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On completely homogeneous L-topological spaces

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Abstract. In this paper we investigate completely homogeneous L-topological spaces. The smallest completely homogeneous L- topology on a set X containing an L- set f is called the principal completely homogeneous L- topology generated by f. Here we also study the principal completely homogeneous L- topological spaces generated by an L- set and characterize completely homogeneous Alexandroff discrete L- topological spaces.

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1. Introduction

In 1965, L. A. Zadeh introduced fuzzy set theory describing fuzziness mathematically for the first time [16]. Based on this notion C. L. Chang introduced fuzzy topology and studied its properties[3]. Later an extensive study of fuzzy topological space has been carried out by several mathematicians and they developed a theory of fuzzy topological spaces. Larson studied the concept of complete homogeneity in topological spaces and characterized all spaces which are minimum and maximum with respect to a topological property [12]. He also determined a characterization of completely homogeneous topological spaces. In [5] T. P. Johnson defined the concept of a completely homogeneous fuzzy topological space in an analogous way and generalized the result of Larson R. E. for fuzzy topological spaces. He characterized all fuzzy topological spaces which are minimum and maximum with respect to a fuzzy topological property. He considered the lattice of completely homogeneous fuzzy topologies also [7]. In [10] the authors extended complete homogeneity to L-topological spaces.

In this paper we continue the study of completely homogeneous L-topological spaces. We characterize Alexandroff discrete completely homogeneous L- topological spaces when L is a complete chain. Also we introduce the principal completely homogeneous L- topological spaces generated by an L-set and obtain certain properties when L is an F-lattice.

If we take $L = \{0, 1\}$, then it is clear that the lattice L^X is isomorphic to the lattice $P(X)$, the power set of X and the L-topologies become crisp. The simplest F-lattice other than $L = \{0, 1\}$ is $L = \{0, a, 1\}$ where a is different from 0 and 1. Also we study the completely homogeneous L- topological spaces and characterize principal completely homogeneous L- topological spaces when $L = \{0, a, 1\}$.

2. Preliminaries

Here we include certain definitions and known results needed for the subsequent development.

Definition 2.1 ([13]). A completely distributive lattice L is called an F-lattice, if L has an order reversing involution $' : L \to L$.

Throughout this paper X stands for a nonempty set, L for an F-lattice with the smallest element 0 and largest element 1 and L^X for the lattice of L-fuzzy sets or L-sets of X. An L- set with constant membership $\alpha \in L$ is denoted by $\underline{\alpha}$. The L-subset x_l with $x \in X$ and $l \in L, l \neq 0$ defined by

$$x_l(y) = \begin{cases} l & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

is called an L- point in X with support x and value l. An L- set x^l, with $x \in X$ and $l \in L, l \neq 1$ is defined by

$$x^l(y) = \begin{cases} l & \text{if } y = x \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.2 ([13]). Let X and Y be two sets and $h : X \to Y$ be a function. Then for any L-set f in X, $h(f)$is an L-set in Y defined by

$$h(f)(y) = \begin{cases} \bigvee\{f(x) : x \in X \;\; h(x) = y\}, & h^{-1}(y) \neq \phi \\ 0, & h^{-1}(y) = \phi. \end{cases}$$

For an L-set g in Y, we define $h^{-1}(g)(x) = g(h(x))$, for all $x \in X$.

Definition 2.3 ([13]). Let X be a nonempty set, L an F-lattice, $\delta \subset L^X$. Then δ is called an L-fuzzy topology or L-topology on X, and (X, δ) is called an L-fuzzy topological space, or L-topological space for short, if δ satisfies the following three conditions:

(i) $\underline{0}, \underline{1} \in \delta$,
(ii) $f \wedge g \in \delta$ for all $f, g \in \delta$,
(iii) $\vee A \in \delta$ for all $A \subset \delta$.

Every element in δ is called an L- open subset in X.

Definition 2.4 ([13]). Let (X, δ) and $(Y, \delta^{'})$ be any two L-topological spaces and h be a mapping from (X, δ) to $(Y, \delta^{'})$.

We say h is an L- continuous function from X to Y, if $h^{-1}(f^{'}) \in \delta$, for every $f^{'}$ in $\delta^{'}$ and h is said to be open, if $h(f) \in \delta^{'}$, for every $f \in \delta$.

A bijection h from (X, δ) onto $(Y, \delta^{'})$ is called an L-homeomorphism if both h and h^{-1} are L- continuous. So a necessary and sufficient condition for a permutation h of a set X to be an L- homeomorphism of (X, δ) on to itself is that $f \in \delta$ if and only if $f \circ h \in \delta$. The set of all L- homeomorphisms of an L- topological space (X, δ) onto itself is a group under composition, which is a subgroup of the group of all permutations on the set X. It is called the group of L- homeomorphisms of (X, δ) and is denoted by $LFH(X, \delta)$.

An L- topological space $(X,\ \delta)$ is called homogeneous if for any two points x and y in X, there exists an L-fuzzy homeomorphism h in $(X,\ \delta)$ such that $h(x) = y$ [4]. Several authors studied homogeneity in L-topological spaces[4, 8, 9, 10]. The order of the $LFH(X, \delta)$ depends on the structure of the L- topological space. For example, if the L-topological space is homogeneous, the order of the group of L-homeomorphisms on $(X,\ \delta)$ is greater than or equal to the cardinality of X. Thus if the number of L-fuzzy homeomorphisms increase, the homogeneity of the L- topological space also increases. The extremity of this happens when the group of L-homeomorphisms equals $S(X)$, the set of all permutations of the set X. Then we say that $(X,\ \delta)$ is a completely homogeneous L- topological space.

Definition 2.5 ([6])**.** An L-topological space (X, δ) is called a completely homogeneous space if every bijection of X onto itself is an L- homeomorphism.

From the definition of completely homogeneous L-topological space we immediately see that indiscrete space $\{\underline{0}, \underline{1}\}$, discrete space 2^X, L- discrete space L^X and L-topological space generated by L- points having same membership value are completely homogeneous L-topological spaces. Also we have every completely homogeneous L-topological space is homogeneous [10].

Notations

- If A is a given set, we will use $|A|$ to denote the cardinality of A.
- If (X, δ) is an L- topological space, then define
  (1) $\overline{\delta} = \delta \setminus \{\underline{1}, \underline{0}\}$,
  (2) $\Lambda_f = \{f(x) : x \in X\}$,
  (3) $\Lambda_\delta = \{f(x) : x \in X, f \in \overline{\delta}\}$,
  (4) $\Lambda_f^X = \{g : X \to \Lambda_f\}$.

3. Completely homogeneous L- topological spaces

In this section we study some properties of completely homogeneous L-topological spaces. Let X be a set and f be an L- set of X. We define for $c \in L \setminus \{1\}$, the set $f_{[c]} = \{x \in X : f(x) > c\}$. Then $f_{[c]}$ is called a c- level of f. Let δ be an L- topology on X. Then the family $T_{[c]} = \{f_{[c]} : f \in \delta\}$ is a topology on X and is called $c-$ level topology of X [11].

The relation between completely homogeneous L-topological space and its level topologies is given in the following theorem.

Theorem 3.1. *Let (X, δ) be an L-topological space which is completely homogeneous, then all the level topologies of δ are completely homogeneous.*

Proof. First we claim that the group of all L- homeomorphisms $LFH(X,\delta)$ of an L-topological space (X,δ) is a subgroup of the group of homeomorphisms of the level topologies. We have $h \in LFH(X,\delta)$ if and only if h is a bijection and both $h(f)$ and $h^{-1}(f)$ are in δ for all $f \in \delta$. Let $(X, T_{[c]})$ be a level topology of (X,δ) and $U \in T_{[c]}$, where $c \in L \setminus \{1\}$. Then $U = f_{[c]}$, for some f in δ and

$$\begin{aligned} h(U) = h(f_{[c]}) &= \{h(x) : f(x) > c\} = \{x : f(h^{-1}(x)) > c\} \\ &= \{x \in x : f \circ h^{-1}(x) > c\} = \{x \in X : h(f)(x) > c\} \\ &= h(f)_{[c]}. \end{aligned}$$

Thus $h(U) \in T_{[c]}$. Similarly,

$$\begin{aligned} h^{-1}(U) = h^{-1}(f_{[c]}) &= \{h^{-1}(x) : f(x) > c\} = \{x : f(h(x)) > c\} \\ &= \{x \in x : f \circ h(x) > c\} = \{x \in X : h^{-1}(f)(x) > c\} \\ &= (h^{-1}(f))_{[c]}. \end{aligned}$$

So h is a homeomorphism on $(X, T_{[c]})$. Hence the group of L- homeomorphisms of an L-topological space is a subgroup of the group of homeomorphisms of the level topologies. Here the group of L- homeomorphisms of (X,δ) is the group of all permutations on X. So the group of homeomorphisms of the level topologies are also the group of all permutations on X. Therefore the level topologies of a completely homogeneous L-topological space $(X,\ \delta)$ are completely homogeneous. □

Remark 3.2. The converse of the theorem 3.1 is not true.

Example 3.3. Let $X = \{a,b,c\}$, $L = \{1, .5, 0\}$ with usual order and δ be the L-topology having base

$$\mathcal{B} = \{f_1, f_2, f_3, g_1, g_2, g_3\},$$

where

$$\begin{aligned} f_1(a) = .5, f_1(b) = 0, f_1(c) = 0, \\ f_2(a) = 0, f_2(b) = .5, f_2(c) = 0, \\ f_3(a) = 0, f_3(b) = 0, f_3(c) = .5, \\ g_1(a) = 1, g_1(b) = .5, g_1(c) = 0, \\ g_2(a) = 0, g_2(b) = 1, g_2(c) = .5, \\ \text{and } g_3(a) = .5, g_3(b) = 0, g_3(c) = 1. \end{aligned}$$

It is easy to verify that $LFH(X,\delta) = \{(a,b,c),(a,c,b), I\}$ where I is the identity permutation on X and the level topologies $T_{[\frac{1}{2}]}$ and $T_{[0]}$ are the discrete topology on X. Here all the level topologies are completely homogeneous but the L-topology is not completely homogeneous.

Corollary 3.4. *Let (X,δ) be an L-topological space and $\Lambda_f \subseteq \{0,c\}$ for every $f \in \delta \setminus \{\underline{1}\}$, where c is a nonzero element in L. Then the group L- homeomorphisms of an L-topological space is equal to the group of homeomorphisms of the level topology $T_{[0]}$.*

Proof. We have the group of L- homeomorphisms $LFH(X,\delta)$ of an L-topological space (X,δ) is a subgroup of the group of homeomorphisms $H(X,T_{[0]})$ of the level topology. Let $h \in H(X,T_{[0]})$ and $f \in \delta$. Then $f_{[0]} \in T_{[0]}$ and consequently $h(f_{[0]})$ and $h^{-1}(f_{[0]})$ are in $T_{[0]}$. But $h(f_{[0]}) = (h \circ f)_{[0]}$. Since f takes only one non zero value, $h(f) \in \delta$. Similarly we can prove that $h^{-1}(f) \in \delta$. Thus h is an L-homeomorphism. So $H(X,T_{[0]}) \subseteq LFH(X,\delta)$. Hence the result holds. □

An immediate consequence of the Theorem 3.1 is the following.

Remark 3.5. Observe that if (X,δ) is an L- topological space and $\Lambda_f \subseteq \{0,c\}$ for every $f \in \delta \setminus \{\underline{1}\}$, where c is a nonzero element in L, then the L-topological space (X,δ) is completely homogeneous if and only if the level topology $T_{[0]}$ is completely homogeneous.

The next definition appears in [1].

Definition 3.6. A nonempty subset R, not containing 0, of L is said to be t-irreducible, if no element of R can be written as the finite meet or arbitrary join of members of $L \setminus R$. The complement of a t-irreducible set is called a t-complete set.

From the definition, it is clear that a subset A of L is said to be t complete, if $0 \in A$, A is closed under finite meet and arbitrary join operations.

Let δ be a completely homogeneous L- topology on X. Then clearly $\{f(x) : x \in X, f \in \delta\}$ is a t-complete subset of L.

Conversely, using t-complete subset of L, we can define more than one completely homogeneous L- topology on X.

Theorem 3.7. *let A be a t-complete subset of L such that $1 \in A$. Define $\delta = \{f \in L^X : f(x) \in A$ for all $x \in X\}$. Then δ is a completely homogeneous L- topology on X.*

Proof. This can be easily verified. □

Theorem 3.8. *Let A be a t-complete subset of L. Define $\delta = \{\underline{l} : l \in A\}$. Then δ is a completely homogeneous L- topology on X.*

Proof. Proof is trivial. □

For any t-complete subset A of L, there exist atleast two completely homogeneous L- topologies on X.

Remark 3.9. If L is a finite chain, then we can define another completely homogeneous L- topology on X, for each t-complete subset A of L defined by $\delta = \{f \in L^X : f(x) \in A \setminus \{0\}$, for all $x \in X\} \cup \{\underline{0}\}$. This is not true in the case of general lattice L. The following example illustrates this.

Example 3.10. Let L be the diamond type lattice $\{0,a,b,1\}$, where $a \wedge b = 0$ and $a \vee b = 1$.

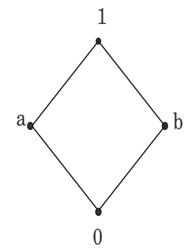

Figure 1. Diamond-type lattice

Then $\delta = \{f \in L^X : f(x) \in A \setminus \{0\} \text{ for all } x \in X\} \cup \{\underline{0}\}$ is not even an L-topological space, where A is L itself.

Let(X, δ) be an L- topology on X. Then the L- topology $\delta^{'}$ on X generated by $\delta \cup \{\underline{\alpha} : \alpha \in L\}$ is called the stratification of δ and $(X, \delta^{'})$ is called the stratification of (X, δ)[13].

Proposition 3.11. *Let (X, δ) be an L-topological space. Then δ is a completely homogeneous L-topology on X if and only if the stratification of δ is completely homogeneous L-topology on X.*

Proof. Let $(X, \delta^{'})$ be the stratification of the space (X, δ). Then every $f \in \delta^{'}$ is of the form $g \vee \underline{\alpha}$ or $g \wedge \underline{\beta}$ where $g \in \delta$ and $\alpha, \beta \in L$. So for any $h \in S(X)$, $h(f) = h(g) \vee \underline{\alpha}$ or $h(g) \wedge \underline{\beta}$. First assume that (X, δ) is a completely homogeneous L- topological space. Then $h(g)$ and $h^{-1}(g)$ are in δ for all $g \in \delta$. So $h(f) \in \delta^{'}$.

Similarly, we can prove that $h^{-1}(f) \in \delta^{'}$. Thus $\delta^{'}$ is a completely homogeneous L-topology on X. Now assume that $\delta^{'}$ is completely homogeneous L-topology on X. Then clearly δ is completely homogeneous L-topology on X. □

In a completely homogeneous topological space, supersets of nonempty open sets are open[14]. But this is not true in the case of an L- topology. See the following example.

Example 3.12. Let $X = \{a, b\}$ and $L = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ with the usual ordering and the involution $'$ given by $l' = 1 - l$ for all $l \in L$.
Define

$$\delta = \{\underline{0}, \underline{1}, a_{\frac{1}{4}}, b_{\frac{1}{4}}, \underline{\frac{1}{4}}, \underline{\frac{1}{2}}, \underline{\frac{3}{4}}, f, g\},$$

where f and g are L- sets of X defined by

$$f(a) = \tfrac{1}{2}, f(b) = \tfrac{3}{4}$$

and

$$g(a) = \tfrac{3}{4}, g(b) = \tfrac{1}{2}.$$

Here δ is a completely homogeneous L- topology. Let h be an L-set defined by

$$h(a) = \frac{1}{2}, h(b) = \frac{1}{4}.$$

Here h is a super set of $a_{\frac{1}{4}}$ but $h \notin \delta$.

Now we prove that super sets of L- open sets having the same range are open in completely homogeneous L- topological space as we see in the next lemma.

Lemma 3.13. *Let (X, δ) be a completely homogeneous L- topological space and $f \in \delta$ such that $f \neq \underline{0}$. Let g be an L- set such that $g \geq f$ and $\Lambda_g \subset \Lambda_f$. Then $g \in \delta$.*

Proof. If $g = f$, there is nothing to prove. So assume that $f < g$. Let $Y = \{x \in X : f(x) < g(x)\}$. Now choose two points y and z such that $y \in Y$ and $z \in X$, where $f(z) = g(y)$. We can choose such a point z since $f \neq \underline{0}$ and range of g is a subset of range of f. Now define $f_y = f \circ h_y$, where h_y is a function from X to X such that

$$h_y(x) = \begin{cases} z & \text{if } x = y \\ y & \text{if } x = z \\ x & \text{for all } x \in X \setminus \{y, z\}. \end{cases}$$

Then $f_y \in \delta$ and hence $f \vee f_y \in \delta$. Observe that

$$f_y(x) = \begin{cases} f(x) & \text{for all } x \in X \setminus \{y, z\} \\ f(z) & \text{if } x = y \\ f(y) & \text{if } x = z. \end{cases}$$

Consequently,

$$(f \vee f_y)(x) = \begin{cases} f(x) & \text{for all } x \in X \setminus \{y, z\} \\ f(y) \vee f(z) & \text{if } x = y \\ f(z) \vee f(y) & \text{if } x = z. \end{cases}$$

$$= \begin{cases} f(x) & \text{for all } x \in X \setminus \{y\} \\ g(y) & \text{if } x = y. \end{cases}$$

Similarly, for each y in Y, we define f_y such that $(f \vee f_y) \in \delta$. Thus $\bigvee_{y \in Y} (f \vee f_y) \in \delta$. Now

$$\bigvee_{y \in Y} (f \vee f_y)(x) = \begin{cases} f(x) & \text{for all } x \in X \setminus Y \\ g(x) & \text{if } x \in Y. \end{cases}$$

$$= g(x) \text{ for all } x \in X.$$

So $\bigvee_{y \in Y} (f \vee f_y) = g$. Hence $g \in \delta$. This completes the proof. □

An L- topological space (X, δ) is said to be Alexandroff discrete L- topological space if $\wedge A \in \delta$ for all $A \subset \delta$.

Next we give a characterization for a completely homogeneous Alexandroff discrete L- topological space when L is a complete chain.

Theorem 3.14. *Let (X, δ) be an Alexandroff discrete L-topological space where L is a complete chain. Then (X, δ) is a completely homogeneous L-topological space on X if and only if $\Lambda_f^X \subseteq \delta$ for all $f \in \delta$.*

Proof. Let (X, δ) be a completely homogeneous Alexandroff discrete L-topological space where L be a complete chain and $f \in \delta$. Define $l_1 = \bigwedge_{l \in \Lambda_f} l$ and

$$x'_{l_i}(y) = \begin{cases} l_i & y = x \\ l_1 & \text{otherwise.} \end{cases}$$

We claim that $x^{'}_{l_i} \in \delta$ for all $l_i \in \Lambda_f$. Here $l_1 \in \Lambda_\delta$. Here we consider two cases.

Case(1): $l_1 \in \Lambda_f$.

Since $l_1 \in \Lambda_f$, there exists an element x_0 in X such that $f(x_0) = l_1$. For each $x \in X \setminus \{x_0\}$, define $f_x = f \circ h_x$, where h_x is a function from X onto itself which maps x to x_0, x_0 to x and keeping all other elements fixed. Then

$$f_x(y) = \begin{cases} l_1 & y = x \\ f(x) & \text{otherwise.} \end{cases}$$

Thus $f_x \in \delta$ for all $x \in X \setminus \{x_0\}$. Let $l \in \Lambda_f$. So there exists an element z in X such that $f(z) = l$. Now

$$\bigwedge_{x \in X \setminus \{z\}} f_x(y) = \begin{cases} l_1 & y \neq z \\ l & y = z. \end{cases}$$
$$= x^{'}_l.$$

Hence we get $x^{'}_l \in \delta$.

Case(2): $l_1 \notin \Lambda_f$.

In this case we construct an L- set using f which takes the value l_1. Fix some x_0 in X and define $f_x = f \circ h_x$ for all $x \in X$, where h_x is a function as defined above. Then $\bigwedge\limits_{x \in X} f_x \in \delta$ and

$$\bigwedge_{x \in X} f_x(y) = \begin{cases} l_1 & y = x_0 \\ f(x) & \text{otherwise.} \end{cases}$$

Thus we get an L- subset of X which takes the value l_1. Now proceeding as in the case (1), we can easily prove that $x^{'}_l \in \delta$. So in both cases any $f \in \Lambda^X_f$ can be expressed as a join of $x^{'}_l$. So $\Lambda^X_f \subseteq \delta$ for all $f \in \delta$.

Conversely, assume that $\Lambda^X_f \subseteq \delta$ for all $f \in \delta$. Then $f \circ h \in \delta$ for all $f \in \delta$. Thus δ is a completely homogeneous L- topology on X. So the result holds. □

An L-topological space (X, δ) is said to be finite if the underlying set X is finite. With the characterization theorem of completely homogeneous Alexandroff discrete L- topological space, we list finite completely homogeneous L- topological spaces when the membership lattice $L = \{0, \frac{1}{2}, 1\}$ with the usual order.

Corollary 3.15. *Let X be a finite set and $L = \{0, \frac{1}{2}, 1\}$. Then the only completely homogeneous L-topologies on X are the following.*

(1) *The indiscrete topology, $\{\underline{1}, \underline{0}\}$.*
(2) $\{\underline{1}, \underline{0}, \underline{\frac{1}{2}}\}$.
(3) *The discrete topology, 2^X.*
(4) $\{\underline{1}\} \cup \{f \in L^X : f(x) \leq \frac{1}{2}$ *for all* $x \in X\}$.
(5) $\{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ *for all* $x \in X\}$.
(6) *The L- discrete topology L^X.*
(7) $\{f \in L^X : f(x) \leq \frac{1}{2}$ *for all* $x \in X\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ *for all* $x \in X\}$.

Proof. Let δ be a completely homogeneous L-topological space on X. If $\Lambda_\delta = \phi$, then δ is the indiscrete topology. If $\Lambda_\delta = \{\frac{1}{2}\}$, then δ is of the form (2). Let $\Lambda_\delta = \{0, \frac{1}{2}\}$. In this case, $\delta = \{\underline{1}\} \cup \{f \in L^X : f(x) \leq \frac{1}{2}$ for all $x \in X\}$.

Similarly, if $\Lambda_\delta = \{0, 1\}$, then δ is the discrete topology. If $\Lambda_\delta = \{\frac{1}{2}, 1\}$, then $\delta = \{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ for all $x \in X\}$. Let $\Lambda_\delta = \{0, \frac{1}{2}, 1\}$. Suppose there exists an L- set f in δ such that $\Lambda_f = \{0, \frac{1}{2}, 1\}$. Then $\delta = L^X$. Otherwise, we get

$$\delta = \{f \in L^X : f(x) \leq \tfrac{1}{2} \text{ for all } x \in X\} \cup \{f \in L^X : f(x) \geq \tfrac{1}{2} \text{ for all } x \in X\}.$$

□

Remark 3.16. It is not possible to drop the chain condition on the lattice from the hypothesis of the Theorem 3.14. The following example illustrates this.

Example 3.17. Let $X = \{a, b\}$ and L be the diamond type lattice. Define

$$\delta = \{\underline{1}, \underline{0}, f_1, f_2\},$$

where

$$f_1(a) = l_1, f_1(b) = l_2$$
$$\text{and } f_2(a) = l_2, f_2(b) = l_1.$$

Here δ is a completely homogeneous L- topology on X, but $\Lambda^X_{f_1} \nsubseteq \delta$.

4. Principal completely homogeneous L- topological space generated by an L- set

In this section, we define principal completely homogeneous L- topological space generated by single L- set and study some of its properties. Also we characterize the principal completely homogeneous L-topological space generated by an L- subset when the membership lattice $L = \{0, \frac{1}{2}, 1\}$ with the usual order.

In this section we use some set theoretic results. Let A and B be two subsets of set X and $|A| = |B|$, it does not necessarily follow that there exists a bijection of X which maps A onto B. In order to exist such a function, we must also know that $|X \setminus A| = |X \setminus B|$. If X is an infinite set, it is possible to choose A and B such that $A \cup B = X$, $A \cap B = \phi$ and $|A| = |X| = |B|$ since for any infinite cardinal number α, we have $\alpha + \alpha = \alpha$ [15].

Definition 4.1. Let $f \in L^X$. Then the smallest completely homogeneous L- topology containing f is called the principal completely homogeneous L- topology generated by f and is denoted by $CHLFT(f)$. Here the L- set f is called the generator of the principal completely homogeneous L- topology.

A completely homogeneous L-topological space (X, δ) is called principal completely homogeneous L-topological space if $\delta = CHLFT(f)$ for some L- set $f \in L^X$.

Observe that the intersection of all completely homogeneous L-topological spaces containing an L- set is completely homogeneous, which asserts the existence of $CHLFT(f)$ for all $f \in L^X$.

An immediate consequence of the Definition 4.1 is the following.

Proposition 4.2. *Let $f \in L^X$ and $\delta = CHLFT(f)$. Then $\mathcal{S} = \{foh : h \in S(X)\}$ form a sub-base for δ.*

Proof. Let δ be a principal completely homogeneous L- topology generated by f. Then $\{foh : h \in S(X)\} \subseteq \delta$. Let $\mathcal{A} = \{\wedge\mathcal{B} : \mathcal{B} \in [\mathcal{S}]^{<\omega} \setminus \{\phi\}\}$, where $[\mathcal{S}]^{<\omega}$ denote the family of all finite subsets of $\mathcal{S}$. We have to show that $\mathcal{A}$ is a base for δ. Clearly $\mathcal{B}$ is contained in δ. Now $\mathcal{B}$ is closed under binary meet and $\vee\mathcal{A} = \underline{1}$, which implies that $\mathcal{B}$ is a base of an L- topology $\delta^{'}$ on X. Every member of $\delta^{'}$ can be expressed as a meet of a subfamily of $\mathcal{B}$ and thus is in δ since $\mathcal{B} \subset \delta$. This means $\delta^{'} \subseteq \delta$ and consequently $\delta^{'} = \delta$. So $\mathcal{S}$ is a sub-base for δ. □

Example 4.3. Let $X = \{a, b, c\}$, $L = \{0, \frac{1}{2}, 1\}$ and $f \in L^X$ defined by

$$f(a) = 0, f(b) = \frac{1}{2} \text{ and } f(c) = 1.$$

Then the L-points $x_{\frac{1}{2}}$ and x_1 belongs to $CHLFT(f)$ for all $x \in X$ and thus $CHLFT(f) = L^X$.

Among the crisp topologies R. E Larson [12] characterized the completely homogeneous topological spaces.

Theorem 4.4. [12] *The only completely homogeneous topologies on X are the following.*

(1) *The indiscrete topology.*
(2) *The discrete topology.*
(3) *Topologies of the form $T_m = \{G \subset X : |X \setminus G| < m\} \cup \{\phi\}$,*
where $\aleph_0 \leq m \leq |X|$.

Obviously the indiscrete topology is generated by the whole set X and the discrete topology is generated by any singleton. So it follows that the indiscrete topology and discrete topology are principal completely homogeneous topologies on X. Now we consider the completely homogeneous topologies of the form $T_m = \{G \subset X : |X \setminus G| < m\} \cup \{\phi\}$, where $\aleph_0 \leq m \leq |X|$.

The next definition appears in [2].

Definition 4.5. The successor of a cardinal m is the least cardinal greater than m. A cardinal is said to be a limit cardinal if it is not the successor of a cardinal.

Theorem 4.6. *Let X be an infinite set and and $T_m = \{G \subset X : |X \setminus G| < m\} \cup \{\phi\}$, where $\aleph_0 \leq m \leq |X|$. Then T_m is a principal completely homogeneous topology on X if and only if m is not a limit cardinal.*

Proof. Assume that m is not a limit cardinal. Then there exists an infinite cardinal $m^{'}$ such that m is the immediate successor of $m^{'}$. Choose a subset A of X such that $|X \setminus A| = m^{'}$. Since T_m is completely homogeneous, it follows that $\{B \subset X : |B| = |A| \text{ and } |X \setminus B| = |X \setminus A|\} \subset T_m$. Also we have super sets of nonempty open sets are open in X. So T_m is a principal completely homogeneous topology generated by A.

Conversely, assume that m is a limit cardinal. Suppose that T_m is generated by subset A of X. Then $|X \setminus A| < m$. Thus there exist an infinite cardinal number $m^{'}$ such that $|X \setminus A| < m^{'} < m$. So $A \in T_{m'}$. Hence the principal completely homogeneous topology generated by A is contained in $T_{m'}$, which is a contradiction. This completes the proof. □

Now we list principal completely homogeneous topologies on an infinite set X.

Corollary 4.7. *The only principal completely homogeneous topologies on X are the following.*
(1) *The indiscrete topology.*
(2) *The discrete topology.*
(3) *Topologies of the form $T_m = \{G \subset X : |X \setminus G| < m\} \cup \{\phi\}$,*
where $\aleph_0 \le m \le |X|$ and m is not a limit cardinal.

Remark 4.8. If X is a finite set, we have the only completely homogeneous topologies on X are the discrete topology and the indiscrete topology. Hence all completely homogeneous topologies on a finite set are principal. But this is not true in L- topology.

Example 4.9. Let $L = \{0, \frac{1}{2}, 1\}$ with usual order and $X = \{a, b\}$. Observe that X and L are finite. Now define

$$\delta = \{\underline{0}, \underline{1}, a_{\frac{1}{2}}, b_{\frac{1}{2}}, \underline{\frac{1}{2}}, a^{\frac{1}{2}}, b^{\frac{1}{2}}\}.$$

Then δ is a completely homogeneous L-topological space. We have

$$CHLFT(a_{\frac{1}{2}}) = \{\underline{0}, \underline{1}, a_{\frac{1}{2}}, b_{\frac{1}{2}}, \underline{\frac{1}{2}}\},$$

$$CHLFT(a^{\frac{1}{2}}) = \{\underline{0}, \underline{1}, a^{\frac{1}{2}}, b^{\frac{1}{2}}, \underline{\frac{1}{2}}\}$$

and

$$\delta = CHLFT(a_{\frac{1}{2}}) \cup CHLFT(a^{\frac{1}{2}}).$$

Thus δ is not a principal completely homogeneous L- topological space.

Remark 4.10. Let X be a finite set. Then every completely homogeneous L-topology on X is principal if and only if $L = \{0, 1\}$.

In following theorems we prove two important properties of the principal completely homogeneous L-topological space generated by an L-set.

Theorem 4.11. *Let X be an infinite set and f be an L- subset of X such that $|f^{-1}(0)| = |X|$. Then $CHLFT(f)$ is Λ_f^X.*

Proof. Let $\delta = CHLFT(f)$, $X_1 = f^{-1}(0)$ and $X_2 = X \setminus X_1$. Given that $|X_1| = |X|$ and so $|X_2| \le |X|$. We consider the following cases.
Case (1): $|X_2| = |X|$.
Here $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \phi$ and $|X_1| = |X| = |X_2|$. It follows that $|X \setminus X_1| = |X \setminus X_2|$. Then there exists a bijection h of X which maps X_1 onto X_2. Since (X, δ) is a completely homogeneous L-topological space, $h^{-1}(f) = f \circ h \in \delta$. Let $g = f \circ h$ and $l \in \Lambda_f$, $l \neq 0$. Then there exists x_0 in X_2 such that $f(x_0) = l$. Note that $g^{-1}(0) = X_2$. Thus we can choose y_0 in X_1 such that $g(y_0) = f(x_0)$. Now consider the L- set $g \circ h_{x_0}$ where h_{x_0} is a function from X onto itself which maps x_0 to y_0, y_0 to x_0 and keeping all the other elements fixed.
Set $f_1 = f$ and $f_2 = g \circ h_{x_0}$. Then

$$(f_1 \wedge f_2)(x) = \begin{cases} l & x = x_0 \\ 0 & \text{otherwise.} \end{cases}$$

Case(2): $|X_2| < |X|$.
Given that $|X_1| = |X|$. In this case, we can find a subset Y_1 of X_1 such that $|X_2| = |Y_1|$. Then $|X \setminus X_2| = |X| = |X \setminus Y_1|$. Thus there exists a bijection h from X onto X such that $h(Y_1) = X_2$. So as in case(1), consider $g = f \circ h$. Now $(f \circ h)(x) = f(h(x)) \neq 0$ for $x \in Y_1$. Now let $l \in \Lambda_f$. Then there exists an element x_0 in X_2 such that $f(x_0) = l$. We have $g^{-1}(0) = X_2$. Choose y_0 in Y_1 such that $g(y_0) = f(x_0)$. Now consider the L- set $g \circ h_{x_0}$, where h_{x_0} is the function as defined in the case (1) above. Set $f_1 = f$ and $f_2 = g \circ h_{x_0}$. Then

$$(f_1 \wedge f_2)(x) = \begin{cases} l & x = x_0 \\ 0 & \text{otherwise.} \end{cases}$$

So in both cases, $x_{0_l} \in \delta$. Since x_0 is arbitrary, $\{x_l : x \in X\} \subseteq \delta$. This is true for all $l \in \Lambda_f$. Hence $\delta = {\Lambda_f}^X$. □

Theorem 4.12. *Let (X, δ) be a completely homogeneous L-topological space where X is an infinite set and $\delta = CHLFT(f)$ for some $f \in L^X$ such that $|f^{-1}(0)| < \alpha$, $\aleph_0 \leq \alpha < |X|$. Then $|g^{-1}(0)| < \alpha$ for all $g \in \delta \setminus \{\underline{0}\}$.*

Proof. Let $g \in \delta \setminus \{\underline{0}\}$. Since $\delta = CHLFT(f)$, by proposition 4.2, $\mathcal{S} = \{f \circ h : h \in S(X)\}$ form a sub-base for δ. So g can be written as $\bigvee_{i \in I} g_i$ where I is an index set and for each $i \in I$, g_i can be written as the meet of finitely many members of $\mathcal{S}$, say $s^i_1 \wedge s^i_2 \ldots \wedge s^i_{r_i}$ where $r_i \in N$ and $s^i_j \in \mathcal{S}$, $1 \leq j \leq r_i$. Note that

$$(s^i_1 \wedge s^i_2 \ldots \wedge s^i_{r_i})^{-1}(0) = (s^i_1)^{-1}(0) \cup (s^i_2)^{-1}(0) \cup \ldots \cup (s^i_{r_i})^{-1}(0).$$

We have $|(s^i_j)(0)| < \alpha$. Since α is an infinite cardinal, we get $|(g_i^{-1})(0)| < \alpha$. Now

$$\begin{aligned} |g^{-1}(0)| &= |(\bigvee_{i \in I} g_i)^{-1}(0)| \\ &= |\bigcap_{i \in I} (g_i^{-1}(0))| < \alpha. \end{aligned}$$

Thus the theorem holds. □

Theorem 4.13. *Let X be an infinite set. Then δ is a completely homogeneous L-topology on X where $\Lambda_\delta = \{0, l\}$, $l \in L \setminus \{0, 1\}$ if and only if δ is one of the following.*

(1) $\{\underline{1}\} \cup \{f \in \{0, l\}^X\}$.
(2) $\{\underline{0}, \underline{1}\} \cup \{f \in \{0, l\}^X : |f^{-1}(0)| < \alpha\}$, *where* $\aleph_0 \leq \alpha \leq |X|$.

Proof. Clearly above two L- topologies are completely homogeneous. Now assume that δ is a completely homogeneous L-topological space and $\Lambda_\delta = \{0, l\}$. Then there exists at least one $f \in \delta$ such that $\{f(x) : x \in X\} = \{0, l\}$. Let $X_1 = f^{-1}(0)$. If $|X_1| = |X|$, then $\delta = \{\underline{1}\} \cup \{f \in \{0, l\}^X\}$. Otherwise $|X_1| < |X|$. In this case, consider the level topology $T_{[0]}$ of δ. Thus $T_{[0]}$ is a completely homogeneous topology on X by Corollary 3.4 and $X \setminus X_1$ is open in δ. So $T_{[0]} = \{A \subset X : |X \setminus A| < \alpha\} \cup \{\phi\}$, where $\aleph_0 \leq \alpha \leq |X|$. Hence $\delta = \{\underline{0}, \underline{1}\} \cup \{f \in \{0, l\}^X : |f^{-1}(0)| < \alpha\}$, where $\aleph_0 \leq \alpha \leq |X|$. □

Now we consider the principal completely homogeneous L-topological space generated by an L- subset when the membership lattice $L = \{0, \frac{1}{2}, 1\}$ with the usual order.

Remark 4.14. Let X be an infinite set and $f \in L^X$ such that $\Lambda_f = \{0, \frac{1}{2}\}$. Then δ is a principal completely homogeneous L- topology generated by f if and only if δ is one of the following.

(1) $\{\underline{1}\} \cup \{f \in L^X : f(x) \leq \frac{1}{2}$ for all $x \in X\}$.

(2) $\{\underline{0}, \underline{1}\} \cup \{f \in \{0, \frac{1}{2}\}^X : |f^{-1}(0)| < \alpha\}$,

where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal.

Theorem 4.15. *Let X be an infinite set and $f \in L^X$ such that $\Lambda_f = \{1, \frac{1}{2}\}$. Then δ is a principal completely homogeneous L- topology on X if and only if δ is one of the following.*

(1) $\{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ *for all* $x \in X\}$.

(2) $\{\underline{0}\} \cup \{f \in \{1, \frac{1}{2}\}^X : |f^{-1}(\frac{1}{2})| < \alpha\}$,

where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal.

Proof. Let $X_1 = f^{-1}(\frac{1}{2})$, $X_2 = f^{-1}(1)$ and $X_1 \cup X_2 = X$. Now consider the following three cases.

Case(1): $|X_1| = |X| = |X_2|$.

In this case we show that δ is of type (1). Since $|X_1| = |X| = |X_2|$, there exists a bijection h from X to X such that h maps X_1 onto X_2. Then

$$(f \circ h)(x) = \begin{cases} 1 & \text{if } x \in X_1 \\ \frac{1}{2} & \text{if } x \in X_2. \end{cases}$$

Now choose two points $x_0 \in X_2$ and $y_0 \in X_1$ and define $h' : X \to X$ as follows.

$$h'(x) = \begin{cases} x_0 & \text{if } x = y_0 \\ y_0 & \text{if } x = x_0 \\ x & \text{otherwise.} \end{cases}$$

Consider $g = f \circ h \circ h'$. Then $g \in \delta$ and

$$g(x) = \begin{cases} \frac{1}{2} & \text{if } x \in (X_2 \setminus \{x_0\}) \cup \{y_0\} \\ 1 & \text{if } x \in (X_1 \setminus \{y_0\}) \cup \{x_0\}. \end{cases}$$

Now let $f_{x_0} = g \wedge f$. Then $f_{x_0} \in \delta$ and

$$f_{x_0} = \begin{cases} 1 & \text{if } x = x_0 \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Thus for each $x \in X$, there is an L- set f_x in δ such that f takes the value 1 at x and $\frac{1}{2}$ at all other points in X.

Let $\mu = \{f \in L^X : f(x) \geq \frac{1}{2}\}$. We claim that $\mu \subseteq \delta$. Consider $f \in \mu$. Let $Y = \{x \in X : f(x) = 1\}$. Then $f = \bigvee_{x \in Y} f_x$ and hence $f \in \delta$. Thus we get $\delta = \mu$.

Case(2): $|X_1| = |X|$ and $|X_2| < |X|$.

Since (X, δ) is completely homogeneous, by applying Lemma 3.13, every $g \geq f$ having the same range as that of f belongs to δ. So any f in L^X such that $|X_1| = |X| = |X_2|$ are in δ and by case (1), $\delta = \mu$.

Case(3): $|X_1| < |X|$ and $|X_2| = |X|$.

Since $\delta = CHLFT(f)$, $\{f \circ h : h \in S(X)\}$ form a subbase for δ, every element g in δ can be written as $g = \bigvee_{i \in I} (\bigwedge_{j=1}^{n} f_{ij})$, where $f_{ij} = f \circ h_{ij}$, $h_{ij} \in S(X)$. Now for each $i \in I$, $(\overset{n}{\underset{j=1}{\wedge}} f_{ij})^{-1}(\frac{1}{2}) = (f_{i1})^{-1}(\frac{1}{2}) \cup (f_{i2})^{-1}(\frac{1}{2}) \cup \ldots f_{in}^{-1}(\frac{1}{2})$. Thus we get $|(\overset{n}{\underset{j=1}{\wedge}} f_{ij})^{-1}(\frac{1}{2})| \leq |X_1|$.

Now

$$\begin{aligned} g^{-1}(\frac{1}{2}) &= (\underset{i\in I}{\vee}(\overset{n}{\underset{j=1}{\wedge}} f_{ij}))^{-1}(\frac{1}{2}) \\ &= \underset{i\in I}{\cap}(\overset{n}{\underset{j=1}{\wedge}} f_{ij})^{-1}(\frac{1}{2}) \\ &= \underset{i\in I}{\cap}\left(\overset{n}{\underset{j=1}{\cup}} f_{ij}^{-1}(\frac{1}{2})\right). \end{aligned}$$

Then

$$\begin{aligned} |g^{-1}(\frac{1}{2})| &= |\underset{i\in I}{\cap}\left(\overset{n}{\underset{j=1}{\cup}} f_{ij}^{-1}(\frac{1}{2})\right)| \\ &\leq |\overset{n}{\underset{j=1}{\cup}} f_{ij}^{-1}(\frac{1}{2})| \\ &\leq |X_1|. \end{aligned}$$

Thus $\delta = \{\underline{0}\} \cup \{f \in \{1, \frac{1}{2}\}^X : |f^{-1}(\frac{1}{2})| < \alpha\}$, where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal.

Conversely, assume δ is one of the form

(1) $\{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ for all $x \in X\}$,

(2) $\{\underline{0}\} \cup \{f \in \{1, \frac{1}{2}\}^X : |f^{-1}(\frac{1}{2})| < \alpha\}$, where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal.

Then it is easy to verify that these are the completely homogeneous L- topologies on X. Thus the theorem holds. □

Theorem 4.16. *Let X be an infinite set and $f \in L^X$ where $\Lambda_f = \{1, \frac{1}{2}, 0\}$. Then δ is a principal completely homogeneous L-topological space generated by f on X if and only if δ is one of the following.*

(1) *The discrete L- topology L^X.*

(2) $\{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < \alpha\}$.

(3) $\{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| = |f^{-1}(\frac{1}{2})| < \alpha\}$.

(4) $\{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < |f^{-1}(\frac{1}{2})| < \alpha\}$,

where $\aleph_0 < \alpha \leq |X|$ and α is not a limit cardinal.

Proof. Obviously the above L- topologies are principal completely homogeneous. Conversely let $f \in L^X$, $X_1 = f^{-1}(0)$, $X_2 = f^{-1}(\frac{1}{2})$ and $X_3 = f^{-1}(1)$. If $|X_1| = |X|$, then by Theorem 4.14, it follows that $\delta = L^X$. Otherwise $|X_1| = \alpha < |X|$. Then by Theorem 4.12, we have

$$|f^{-1}(0)| < \alpha \text{ for every } f \in \delta \setminus \{\underline{0}\}, \tag{4.1}$$

where $\aleph_0 \leq \alpha \leq |X|$. Since $|X_1| < |X|$, $|X_2 \cup X_3| = |X|$.

Case(1): $|X_2| = |X|$ and $|X_3| \leq |X|$.

By Lemma 3.13, there exist an L- subset g in δ such that $|g^{-1}(\frac{1}{2})| = |g^{-1}(1)| = |X|$

and $X = g^{-1}(\frac{1}{2}) \cup g^{-1}(1)$. Then by Theorem 4.15, it follows that

$$\{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2} \text{ for all } x \in X\} \subset \delta. \tag{4.2}$$

From equations (4.1) and (4.2), we get $\delta = \{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < \alpha\}$, where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal.

Case(2): $|X_2| < |X|$ and $|X_3| = |X|$.
If $|X_1| < |X_2|$. In this case $\delta = \{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < |f^{-1}(\frac{1}{2})| < \alpha\}$, where $\aleph_0 < \alpha \leq |X|$. Suppose that $|X_1| = |X_2|$. Then $\delta = \{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| = |f^{-1}(\frac{1}{2})| < \alpha\}$, where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal. □

Thus we determined the principal completely homogeneous L- topological space generated by an L- set when $L = \{0, \frac{1}{2}, 1\}$.

We conclude this section by listing all principal completely homogeneous L- topologies on X when $L = \{0, \frac{1}{2}, 1\}$.

Corollary 4.17. *Let X be an infinite set. Then the only principal completely homogeneous L- topologies on X when $L = \{0, \frac{1}{2}, 1\}$ are the following.*
(1) *The trivial L- topology,* $\{\underline{0}, \underline{1}\}$.
(2) $\{\underline{0}, \underline{\frac{1}{2}}, \underline{1}\}$.
(3) *The discrete crisp topology,* 2^X.
(4) *L- topologies of the form* $T = \{\chi_G : |X \setminus G| < \alpha\} \cup \{\underline{0}\}$.
(5) $\{\underline{1}\} \cup \{f \in L^X : f(x) \leq \frac{1}{2}$ *for all* $x \in X\}$.
(6) $\{\underline{0}, \underline{1}\} \cup \{f \in \{0, \frac{1}{2}\}^X : |f^{-1}(0)| < \alpha\}$.
(7) $\{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ *for all* $x \in X\}$.
(8) $\{\underline{0}\} \cup \{f \in \{1, \frac{1}{2}\}^X : |f^{-1}(\frac{1}{2})| < \alpha\}$.
(9) *The discrete L- topology* L^X.
(10) $\{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| = |f^{-1}(\frac{1}{2})| < \alpha\}$.
(11) $\{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < |f^{-1}(\frac{1}{2})| < \alpha\}$,
where $\aleph_0 < \alpha \leq |X|$ and α is not a limit cardinal.

Proof. Proof follows from Theorems 4.15, 4.16 and Remark 4.14. □

5. Conclusion

We have studied completely homogeneous L- topological spaces and identified some of its properties. We obtained a characterization for Alexandroff discrete completely homogeneous L- topological spaces when L is a complete chain. Also the concept of principal completely homogeneous L- topological spaces is introduced.

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