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Extensions of fuzzy ideals of semirings

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Abstract. In this paper we introduce the notions of extension of fuzzy ideal of a semiring S, fuzzy 3-weakly completely prime ideal of S and study the relationship between fuzzy weakly completely prime ideals, fuzzy 3-weakly completely prime ideals by means of the extensions of fuzzy ideals of semiring.

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1. Introduction

In [3, 4, 5, 6], T. K. Dutta and B. K. Biswas introduced and studied some properties of fuzzy prime, fuzzy semiprime, fuzzy completely prime ideals in semiring . In [7], Xiang-Yun Xie introduced the concept of extension of fuzzy ideals of semigroups and in [8], Xie and Yan generalised the concept on ordered semigroups. The aim of this paper is to introduce the extension of fuzzy ideals of semirings as Xiang-Yun Xie did in case of semigroups. We also want to see how far the results of Xiang-Yun Xie in case of semigroups are valid in case of semirings. In this paper we introduce the concepts of extension of fuzzy ideal, fuzzy 3-weakly completely prime ideal in semiring and study the relationship between fuzzy weakly completely prime, fuzzy 3-weakly completely prime by means of the extensions of fuzzy ideals of semiring. Finally we have shown that if μ is a fuzzy semiprime ideal of a commutative semiring S then μ is the infimum of all fuzzy weakly completely prime ideal of S containing μ.

2. Preliminaries

Definition 2.1. A nonempty set S is said to form a semiring with respect to two binary compositions, addition(+) and multiplication(.) defined on it, if the following conditions are satisfied :

(i) $(S, +)$ is a commutative semigroup with zero ('0'),

(ii) $(S, .)$ is a semigroup,

(iii) for any three elements $a, b, c \in S$, the left distributive law $a.(b+c) = a.b+a.c$ and the right distributive law $(b + c).a = b.a + c.a$ both ,

(iv) $s.0 = 0.s = 0$, for all $s \in S$.

Definition 2.2 ([2])**.** A nonempty subset I of a semiring S is called an ideal if

(i) $a, b \in I$ implies $a + b \in I$,

(ii) $a \in I$, $s \in S$ implies $s.a \in I$ and $a.s \in I$.

Definition 2.3 ([1])**.** An ideal I of a semiring S is called a k-ideal if $b \in S$, $a+b \in I$ and $a \in I$ implies $b \in I$.

Definition 2.4 ([3])**.** Let μ be a nonempty fuzzy subset of a semiring S (i.e. $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a fuzzy left [fuzzy right] ideal of S if

(i) $\mu(x + y) \geq min[\mu(x), \mu(y)]$,

(ii) $\mu(xy) \geq \mu(y)$ [resp. $\mu(xy) \geq \mu(x)$], $\forall x, y \in S$.

A fuzzy ideal of a semiring S is a nonempty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S.

Definition 2.5 ([4])**.** A fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of a semiring S is said to be a fuzzy left k-ideal (resp. fuzzy right k-ideal, fuzzy k-ideal) of S if

$$\mu(x) \geq min[\mu(x + y), \mu(y)],\ \forall x, y \in S.$$

Definition 2.6 ([4])**.** Let S be a semiring and μ_1, μ_2 be two fuzzy ideals of S. Then composition of μ_1 and μ_2, denoted by $\mu_1 o \mu_2$ and is defined by

$$\mu_1 o \mu_2(x) = \begin{cases} \sup\limits_{x=uv} [min[\mu_1(u), \mu_2(v)]], \\ 0 \text{ if } x \text{ is not expressible as } x = uv \text{ for any } u, v \in S. \end{cases}$$

Definition 2.7 ([3])**.** A fuzzy ideal μ of a semiring S is called a fuzzy prime ideal if μ is not a constant function and for any two fuzzy ideals μ_1 and μ_2 of S, $\mu_1 o \mu_2 \subseteq \mu$ implies that either $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.

Definition 2.8 ([5])**.** A fuzzy ideal μ of a semiring S is said to be a fuzzy semiprime ideal if μ is not a constant function and for any fuzzy ideal θ of S, $\theta o \theta \subseteq \mu$ implies that $\theta \subseteq \mu$.

Throughout this paper $\mathbb{Z}_0^+$ denotes the semiring of all nonnegative integers with respect to the usual addition and multiplication of integers.

3. Extension of fuzzy ideals

Definition 3.1. Let S be a semiring, μ be a fuzzy subset of S and $s \in S$. The fuzzy subset $< s, \mu >: S \to [0, 1]$, defined by $< s, \mu > (x) = \mu(sx)$, is called extension of μ by s.

Example 3.2. Let $S = \mathbb{Z}_0^+$. We define a fuzzy subset μ of S as follows:

$$\mu(0) = 1,\ \mu(n) = \begin{cases} .5, & \text{if } n \text{ is even,} \\ .3, & \text{if } n \text{ is odd.} \end{cases}$$

Then $< 2, \mu > (n) = \begin{cases} 1, & \text{if } n = 0, \\ .5, & \text{if } n \neq 0 \end{cases}$ is the extension of μ by 2.

Here extension of μ by 2 is also a fuzzy subset of S takes only the values which takes μ for the integers of multiple of 2.

Proposition 3.3. *Let S be a commutative semiring. If μ is a fuzzy ideal of S and $s \in S$, then the extension of μ by s is a fuzzy ideal of S.*

Proof. Obviously $< s, \mu >$ is a fuzzy subset of S.
Let $x, y \in S$. Then

$$\begin{aligned} < s, \mu > (x + y) &= \mu(s(x + y)) \\ &= \mu(sx + sy) \\ &\geq min[\mu(sx), \mu(sy)] \\ &= min[< s, \mu > (x), < s, \mu > (y)]. \end{aligned}$$

Thus

$$< s, \mu > (x + y) \geq min[< s, \mu > (x), < s, \mu > (y)].$$

Also

$$< s, \mu > (xy) = \mu(sxy) \geq \mu(sx) =< s, \mu > (x)$$

and

$$< s, \mu > (xy) = \mu(sxy) = \mu(syx) \geq \mu(sy) =< s, \mu > (y).$$

So $< s, \mu >$ is a fuzzy ideal of S. □

The converse of the above proposition may not be true. This follows from the following example.

Example 3.4. We define a fuzzy subset $\mu : \mathbb{Z}_0^+ \to [0, 1]$ as follows:

$$\mu(0) = 1,\ \mu(n) = \begin{cases} .5, & \text{if } 1 \leq n \leq 4, \\ .2, & \text{if } n > 4. \end{cases}$$

Then μ is not a fuzzy ideal of $\mathbb{Z}_0^+$, as $\mu(3 + 4) \ngeq \min\{\mu(3), \mu(4)\}$.
Now $< 5, \mu > (n) = \begin{cases} 1, & \text{if } n = 0, \\ .2, & \text{if } n \neq 0. \end{cases}$
Clearly, $< 5, \mu >$ is a fuzzy ideal of $\mathbb{Z}_0^+$.

Proposition 3.5. *Let S be a commutative semiring. If μ is a fuzzy k-ideal of S and $s \in S$, then the extension of μ by s is a fuzzy k-ideal of S.*

Proof. By Proposition 3.3, $< s, \mu >$ is a fuzzy ideal of S. Since μ is a fuzzy k-ideal of S, $\mu(sx) \geq min[\mu(sx + sy), \mu(sy)]$ for all $x, y \in S$. Then

$$< s, \mu > (x) \geq min[< s, \mu > (x + y), < s, \mu > (y)]$$

for all $x, y \in S$.
Thus $< s, \mu >$ is a fuzzy k-ideal of S. □

Here also the converse may not be true. This follows from the following example.

Example 3.6. We define a fuzzy subset $\mu : \mathbb{Z}_0^+ \to [0, 1]$ of $\mathbb{Z}_0^+$ as follows:

$$\mu(0) = 1,\ \mu(n) = \begin{cases} .2, & \text{if } 1 \leq n \leq 7, \\ .5, & \text{if } n > 7. \end{cases}$$

Then clearly, μ is a fuzzy ideal of $\mathbb{Z}_0^+$.
Here μ is not a fuzzy k-ideal of $\mathbb{Z}_0^+$, as $\mu(3) \ngeq \min\{\mu(3+9), \mu(9)\}$.
Now $< 8, \mu > (n) = \begin{cases} 1, & \text{if } n = 0, \\ .5, & \text{if } n \neq 0. \end{cases}$
By proposition 3.3, $< 8, \mu >$ is a fuzzy ideal of $\mathbb{Z}_0^+$.
Let $x, y \in \mathbb{Z}_0^+$.
Case-1: If $x = 0$, then $< 8, \mu > (x) = 1 \geq \min[< 8, \mu > (x+y), < 8, \mu > (y)]$.
Case-2: If $x \neq 0$, then $< 8, \mu > (x+y) = .5$. So

$$< 8, \mu > (x) = .5 = \min[< 8, \mu > (x+y), < 8, \mu > (y)].$$

Thus $< 8, \mu > (x) \geq min[< 8, \mu > (x+y), < 8, \mu > (y)]$, $\forall x, y \in \mathbb{Z}_0^+$. Hence the extension of the fuzzy ideal μ by 8 is a fuzzy k-ideal of $\mathbb{Z}_0^+$.

Definition 3.7. If μ is a fuzzy subset of S, where S is a semiring, we define

$$supp\mu = \{s \in S \ : \mu(s) > 0\}.$$

Proposition 3.8. *Let S be a semiring, μ be a fuzzy ideal of S and $s \in S$. Then*
(1) $\mu \subseteq < s, \mu >$.
(2) $< s^n, \mu > \subseteq < s^{n+1}, \mu >$ *for every natural number n.*
(3) *If $\mu(s) > 0$, then supp $< s, \mu > = S$.*

Proof. (1) Since μ is a fuzzy ideal of S, we have

$$< s, \mu > (x) = \mu(sx) \geq \mu(x), \forall x \in S.$$

Thus $\mu \subseteq < s, \mu >$.
(2) For all natural number n and $\forall x \in S$,

$$< s^{n+1}, \mu > (x) = \mu(s^{n+1}x) = \mu(ss^n x) \geq \mu(s^n x) = < s^n, \mu > (x).$$

Then $< s^n, \mu > \subseteq < s^{n+1}, \mu >$.
(3) Let $x \in S$. Then $< s, \mu > (x) = \mu(sx) \geq \mu(s) > 0$. Thus $x \in supp < s, \mu >$. So $supp < s, \mu > = S$. □

Proposition 3.9 ([3]). *If μ is a fuzzy subset of a semiring S then μ is a fuzzy prime ideal of S if and only if $Im\mu = \{1, \alpha\}$,*
where $\alpha \in [0, 1)$ and $\mu_0 = \{x \in S : \mu(x) = \mu(0)\}$ is a fuzzy prime ideal of S.

Proposition 3.10 ([5]). *If μ is a fuzzy semiprime ideal of a commutative semiring S, then $\mu(x^2) = \mu(x)$ for all $x \in S$.*

In the proposition 3.8(2), we have shown that for any fuzzy ideal μ of a semiring S, $< x, \mu > \subseteq < x^2, \mu >$, $\forall x \in S$. In the next proposition, we show that if S is a commutative semiring and μ is a fuzzy semiprime ideal of S then extensions of fuzzy ideal μ by x and x^2 are equal for all $x \in S$.

Proposition 3.11. *If S is a commutative semiring and μ is a fuzzy semiprime ideal of S, then $< x, \mu > = < x^2, \mu >$, $\forall x \in S$.*

Proof. It is clear that

$$< x^2, \mu > (y) = \mu(x^2y) = \mu(xxy) \geq \mu(xy) =< x, \mu > (y), \forall y \in S.$$

Then $< x, \mu >\subseteq< x^2, \mu >$. On one hand,

$$\begin{array}{rcll} \mu(xy) & = & \mu(xy)^2, & \text{as } \mu \text{ is semiprime} \\ & = & \mu(x^2y^2), & \text{as } S \text{ is commutative} \quad . \\ & \geq & \mu(x^2y) & \end{array}$$

Thus $< x, \mu > (y) \geq< x^2, \mu > (y)$, $\forall y \in S$. So $< x^2, \mu >\subseteq< x, \mu >$. Hence $< x, \mu >=< x^2, \mu >$, $\forall x \in S$. □

Definition 3.12. Let S be a semiring , A be a subset of S and $x \in S$. Define

$$< x, A >= \{s \in S : xs \in A\}.$$

Proposition 3.13. *Let S be a semiring and A be a nonempty subset of S. Then $< s, \lambda_A >= \lambda_{<s,A>}$ for all $s \in S$.*

The proof is trivial and hence we omit it.

Definition 3.14. Let S be a semiring and μ be a fuzzy ideal of S. μ is called a fuzzy weakly completely prime ideal if $\mu(x_1x_2) = max\{\mu(x_1), \mu(x_2)\}$ for all $x_1, x_2 \in S$.

Proposition 3.15 ([3])**.** *If μ is a fuzzy completely prime ideal of a semiring S, then μ_0 is a completely prime ideal of S.*

Proposition 3.16. *Every fuzzy completely prime ideal of a semiring S is fuzzy weakly completely prime.*

Proof. Let S be a semiring and μ be a fuzzy completely prime ideal of S. Then by Proposition 3.15, μ_0 is a fuzzy completely prime ideal of S and $Im\mu = \{1, \alpha\}$, where $\alpha \in [0, 1)$. Let $x_1, x_2 \in S$.

Case-1: Let $\mu(x_1x_2) = 1$. Then $x_1x_2 \in \mu_0$. Since μ_0 is a completely prime ideal of S, $x_1 \in \mu_0$ or $x_2 \in \mu_0$. Thus $\mu(x_1) = 1$ or $\mu(x_2) = 1$. So $max\{\mu(x_1), \mu(x_2)\} = 1$. Hence $\mu(x_1x_2) = max\{\mu(x_1), \mu(x_2)\}$.

Case-2: Let $\mu(x_1x_2) \neq 1$. Then $\mu(x_1x_2) = \alpha$. Thus $\mu(x_1) = \alpha$ and $\mu(x_2) = \alpha$, otherwise either $\mu(x_1) = 1$ or $\mu(x_2) = 1$. This implies that $x_1x_2 \in \mu_0$, i.e., $\mu(x_1x_2) = 1$, a contradiction. So $max\{\mu(x_1), \mu(x_2)\} = \alpha$. Hence $\mu(x_1x_2) = max\{\mu(x_1), \mu(x_2)\}$, i.e., $\mu(x_1x_2) = max\{\mu(x_1), \mu(x_2)\}$, for all $x_1, x_2 \in S$. Therefore μ is a fuzzy weakly completely prime ideal of S. □

Converse of the above proposition may not be true. This follows from the following example.

Example 3.17. Let $S = \mathbb{Z}_0^+$. We define a fuzzy subset μ of S as follows:

$$\mu(0) = 1,\ \mu(n) = \begin{cases} .5, & \text{if } n \quad \text{is even,} \\ .3, & \text{if } n \quad \text{is odd.} \end{cases}$$

Now μ is a fuzzy ideal of S. Here μ is not a fuzzy prime ideal, as $|Im\mu| = 3$. Then μ is not a fuzzy completely prime ideal of S. Let $x_1, x_2 \in S$.

Case-1: Let either $x_1 = 0$ or $x_2 = 0$. Then $\mu(x_1) = 1$ or $\mu(x_2) = 1$ and $x_1x_2 = 0$ which implies that $\mu(x_1x_2) = 1$. Thus $\mu(x_1x_2) = max\{\mu(x_1), \mu(x_2)\}$.

Case-2: Let at least one of x_1 or x_2 is a nonzero even positive integer and $x_1 \neq 0, x_2 \neq 0$. Then x_1x_2 is even. Thus $\mu(x_1x_2) = .5$ and $\mu(x_1x_2) = max\{\mu(x_1), \mu(x_2)\}$.

Case-3: Let both x_1 and x_2 are odd. Then x_1x_2 is odd. Thus $\mu(x_1) = .3$, $\mu(x_2) = .3$ and $\mu(x_1x_2) = .3$. So $\mu(x_1x_2) = max\{\mu(x_1), \mu(x_2)\}$. Hence μ is a fuzzy weakly completely prime ideal of S.

Proposition 3.18. *Let S be a commutative semiring and μ be a fuzzy weakly completely prime ideal of S such that $Im\mu = \{1, \alpha\}$. Then μ is a fuzzy completely prime ideal of S.*

Proof. Suppose $x_1Sx_2 \subseteq \mu_0$, where $x_1, x_2 \in S$. Then $x_1sx_2 \in \mu_0$, $\forall s \in S$. Thus $x_1^2x_2 \in \mu_0$. So $\mu(x_1^2x_2) = 1$. Since μ is a fuzzy weakly completely prime ideal of S, $\mu(x_1^2x_2) = max\{\mu(x_1), \mu(x_2)\} = 1$. Hence either $\mu(x_1) = 1$ or $\mu(x_2) = 1$, which implies that $x_1 \in \mu_0$ or $x_2 \in \mu_0$. Therefore μ_0 is a prime ideal of S. Therefore by proposition 3.9, μ is a fuzzy prime ideal of S. Since S is commutative, μ is a fuzzy completely prime ideal of S. □

Theorem 3.19. *Let S be a semiring and μ be a fuzzy weakly completely prime ideal of S. If $x \in S$ is such that $\mu(x) = \inf_{y \in S} \mu(y)$, then $< x, \mu >= \mu$.*

Proof. Clearly $\inf_{y \in S} \mu(y)$ exists in $[0, 1]$. Let $y \in S$. Then

$$< x, \mu > (y) = \mu(xy). \tag{3.1}$$

Since μ is a fuzzy weakly completely prime ideal of S, $\mu(xy) = max\{\mu(x), \mu(y)\}$. Then

$$\text{either } \mu(xy) = \mu(x) \text{ or } \mu(xy) = \mu(y). \tag{3.2}$$

Let $\mu(xy) \neq \mu(y)$. Then by (3.2),

$$\mu(xy) = \mu(x). \tag{3.3}$$

Since μ is a fuzzy ideal of S, by Proposition 3.8, $\mu \subseteq < x, \mu >$. Thus by (3.3),

$$\mu(y) \leq < x, \mu > (y) = \mu(xy) = \mu(x).$$

Also since $\mu(x) = \inf_{y \in S} \mu(y)$, $\mu(x) \leq \mu(y)$. So $\mu(x) = \mu(y)$ and $\mu(xy) = \mu(y)$, a contradiction. Hence $\mu(xy) = \mu(y)$, i.e., $< x, \mu > (y) = \mu(y)$, $\forall y \in S$. Therefore $< x, \mu >= \mu$. □

Proposition 3.20. *Let S be a semiring and μ be a fuzzy completely prime ideal of S. If $x \in S$ is such that $x \notin \mu_0$, then $< x, \mu >= \mu$.*

Proof. Since μ is a fuzzy completely prime ideal of S, μ_0 is a completely prime ideal of S. Suppose $Im\mu = \{1, \alpha\}$. Let $s \in S$.

case-1: Let $s \in \mu_0$. Then $xs \in \mu_0$. Thus $< x, \mu > (s) = \mu(xs) = 1 = \mu(s)$.

case-2: Let $s \notin \mu_0$. Then $xs \notin \mu_0$, as μ_0 is a completely prime ideal of S. Thus $< x, \mu > (s) = \mu(xs) = \alpha = \mu(s)$. So $< x, \mu > (s) = \mu(s)$, $\forall s \in S$. Hence $< x, \mu >= \mu$. □

Proposition 3.21. [3] *Let I be a prime ideal of a semiring S. Then the characteristic function λ_I is a fuzzy prime ideal of S.*

Proposition 3.22. *Let S be a commutative semiring and I be an ideal of S. If I is a prime ideal of S then for $x \in S$ such that $x \notin I$, $< x, \lambda_I >= \lambda_I$.*

Proof. Since I is a prime ideal of S, by Proposition 3.21, λ_I is a fuzzy prime ideal of S. Now $x \notin I$, $\lambda_I(x) = 0$. Then $x \notin (\lambda_I)_0$. Thus by Proposition 3.20, $< x, \lambda_I >= \lambda_I$. □

Proposition 3.23. *Let S be a commutative semiring and μ be a fuzzy prime ideal of S. Then $< x, \mu >$ is a fuzzy prime ideal of S for $x \in S$ such that $x \notin \mu_0$.*

Proof. Since S is commutative, μ is a fuzzy completely prime ideal of S. Then by Proposition 3.20, $< x, \mu >= \mu$. Thus $< x, \mu >$ is a fuzzy prime ideal of S. □

Proposition 3.24. *Let S be a semiring and μ be a fuzzy prime ideal of S. If $x \in \mu_0$, then $< x, \mu >= \lambda_S$.*

Proof. Since $x \in \mu_0$, $xs \in \mu_0$ for all $s \in S$. Then $< x, \mu > (s) = \mu(xs) = 1 = \lambda_S(s)$, for all $s \in S$. Thus $< x, \mu >= \lambda_S$. □

Theorem 3.25. *Let S be a commutative semiring and μ be a fuzzy subset of S such that $< s, \mu >= \mu$ for every $s \in S$. Then μ is constant.*

Proof. Let $x, y \in S$. Then $< x, \mu >= \mu$ and $< y, \mu >= \mu$. Thus

$$\mu(y) =< x, \mu > (y) = \mu(xy) = \mu(yx) =< y, \mu > (x) = \mu(x).$$

So μ is constant. □

Theorem 3.26. *Let S be a semiring, μ be a fuzzy ideal of S and $Im\mu = \{1, \alpha\}$. Suppose $< y, \mu >= \mu$ for all those $y \in S$ for which $\mu(y) = \alpha$. Then μ is a fuzzy weakly completely prime ideal of S.*

Proof. Let $x_1, x_2 \in S$. Since μ is a fuzzy ideal of S, we have

$$\mu(x_1x_2) \geq \mu(x_1) \text{ and } \mu(x_1x_2) \geq \mu(x_2). \tag{3.4}$$

Case-1: Suppose $\mu(x_1x_2) = \mu(x_1)$. Then by (3.4), $\mu(x_1) \geq \mu(x_2)$. Thus

$$max\{\mu(x_1), \mu(x_2)\} = \mu(x_1) = \mu(x_1x_2).$$

Case-2: Suppose $\mu(x_1x_2) \neq \mu(x_1)$. Then $\mu(x_1)$ can not be a maximal element of $\mu(S)$, otherwise $\mu(x_1) = 1 = \mu(x_1x_2)$, a contradiction. Thus $\mu(x_1) = \alpha$ and by hypothesis, $< x_1, \mu >= \mu$. So $< x_1, \mu > (x_2) = \mu(x_2)$, i.e., $\mu(x_1x_2) = \mu(x_2)$. Hence $\mu(x_2) = \mu(x_1x_2) \geq \mu(x_1)$ and thus $\mu(x_1x_2) = max\{\mu(x_1), \mu(x_2)\}$. Therefore μ is a fuzzy weakly completely prime ideal. □

Proposition 3.27. *Let S be a commutative semiring and μ be a fuzzy weakly completely prime ideal of S, then $< x, \mu >$ is a fuzzy weakly completely prime ideal of S for every $x \in S$.*

Proof. Since μ is a fuzzy ideal of a commutative semiring S, by Proposition 3.3, $< x, \mu >$ is a fuzzy ideal of S for every $x \in S$. Let $y, z \in S$. Then

$$\begin{aligned} < x, \mu > (yz) &= \mu(xyz) \\ &= max\{\mu(xy), \mu(z)\} \\ &= max\{\mu(x), \mu(y), \mu(z)\} \\ &= max\{\mu(xy), \mu(xz)\} \\ &= max\{< x, \mu > (y), < x, \mu > (z)\} \end{aligned}$$

Thus $< x, \mu >$ is a fuzzy weakly completely prime ideal of S. □

Proposition 3.28. [5] *If $I(\neq S)$ is a semiprime ideal of a semiring S, then the characteristic function λ_I of I is a fuzzy semiprime ideal.*

Proposition 3.29. *Let S be a commutative semiring and μ be a fuzzy semiprime ideal of S. Then $< x, \mu >$ is a fuzzy semiprime ideal of S for every $x \in S$.*

Proof. Let $x \in S$. As μ is a fuzzy ideal of S, by Proposition 3.3, $< x, \mu >$ is a fuzzy ideal of S. Let $y \in S$. Then $< x, \mu > (y^2) = \mu(xy^2) \geq \mu(xy) =< x, \mu > (y)$.
Again

$$\begin{aligned} < x, \mu > (y) &= \mu(xy) \\ &= \mu(xy)^2, && \text{as } \mu \text{ is semiprime} \\ &= \mu(x^2y^2), && \text{as } S \text{ is commutative} \\ &\geq \mu(xy^2) \\ &= < x, \mu > (y^2). \end{aligned}$$

Thus $< x, \mu > (y^2) =< x, \mu > (y)$, $\forall y \in S$. So $< x, \mu >$ is a fuzzy semiprime ideal of S. □

Corollary 3.30. *Let S be a commutative semiring, $\{\mu_i\}_{i\in\Lambda}$ be a nonempty family of fuzzy semiprime ideals of S and $\mu = \inf\limits_{i\in\Lambda} \mu_i$. Then for any $x \in S$, $< x, \mu >$ is a fuzzy semiprime ideal of S.*

Proof. Obviously μ is a fuzzy subset of S. Let $x, y \in S$. Then

$$\begin{aligned} \mu(x + y) &= \inf_{i\in\Lambda} \mu_i(x + y) \\ &\geq \inf_{i\in\Lambda} min\{\mu_i(x), \mu_i(y)\} \\ &= min\{\inf_{i\in\Lambda} \mu_i(x), \inf_{i\in\Lambda} \mu_i(y)\} \\ &= min\{\mu(x), \mu(y)\}. \end{aligned}$$

Also,

$$\begin{aligned} \mu(xy) &= \inf_{i\in\Lambda} \mu_i(xy) \\ &\geq \inf_{i\in\Lambda} \mu_i(x) \\ &= \mu(x). \end{aligned}$$

Similarly $\mu(xy) \geq \mu(y)$. Thus μ is a fuzzy ideal of S.
Now

$$\begin{aligned} \mu(y^2) &= \inf_{i\in\Lambda} \mu_i(y^2) \\ &= \inf_{i\in\Lambda} \mu_i(y), && \text{as each } \mu_i \text{ is semiprime} \\ &= \mu(y). \end{aligned}$$

So μ is a fuzzy semiprime ideal of S. Hence by Proposition 3.29, $< x, \mu >$ is a fuzzy semiprime ideal for all $x \in S$. □

Corollary 3.31. *Let S be a commutative semiring and $\{P_i\}_{i\in\Lambda}$ be a nonempty family of semiprime ideals of S and $P = \bigcap\limits_{i\in\Lambda} P_i \neq \phi$. Then $< x, \lambda_P >$ is a fuzzy semiprime ideal of S for every $x \in S$.*

Proof. Obviously $P = \bigcap\limits_{i\in\Lambda} P_i$ is a semiprime ideal of S. Then by Proposition 3.28, λ_P is a fuzzy semiprime ideal of S. Thus by Proposition 3.29, $< x, \lambda_P >$ is a fuzzy semiprime ideal of S for every $x \in S$. □

Corollary 3.32. *Let S be a commutative semiring and μ be a fuzzy weakly completely prime ideal of S. If μ is not constant then μ is not a maximal fuzzy weakly completely ideal of S.*

Proof. Since μ is a weakly fuzzy completely prime ideal of S, by Proposition 3.27, for each $x \in S$, $< x, \mu >$ is a fuzzy weakly completely prime ideal of S. Also there exists $x \in S$ such that $\mu \subset < x, \mu >$, otherwise $\mu =< x, \mu >$, $\forall x \in S$ which implies that μ is constant (by Theorem 3.25), a contradiction. Then μ is not a maximal fuzzy weakly completely ideal of S. $\square$

Theorem 3.33. *Let S be commutative semiring and μ be a fuzzy semiprime ideal of S, then*
$\mu = inf\{< x, \mu >: x \in S\}$.

Proof. By Proposition 3.8, $\mu \subseteq < x, \mu >, \forall x \in S$.
Let μ_1 be any fuzzy ideal of S such that $\mu_1 \subseteq < x, \mu >$, $\forall x \in S$. Let $y \in S$. Then

$$\begin{aligned} \mu_1(y) &\leq < y, \mu > (y) \\ &= \mu(y^2) \\ &= \mu(y), \qquad \text{as } \mu \text{ is semiprime.} \end{aligned}$$

Thus $\mu_1 \subseteq \mu$. So $\mu = inf\{< x, \mu >: x \in S\}$. $\square$

4. Fuzzy 3-weakly completely prime ideal

Definition 4.1. Let S be a semiring. A fuzzy ideal μ of S is called fuzzy 3-weakly completely prime (3-WCP) ideal if for any $x_1, x_2, x_3 \in S$,

$$\begin{aligned} \mu(x_1x_2x_3) &= max\{\mu(x_1x_2), \mu(x_1x_3)\} \\ &= max\{\mu(x_2x_3), \mu(x_2x_1)\} \\ &= max\{\mu(x_3x_1), \mu(x_3x_2)\}. \end{aligned}$$

Example 4.2. Let $S = 2\mathbb{Z}_0^+$. We define a fuzzy subset μ of S as follows:

$$\mu(0) = 1,\ \mu(2) = .3 \text{ and } \mu(2n) = .5, \text{ for } n \geq 2.$$

Obviously μ is a fuzzy ideal of S. Let $x_1,\ x_2,\ x_3 \in S$.

Case-1: If at least one of $x_1,\ x_2$ or x_3 is 0, then

$$\begin{aligned} \mu(x_1x_2x_3) = 1 &= max\{\mu(x_1x_2), \mu(x_1x_3)\} \\ &= max\{\mu(x_2x_3), \mu(x_2x_1)\} \\ &= max\{\mu(x_3x_1), \mu(x_3x_2)\}. \end{aligned}$$

Case-2: If each of $x_1,\ x_2$ and x_3 are nonzero, then each of $x_1x_2x_3,\ x_1x_2,\ x_1x_3,\ x_2x_3,\ x_2x_1,\ x_3x_1,\ x_3x_2$ are greater than or equal to 4. Thus

$$\begin{aligned} \mu(x_1x_2x_3) = .5 &= max\{\mu(x_1x_2), \mu(x_1x_3)\} \\ &= max\{\mu(x_2x_3), \mu(x_2x_1)\} \\ &= max\{\mu(x_3x_1), \mu(x_3x_2)\}. \end{aligned}$$

Thus μ is a fuzzy 3-WCP ideal of S.

Proposition 4.3. *Let S be a semiring and μ be a fuzzy ideal of S. If μ is a fuzzy weakly completely prime ideal of S then μ is a fuzzy 3-WCP ideal of S.*

Proof. Let $x_1, x_2, x_3 \in S$. Since μ is a fuzzy weakly completely prime ideal of S, $\mu(x_ix_j) = \mu(x_jx_i)$, $\forall i, j = 1, 2, 3$.
Now

$$\begin{aligned}\mu(x_1x_2x_3) &= max\{\mu(x_1), \mu(x_2x_3)\}\\ &\leq max\{\mu(x_1x_2), \mu(x_2x_3)\}\\ &\leq max\{\mu(x_1x_2x_3), \mu(x_1x_2x_3)\}\\ &= \mu(x_1x_2x_3).\end{aligned}$$

Then

$$\begin{aligned}\mu(x_1x_2x_2) &= max\{\mu(x_1x_2), \mu(x_2x_3)\}\\ &= max\{\mu(x_2x_1), \mu(x_2x_3)\}.\end{aligned}$$

Also

$$\begin{aligned}\mu(x_1x_2x_3) &= max\{\mu(x_1x_2), \mu(x_3)\}\\ &= max\{\mu(x_2x_1), \mu(x_3)\}\\ &= \mu(x_2x_1x_3)\end{aligned}$$

and

$$\begin{aligned}\mu(x_1x_2x_3) &= max\{\mu(x_1x_2), \mu(x_3)\}\\ &\leq max\{\mu(x_1x_2), \mu(x_1x_3)\}\\ &= max\{\mu(x_2x_1), \mu(x_1x_3)\}\\ &\leq max\{\mu(x_2x_1x_3), \mu(x_2x_1x_3)\}\\ &= \mu(x_2x_1x_3)\\ &= \mu(x_1x_2x_3).\end{aligned}$$

Thus $\mu(x_1x_2x_3) = max\{\mu(x_1x_2), \mu(x_1x_3)\}$.
Again

$$\begin{aligned}\mu(x_1x_2x_3) &= max\{\mu(x_1), \mu(x_2x_3)\}\\ &\leq max\{\mu(x_3x_1), \mu(x_2x_3)\}\\ &= max\{\mu(x_1x_3), \mu(x_2x_3)\}\\ &\leq max\{\mu(x_2x_1x_3), \mu(x_1x_2x_3)\}\\ &= max\{\mu(x_1x_2x_3), \mu(x_1x_2x_3)\}\\ &= \mu(x_1x_2x_3).\end{aligned}$$

So

$$\begin{aligned}\mu(x_1x_2x_3) &= max\{\mu(x_3x_1), \mu(x_2x_3)\}\\ &= max\{\mu(x_3x_1), \mu(x_3x_2)\}.\end{aligned}$$

Hence μ is a fuzzy 3-WCP ideal of S. □

Remark 4.4. If a semiring S contains the multiplicative identity then the notions of fuzzy weakly completely prime ideal and fuzzy 3-WCP ideal coincide.

The converse of the proposition 4.3 is not true in general. This follows from the following example.

Example 4.5. Let $S = 2\mathbb{Z}_0^+$. We define a fuzzy subset μ of S as follows:

$$\mu(0) = 1,\ \mu(2) = .3 \text{ and } \mu(2n) = .5, \text{ for } n \geq 2.$$

By example 4.2, μ is a fuzzy 3-WCP ideal of S. But μ is not a fuzzy weakly completely prime ideal of S, as $\mu(2.2) \neq \max\{\mu(2), \mu(2)\}$.

Theorem 4.6. *Let S be a commutative semiring and μ be a fuzzy ideal of S. Then μ is a fuzzy 3-WCP ideal of S if and only if any extension of μ by x, where $x \in S$ is a fuzzy weakly completely prime ideal of S.*

Proof. Suppose μ is fuzzy 3-WCP ideal of S and let $x \in S$. Then

$$
\begin{aligned}
< x, \mu > (x_1x_2) &= \mu(xx_1x_2) \\
&= max\{\mu(xx_1), \mu(xx_2)\}] \\
&= max\{< x, \mu > (x_1), < x, \mu > (x_2)\}, \forall x_1, x_2 \in S.
\end{aligned}
$$

Thus $< x, \mu >$ is fuzzy weakly completely prime fuzzy ideal of S for every $x \in S$.

Suppose $< x, \mu >$ is a fuzzy weakly completely prime ideal of S for every $x \in S$. Then

$$
\begin{aligned}
\mu(x_1x_2x_3) = < x_1, \mu > (x_2x_3) &= max\{< x_1, \mu > (x_2), < x_1, \mu > (x_3)\} \\
&= max\{\mu(x_1x_2), \mu(x_1x_3)\}.
\end{aligned}
$$

Since S is commutative,

$$
\begin{aligned}
\mu(x_1x_2x_3) &= max\{\mu(x_2x_3), \mu(x_2x_1)\} \\
&= max\{\mu(x_3x_1), \mu(x_3x_2)\}.
\end{aligned}
$$

Thus μ is a fuzzy 3-WCP ideal of S. □

Corollary 4.7. *Let S be a commutative semiring with unity 1 and μ be a fuzzy 3-WCP ideal of S. Then μ is a fuzzy weakly completely prime ideal of S.*

Proof. Now $< 1, \mu > (x) = \mu(1x) = \mu(x), \forall x \in S$. Then $< 1, \mu >= \mu$. Since μ is a fuzzy 3-WCP, by Theorem 4.6, $< 1, \mu >$ is a fuzzy weakly completely prime ideal of S. Thus

$$
\begin{aligned}
< 1, \mu > (xy) &= max\{< 1, \mu > (x), < 1, \mu > (y)\}, \\
\text{i.e., } \mu(xy) &= max\{\mu(x), \mu(y)\}, \forall x, y \in S.
\end{aligned}
$$

Hence μ is fuzzy weakly completely prime ideal of S. □

Corollary 4.8. *Let S be a commutative semiring and μ be a fuzzy ideal of S. If μ is a fuzzy 3-WCP and fuzzy semiprime ideal of S then μ is the infimum of all fuzzy weakly completely prime ideal of S containing μ.*

Proof. Follows from Theorem 3.33 and Theorem 4.6. □

5. Conclusions

In this paper we study the relationship between fuzzy weakly completely prime, fuzzy 3-weakly completely prime by means of the extensions of fuzzy ideals of semiring. It can be investigated the other interrelations between the fuzzy ideals and extension of fuzzy ideals in case of h-fuzzy ideals, irreducible fuzzy ideals, fuzzy bi-ideals, fuzzy quasi-ideals etc.

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References

[1] D. R. LaTorre, On h-ideals and k-ideals in hemirings, Publ. Math. Debrecen 12 (1965) 219–226.
[2] Jonathan S. Golan, Semirings and their Applications, Kluwer Acad. Publ., Dordrecht 1999.
[3] T. K. Dutta and B. K. Biswas, Fuzzy Prime Ideals of A Semirirng Bull. Malays. Math. Sci. Soc. 17 (2) (1994) 9–16.
[4] T. K. Dutta and B. K. Biswas, Fuzzy k-Ideals of Semirirngs, Bull. calcutta Math. Soc. 87 (1995) 91–96.
[5] T. K. Dutta and B. K. Biswas, On Fuzzy Semiprime Ideals of A Semirirng, J. Fuzzy Math. 8 (3) (2000) 1–7.
[6] T. K. Dutta and B. K. Biswas, On Completely Fuzzy Semiprime Ideals of a Semirirng, J. Fuzzy Math. 8 (3) (2000) 577–581.

[7] Xiang-Yun Xie, Fuzzy Ideals Extensions of Semigroups, Soochow J. Math. 27 (2) (2001) 125–138.
[8] Xiang-Yun Xie and Feng Yan, Fuzzy Ideals Extensions of Ordered Semigroups, Lobachevskii J. Math. 19 (2005) 29–40.

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