_0$. Then the following statements are equivalent:*

(1) (X,τ) is r-GRF-regular,

(2) if $x_t \in \alpha$, for each r-grfo $\alpha \in I^X$, there exists $\beta \in I^X$ with $\tau(\beta) \geq r$ such that $x_t \in \beta \leq RC_\tau(\beta, r) \leq \alpha$.

(3) if $x_t\overline{q}\alpha$, for each r-grfc $\alpha \in I^X$, there exists $\beta_i \in I^X$ with $\tau(\beta_i) \geq r$, for $i \in \{1,2\}$ such that $x_t \in \beta_1,\ \alpha \leq \beta_2$ and $RC_\tau(\beta_1, r)\overline{q}RC_\tau(\beta_2, r)$.

Proof. (1)⇒(2) Let $x_t \in \alpha$ for each r-grfo α. Then $x_t\overline{q}(\overline{1}-\alpha)$ for r-grfc $(\overline{1}-\alpha)$. Since (X,τ) is r-GRF-regular, there exist $\beta,\gamma \in I^X$ with $\tau(\beta) \geq r,\ \tau(\gamma) \geq r$ such that $x_t \in \beta,\ \overline{1}-\alpha \leq \gamma$ and $\beta\overline{q}\gamma$. It implies $x_t \in \beta \leq \overline{1}-\gamma \leq \alpha$. Since $\tau(\gamma) \geq r$, $x_t \in \beta \leq RC_\tau(\beta, r) \leq \alpha$.

(2)⇒(3) Let $x_t\overline{q}\alpha$ for each r-grfc. Then $x_t \in \overline{1}-\alpha$ for r-grfo $\overline{1}-\alpha$. By (2), there exists $\beta \in I^X$ with $\tau(\beta) \geq r$ such that $x_t \in \beta \leq RC_\tau(\beta, r) \leq \overline{1}-\alpha$. Since $\tau(\beta) \geq r$, β is r-grfo and $x_t \in \beta$. Again, by (2), there exists $\beta_1 \in I^X$ with $\tau(\beta_1) \geq r$ such that $x_t \in \beta_1 \leq RC_\tau(\beta_1, r) \leq \beta \leq RC_\tau(\beta, r) \leq \overline{1}-\alpha$. It implies $\alpha \leq (\overline{1}-RC_\tau(\beta, r) = RI_\tau(\overline{1}-\beta, r)) \leq \overline{1}-\beta$. Put $\beta_2 = RI_\tau(\overline{1}-\beta, r)$. Then $\tau(\beta_2) \geq r$. So, $RC_\tau(\beta_2, r) \leq \overline{1}-\beta \leq \overline{1}-RC_\tau(\beta_1, r)$, that is, $RC_\tau(\beta_1, r)\overline{q}RC_\tau(\beta_2, r)$.

(3)⇒(1) It is trivial. □

Theorem 7.3. *Let (X,τ) be fts and $r \in I_0$. Then the following statements are equivalent:*

(1) (X,τ) is r-GRF-normal,

(2) if $\gamma \leq \alpha$, for each r-grfc set $\gamma \in I^X$ and r-grfo set $\alpha \in I^X$, there exists $\beta \in I^X$ with $\tau(\beta) \geq r$ such that $\gamma \leq \beta \leq RC_\tau(\beta, r) \leq \alpha$,

(3) If $\alpha_1\overline{q}\alpha_2$, for each r-grfc sets $\alpha_i \in I^X$, for $i \in \{1,2\}$, there exists $\beta_i \in I^X$ with $\tau(\beta_i) \geq r$ such that $\alpha_i \leq \beta_i$ and $RC_\tau(\beta_1, r)\overline{q}RC_\tau(\beta_2, r)$.

Proof. It is similarly proved as in Theorem 7.2. □

Theorem 7.4. *Let (X,τ) be a fts. Then the following statements hold.*

(1) Every r-GRF-regular (r-GRF-normal) space is r-GFR_2 (r-GFR_3).

(2) Every r-GRF-regular (r-GRF-normal) space is r-FR_2 (r-FR_3).

(3) A fts (X,τ) is r-GRF-regular (r-GRF-normal) space iff it is r-GFR_2 (r-GFR_3) and $FRT_{1/2}$.

(4) A fts (X,τ) is r-GRF-regular (r-GRF-normal) space iff it is r-FR_2 (r-FR_3) and $FRT_{1/2}$.

Proof. (1) For $x_t\overline{q}\alpha$ with r-grfc set $\alpha \in I^X$, α is r-gfc. Since (X,τ) is r-GRF-regular, there exist $\beta_i \in I^X$ with $\tau(\beta_i) \geq r$, for $i \in \{1,2\}$ such that $x_t \in \beta_1$, $\alpha \leq \beta_2$ and $\beta_1\overline{q}\beta_2$. Then (X,τ) is r-GFR_2.

(2) For $x_t\overline{q}\alpha$ with r-frc set $\alpha \in I^X$, α is r-grfc. Since (X,τ) is r-GRF-regular, there exist $\beta_i \in I^X$ with $\tau(\beta_i) \geq r$, for $i \in \{1,2\}$ such that $x_t \in \beta_1$, $\alpha \leq \beta_2$ and $\beta_1\overline{q}\beta_2$. Then (X,τ) is r-FR_2.

(3) $(\Rightarrow)$ Let (X,τ) be r-GRF-regular. By (1). We only show that (X,τ) is $FRT_{1/2}$. If $\alpha \in \{\overline{0},\overline{1}\}$, then α is r-grfc and $\tau(\alpha) = \overline{1}$. Let $\alpha \notin \{\overline{0},\overline{1}\}$ be r-grfc. Then for $x_t \in \overline{1}-\alpha$ with r-grfc α, by Theorem 7.2(2), there exists $\beta_{x_t} \in I^X$ with $\tau(\beta_{x_t}) \geq r$ such that $x_t \in \beta_{x_t} \leq RC_\tau(\beta_{x_t},r) \leq \overline{1}-\alpha$. Thus

$$\overline{1}-\alpha = \bigvee_{x_t\in\overline{1}-\alpha}\{\beta_{x_t} |\ RC_\tau(\beta_{x_t},r) \leq \overline{1}-\alpha,\ \tau(\beta_{x_t}) \geq r\}.$$

So $\tau(\overline{1}-\alpha) \geq r$. Hence (X,τ) is $FRT_{1/2}$.

$(\Leftarrow)$ It is easily proved.

(4) $(\Rightarrow)$ It is easily proved from (2) and (3).

$(\Leftarrow)$ It is easily proved. □

Theorem 7.5. *Let (X,τ) be a fts. Then following statements hold.*

(1) *If (X,τ) is r-GRF-regular, then it is r-FR_0.*
(2) *If (X,τ) is r-GRF-regular, then it is r-$FT_{2\frac{1}{2}}$.*
(3) *If (X,τ) is r-GRF-regular, then it is r-FT_3.*
(4) *If (X,τ) is r-GRF-normal and r-FR_0, then it is r-GRF-regular.*
(5) *If (X,τ) is r-GRF-normal and r-FR_0, then it is r-FT_4.*

Proof. (1) Let $x_t\overline{q}RC_\tau(y_s,r)$, for any distinct fuzzy points $x_t, y_s \in Pt(X)$. Since $RC_\tau(y_s,r)$ is r-grfc and (X,τ) is r-GRF-regular, there exist $\beta_i \in I^X$ with $\tau(\beta_i) \geq r$, for $i \in \{1,2\}$ such that $x_t \in \beta_1$, $y_s \in RC_\tau(y_s,r) \leq \beta_2$ and $\beta_1\overline{q}\beta_2$. It implies $x_t \in \beta_1 \leq \overline{1}-\beta_2 \leq \overline{1}-RC_\tau(y_s,r) \leq \overline{1}-y_s$. Then, $RC_\tau(x_t,r) \leq \overline{1}-y_s$, that is, $y_s\overline{q}RC_\tau(x_t,r)$. Thus (X,τ) is r-FR_0.

(2) Let $x_t\overline{q}y_s$, for any distinct fuzzy points $x_t, y_s \in Pt(X)$. Since (X,τ) is r-GRF-regular, by (1) and since every r-frc set is r-grfc set, y_s is r-grfc. Then by Theorem 7.2(3), there exist $\beta_i \in I^X$ with $\tau(\beta_i) \geq r$, for $i \in \{1,2\}$ such that $x_t \in \beta_1$, $y_s \in \beta_2$ and $RC_\tau(\beta_1,r)\overline{q}RC_\tau(\beta_2,r)$. Thus (X,τ) is r-$FT_{2\frac{1}{2}}$.

(3) Let (X,τ) is r-GRF-regular. By (2) and Theorem 7.4(2), (X,τ) is r-$FT_{2\frac{1}{2}}$ and FR_2. Since r-$FT_{2\frac{1}{2}}$ implies r-FT_1, (X,τ) is r-FT_3.

(4) Let $x_t\overline{q}\alpha$ for each r-grfc α. Since (X,τ) is r-FR_0, x_t is r-grfc. Since (X,τ) is r-GRF-normal, there exist $\beta_i \in I^X$ with $\tau(\beta_i) \geq r$, for $i \in \{1,2\}$ such that $x_t \in \beta_1$, $\alpha \in \beta_2$ and $\beta_1\overline{q}\beta_2$. Then (X,τ) is r-GRF-regular.

(5) Let (X,τ) be r-GRF-normal and r-FR_0. Since r-GRF-regular implies r-$FT_{2\frac{1}{2}}$ implies r-FT_1, by (4), (X,τ) is r-FT_1. Then by Theorem 7.4(2), (X,τ) is r-FT_4. □

Theorem 7.6. *If $f : (X,\tau_1) \to (Y,\tau_2)$ be F-continuous, GRF-irresolute closed and injective map and (Y,τ_2) is r-GRF-regular (resp. r-GRF-normal), then (X,τ_1) is r-GRF-regular (resp. r-GRF-normal).*

Proof. Let $x_t\overline{q}\alpha$ for each r-grfc set $\alpha \in I^X$. Since f is GRF-irresolute closed, $f(\alpha)$ is r-grfc. Since f is injective, $x_t\overline{q}\alpha$ implies $f(x_t)\overline{q}f(\alpha)$. Since (Y,τ_2) is r-GRF-regular, there exists $\beta_i \in I^X$ with $\tau_2(\beta_i) \geq r$ and $i \in \{1,2\}$ such that $f(x_t) \in \beta_1$, $f(\alpha) \leq \beta_2$ and $\beta_1\overline{q}\beta_2$. Since f is F-continuous, $x_t \in f^{-1}(\beta_1)$, $\alpha \leq f^{-1}(\beta_2)$ with $\tau_2(f^{-1}(\beta_i)) \geq$

r and $i \in \{1,2\}$ and $f^{-1}(\beta_1)\overline{q}f^{-1}(\beta_2)$. Then (X,τ_1) is r-GRF-regular. Other case is similarly proved. □

Theorem 7.7. *If $f : (X,\tau_1) \to (Y,\tau_2)$ be GRF-irresolute, F-open, F-closed and surjective map and (X,τ_1) is r-GRF-regular (resp. r-GRF-normal), then (Y,τ_2) is r-GRF-regular (resp. r-GRF-normal).*

Proof. Let $y_s \in \beta$ for each r-grfo $\beta \in I^Y$. Since f is GRF-irresolute and surjective, then there exists $x \in f^{-1}(\{y\})$ such that $x_s \in f^{-1}(\beta)$ with r-grfo set $f^{-1}(\beta)$. Since (X,τ_1) is r-GRF-regular, by Theorem 7.2(2), there exists $\gamma \in I^X$ with $\tau_1(\gamma) \geq r$ such that $x_s \in \gamma \leq RC_{\tau_1}(\gamma,r) \leq f^{-1}(\beta)$. It implies $y_s \in f(\gamma) \leq f(RC_{\tau_1}(\gamma,r)) \leq \beta$. Since f is F-open and F-closed, $\tau_2(f(\gamma)) \geq r$ and $\tau_2(\overline{1} - f(RC_{\tau_1}(\gamma,r))) \geq r$. Then $y_s \in f(\gamma) \leq RC_{\tau_2}(f(\gamma),r) \leq RC_{\tau_2}(f(RC_{\tau_1}(\gamma,r)),r) \leq \beta$. Thus (Y,τ_2) is r-GRF-regular. Other case is similarly proved. □

Example 7.8. Let $X = \{a,b,c\}$ be a set. We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \{\overline{0},\overline{1}\}, \\ \frac{1}{3} & \text{if } \alpha = \chi_{\{a,b\}}, \\ \frac{1}{2} & \text{if } \alpha = c_1, \\ 0 & \text{otherwise.} \end{cases}$$

We can obtain a fuzzy closure operator $C_\tau : I^X \times I_0 \to I^X$ as follows:

$$C_\tau(\alpha,r) = \begin{cases} \overline{0} & \text{if } \alpha = \overline{0}, r \in I_0, \\ \chi_{\{a,b\}} & \text{if } \overline{0} \neq \alpha \leq \chi_{\{a,b\}}, 0 < r \leq 1/2, \\ c_1 & \text{if } \overline{0} \neq \alpha \leq c_1, 0 < r \leq 1/3, \\ \overline{1} & \text{otherwise.} \end{cases}$$

(1) For $0 < r \leq 1/3$, Since $\chi_{\{a,b\}}\overline{q}c_1$ with $\tau(\chi_{\{a,b\}}) \geq r$ and $\tau(c_1) \geq r$, (X,τ) is r-FR_3. Also, for $0 < r \leq 1/3$, Since $x_t\overline{q}(C_\tau(c_s,r) = c_1)$, for $x \in \{a,b\}$ iff $c_s\overline{q}(C_\tau(x_t,r) = \chi_{\{a,b\}})$, (X,τ) is r-FR_0.

(2) For $0 < r \leq 1/3$, since (X,τ) is r-FR_3 and r-FR_0 from (1), (X,τ) is r-FR_2.

(3) For $0 < r \leq 1/3$, $\chi_{\{a,b\}}$ and $b_{0.5} \vee c_1$ are r-grfc sets. Let $\chi_{\{a,b\}}\overline{q}b_{0.5} \vee c_1$. For each $\alpha,\beta \in \{\chi_{\{a,b\}},\overline{1}\}$ with $\chi_{\{a,b\}} \in \alpha$ and $b_{0.5} \vee c_1 \in \beta$, we give $\alpha q \beta$. Then (X,τ) is neither r-GRF-normal nor r-GRF-regular. Thus by (1), (2) and (3), the converse of Theorem 7.4(2) is not true. Also, by (1), (2) and (3), the converse of Theorem 7.5(1) is not true.

Example 7.9. Let $X = \{a,b,c\}$ and $\beta,\gamma \in I^X$ defined as follows: $\beta(a) = 0.5, \beta(b) = 0.5, \beta(c) = 0.5; \gamma(a) = 0.5, \gamma(b) = 0.6, \gamma(c) = 0.6$. Define fuzzy topology $\tau = I^X \to I$ as follows: $\tau(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \{\overline{0},\overline{1}\}, \\ \frac{1}{3} & \text{if } \alpha = \gamma, \\ \frac{1}{2} & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$

For $0 < r \leq 1/3$, (X,τ) is GFR_3. Also, for $0 < r \leq 1/3$, β and $\overline{1} - \beta$ are r-grfc sets. Let $\beta\overline{q}(\overline{1} - \beta)$. Since $\beta \leq \beta$ and $\overline{1} - \beta \leq \gamma$ with $\tau(\beta) \geq r$ and $\tau(\gamma) \geq r$, we give

$\beta q \gamma$. Hence (X, τ) is not r-GRF-normal. Then, the converse of Theorem 7.4(1) is not true.

8. Conclusions

Šostak's fuzzy topology has been recently of major interest among fuzzy topologies. In section 2, we have introduced r-generalized regular fuzzy closed (open) sets in fuzzy topological spaces of Šostak's and studied some of its fuzzy set theoritic properties. In sections 3 and 4, we have also introduced generalized regular fuzzy continuous (irresolute) functions and generalized regular fuzzy contra continuity in Šostak's fuzzy topological spaces. Further, we have examined interrelationship of r-generalized regular fuzzy closed sets and other r-generalizations of closed sets in Šostak's topological spaces.

Acknowledgements. The Second author sincerely acknowledges the financial support from U. G. C. New Delhi, grant no. F4-1/2006(BSR)/7-254/2009(BSR)-22.10.2013. India in the form of UGC-BSR Fellowship.

References

[1] S. E. Abbas and I. M. Taha, Weaker forms of fuzzy contra continuity in fuzzy topological spaces, J. Fuzzy Math. 20 (4) (2012) 857–876.
[2] G. Balasubramanian and P. Sundaram, On some generalizations of fuzzy continuous functions, Fuzzy Sets and Systems 86 (1997) 93–100.
[3] B. Bhattacharya and J. Chakraborty, Generalized regular fuzzy closed sets and their Applications, J. Fuzzy Math. 23 (1) (2015) 227–239.
[4] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[5] K. C. Chattopadhyay and S. K. Samanta, Fuzzy topology, Fuzzy Sets and Systems 54 (1993) 207–212.
[6] K. C. Chattopadhyay, R. N. Hazra and S. K. Samanta, Gradation of openness Fuzzy Sets and Systems 49 (1992) 237–242.
[7] U. Hohle, Upper semicontinuous fuzzy sets and applications, J. Math. Anall. Appl. 78 (1980) 659–673.
[8] U. Hohle and A. P. Šostak, A general theory of fuzzy topological spaces, Fuzzy Sets and Systems 73 (1995) 131–149.
[9] U. Hohle and A. P. Šostak, Axiomatic foundations of fixed-basis fuzzy topology, The Handbooks of fuzzy sets series 3, Kluwer academic publishers, Dordrecht (Chapter 3) 1999.
[10] A. Kandil and M. E. El-Shafee, Regularity axioms in fuzzy topological spaces and FR_i-proximities, Fuzzy Sets and Systems 27 (1988) 217–231.
[11] Y. C. Kim, Separation axioms in smooth fuzzy topological spaces, J. of Korea Fuzzy Logic and Intelligent Systems 9 (1) (1999) 57–62.
[12] Y. C. Kim and J. M. Ko, R-generalized fuzzy closed sets, J. Fuzzy Math. 12 (1) (2004) 7–21.
[13] Y. C. Kim and J. W. Park, Some properties of r-generalized fuzzy closed sets, Far East J. of Math. Science 7 (3) (2002) 253–268.
[14] T. Kubiak, On fuzzy topologies, Ph.D. Thesis, A. Mickiewicz, Poznan 1985.
[15] T. Kubiak and A. P. Šostak, Lower set-valued fuzzy topologies, Quaestions Math. 20 (3) (1997) 423–429.
[16] N. Levine, Generalized closed sets in topology, Rend. Cirec. Math. Palermo 19 (1970) 89–96.
[17] J. H. Park and J. K. Park, On regular generalized fuzzy closed sets and generalization of fuzzy continuous functions, Indian. J. Pure. Appl. Math. 34 (7) (2003) 1013–1024.
[18] Seok Jong Lee and Eun Pyo Lee, Fuzzy r-regular open sets and fuzzy almost r-continuous maps, Bull. Korean Math. Soc. 39 (3) (2002) 441–453.

[19] A. P. Šostak, On a fuzzy topological structure, Rend. Circ. Matem. Palermo Ser II. 11 (1986) 89–103.
[20] A. P. Šostak, Two decades of fuzzy topology : Basic ideas, Notion and results, Russian Math. Surveys 44 (6) (1989) 125–186.
[21] A. P. Šostak, Basic structures of fuzzy topology, J. Math. Sci. 78 (6) (1996) 662–701.

A. Vadivel (avmaths@gmail.com)
Department of Mathematics, Annamalai University, Annamalainagar, Tamil Nadu-608 002, India

E. Elavarasan (maths.aras@gmail.com)
Research Scholar, Department of Mathematics, Annamalai University, Annamalainagar, Tamil Nadu-608 002, India