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Topology of soft double sets

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Abstract. The aims of this paper first, is to define and study soft double sets and some of its basic properties. Second, is to define and study soft double topological space and some properties related to it. Third, is to define and study the mappings on soft double topological spaces and some properties. Finally, is to define and study soft double continuity between soft double topological spaces.

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1. Introduction

Atanassov [3, 4, 5, 6] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker [8] generalized topological structures in intuitionistic fuzzy case. The concept of intuitionistic fuzzy topological spaces and the topology on intuitionistic sets was first given by Coker [7, 9].
Flou set stems from linguistic considerations of Yves Gentilhomme [12] about the vocabulary of a natural language. The mathematical definition of flou sets and binary operations on it are introduced by E. E. Kerre [21].
In 2005, the suggestion of J. G. Garcia et al. [10] that double set is a more appropriate name than flou set, and double topology for the flou topology.
In 2007, Kandil et al. [20] proved the $1-1$ correspondence mapping f between the set of all flou (double) sets and the set of all intuitionistic sets defined as: $f(A_1, A_2) = (A_1, A_2^c)$, A_2^c is the complement of A_2 . Kandil et al. [19, 20] introduced the concept of double sets, double topological spaces and continuous functions between these spaces. They also introduced separation axioms in double topological spaces. In 2009, Kandil et al. [19] introduced the notion of double-compact topological space and studied some fundamental properties of this notion. In 2014, Kandil

et al. [18] introduced some types of compactness in double topological spaces.
There are some theories: the theory of rough sets [27], the theory of vague sets [11], and the theory of fuzzy sets [31], which can be regarded as mathematical tools for dealing with uncertainties. However, all these theories have their own difficulties. The main reason for these difficulties is, possible, the inadequacy of the parametrization tool of the theory as it was mentioned by Molodtsov [26]. In [26, 25] Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, operations research, probability, theory of measurement etc. After presentation of the operations of soft sets [24], the properties and applications of soft set theory have been studied increasingly [3, 23, 24, 28].
Recently, in 2011, Shabir and Naz [29] initiated the study of soft topological spaces. They defined soft topology on the collection of soft sets over X. Consequently, they defined basic notions of soft topological spaces. Hussain and Ahmad [13] investigated the properties of soft nbds and soft closure operator. Some soft topological properties were introduced in [1, 2, 14, 15, 16, 17].

In this paper we give an overview of double (soft) sets and some of related topics, we introduce the notions of soft double sets, soft double points, quasi-coincident relation on soft double sets, soft double topological spaces and we investigate the fundamental properties of these notions. Also we define a soft double closure(resp. interior) operator, soft double neighborhoods and soft double continuous mappings. Moreover, some basic properties of these notions have obtained.

2. Preliminaries

In this section, we collect some definitions and theorems which will be needed in the sequel. For more details see [13, 19, 20, 22, 24, 26, 29, 30, 32].

Definition 2.1 ([20]). Let X be a nonempty set.

(i) A double set $\underline{A}$ is an ordered pair $(A_1, A_2) \in P(X) \times P(X)$ such that $A_1 \subseteq A_2$.

(ii) $D(X) = \{(A_1, A_2) \in P(X) \times P(X), A_1 \subseteq A_2\}$ is the family of all double sets on X.

(iii) Let $\eta_1, \eta_2 \subseteq P(X)$. The product of η_1 and η_2, denoted by $\eta_1 \times \eta_2$, defined by:

$$\eta_1 \times \eta_2 = \{(A_1, A_2) \in \eta_1 \times \eta_2 : A_1 \subseteq A_2\}.$$

(iv) The double set $\underline{X} = (X, X)$ is called the universal double set.

(v) The double set $\underline{\varnothing} = (\varnothing, \varnothing)$ is called the empty double set.

Definition 2.2 ([20]). Let $\underline{A} = (A_1, A_2), \underline{B} = (B_1, B_2) \in D(X)$.

(i) $\underline{A} = \underline{B} \Leftrightarrow A_i = B_i, i = 1, 2$.

(ii) $\underline{A} \subseteq \underline{B} \Leftrightarrow A_i \subseteq B_i, i = 1, 2$.

(iii) $\underline{A} \cup \underline{B} = (A_1 \cup B_1, A_2 \cup B_2)$.

(iv) $\underline{A} \cap \underline{B} = (A_1 \cap B_1, A_2 \cap B_2)$.

(v) $\underline{A}^c = (A_2^c, A_1^c)$, where $\underline{A}^c$ is the complement of $\underline{A}$.

(vi) $\underline{A} \setminus \underline{B} = (A_1 \setminus B_2, A_2 \setminus B_1)$.

(vii) Let $x \in X$. Then, the double sets $\underline{x}_1 = (\{x\}, \{x\})$ and $\underline{x}_{0.5} = (\varnothing, \{x\})$ are said to be double points in X.

The family of all double points we denoted by $DP(X)$, i.e., $DP(X) = \{x_t : x \in X, t \in \{1, 0.5\}\}$. (viii) $\underline{x}_1 \underline{\in} \underline{A} \Leftrightarrow x \in A_1$ and $\underline{x}_{0.5} \underline{\in} \underline{A} \Leftrightarrow x \in A_2$.

Definition 2.3 ([19]). Two double sets $\underline{A}$ and $\underline{B}$ are said to be a quasi-coincident, denoted by $\underline{A}q\underline{B}$, if $A_1 \cap B_2 \neq \varnothing$ or $A_2 \cap B_1 \neq \varnothing$. $\underline{A}$ is called a not quasi-coincident with $\underline{B}$, denoted by $\underline{A} \not{q}\underline{B}$, if $A_1 \cap B_2 = \varnothing$ and $A_2 \cap B_1 = \varnothing$.

Definition 2.4 ([20]). Consider two ordinary sets X and Y. Let f be a mapping from X into Y. Then, the image of a double set $\underline{A}$ *in* $D(X)$ defined by:

$$f(\underline{A}) = (f(A_1), f(A_2)).$$

Also the inverse image of a double set $\underline{B} \in D(Y)$ defined by:

$$f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(B_2)).$$

Definition 2.5 ([20]). Let X be a non-empty set. The family η of double sets in X is called a double topology on X if it satisfies the following axioms:

(i) $\underline{\varnothing}, \underline{X} \in \eta$,

(ii) if $\underline{A}, \underline{B} \in \eta$, then $\underline{A} \sqcap \underline{B} \in \eta$,

(iii) If $\{\underline{A}_s : s \in S\} \subseteq \eta$, then $\underline{\bigcup}_{s\in S}\underline{A}_s \in \eta$.

The pair (X, η) is called a double topological space.

Definition 2.6 ([20]). Let (X, η) be a double topological space. A double set $\underline{O}_{\underline{x}_t}$ is called an open neighborhood of the double point $\underline{x}_t (t \in \{1, 0.5\})$ if $\underline{O}_{\underline{x}_t} \in \eta$ and $\underline{x}_t \underline{\in} \underline{O}_{\underline{x}_t}$. The family of all double neighborhoods of the point $\underline{x}_t$ will be denoted by $\underline{N}(\underline{x}_t)$.

Definition 2.7 ([20]). Let (X, η) be a double topological space and $\underline{A} \in D(X)$. The double closure of $\underline{A}$, denoted by $cl_\eta(\underline{A})$ or $\overline{\underline{A}}$, defined by:

$$\overline{\underline{A}} = \underline{\cap}\{\underline{B} : \underline{B} \in \eta^c \text{ and } \underline{A} \ \subseteq\ \underline{B}\}.$$

Definition 2.8 ([20]). Let (X, η) be a double topological space and $\underline{A} \in D(X)$. The double interior of $\underline{A}$, denoted by $int_\eta(\underline{A})$ or $\underline{A}^o$, defined by:

$$\underline{A}^o = \underline{\cup}\{\underline{B} : \underline{B} \in \eta \text{ and } \underline{B} \subseteq \underline{A}\}.$$

Definition 2.9 ([20]). Let $f : X \to Y$ be a mapping and let (X, η) and (Y, η^*) be double topological spaces. Then, f is called a D-continuous if $f^{-1}(\underline{B}) \in \eta$, whenever $\underline{B} \in \eta^*$.

Secondly, basic concepts about soft sets will be given:

Definition 2.10 ([26]). Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non-empty subset of E. A soft set F_A over the universal X is a mapping from the parameter set E to $P(X)$ with support A i.e., $F_A : E \longrightarrow P(X)$. In other words a soft set over X is a parametrized family of subsets of X, where $F_A(e) \neq \varnothing$ if $e \in A \subseteq E$ and $F_A(e) = \varnothing$ if $e \notin A$.
Note that a soft set can be written in the following form, $F_A = \{(e, F_A(e)) : e \in A \subseteq E, F_A : E \longrightarrow P(X)\}$.

In this paper we use the notation F_E for any soft subset, where $F_E(e) \neq \varnothing$ $\forall e \in A$ and $F_E(e) = \varnothing$, $\forall e \notin A$.

Definition 2.11 ([24]). Let $F_E, G_E \in S(X, E)$.

(i) F_E is said to be a null soft set, denoted by Φ, if $F_E(e) = \varnothing,\ \forall e \in E$.

(ii) F_E is called absolute soft set, denoted by X_E, if $F_E(e) = X,\ \forall e \in E$.

(iii) F_E is called a soft subset of G_E, denoted by $F_E \sqsubseteq G_E$, if

$$F_E(e) \subseteq G_E(e),\ \forall e \in E.$$

(iv) F_E and G_E are called equal, denoted by $F_E = G_E$, if $F_E \sqsubseteq G_E$ and $G_E \sqsubseteq F_E$.

(v) The union of F_E and G_E is a soft set H_E defined by

$$H_E(e) = F_E(e) \cup G_E(e),\ \forall e \in E.$$

(vi) The intersection of F_E and G_E is a soft set H_E defined by

$$H_E(e) = F_E(e) \cap G_E(e),\ \forall e \in E.$$

(vii) The difference of F_E and G_E is a soft set H_E defined by

$$H_E(e) = F_E(e) - G_E(e),\ \forall e \in E.$$

We write $H_E = F_E - G_E$.

(viii) The complement of F_E, denoted by F_E^c, defined by

$$F_E^c(e) = X - F_E(e),\ \forall e \in E.$$

Definition 2.12 ([32]). Let $F_E \in S(X, E)$. Then, F_E is called a soft point over X, if there exist $e \in E$ and $x \in X$ such that

$$F(\varepsilon) = \begin{cases} \{x\}, & \text{if } \varepsilon = e; \\ \varnothing, & \text{if } \varepsilon \in E - \{e\}. \end{cases}$$

and F_E denoted by x_e.

Definition 2.13 ([32]). The soft point x_e is said to be belonging to the soft set G_E, denoted by $x_e \widetilde{\in} G_E$, if for the element $e \in E,\ F_E(e) \sqsubseteq G_E(e)$.

Proposition 2.14 ([30]). *The union of any collection of soft points can be considered as a soft set and every soft set can be expressed as union of all soft points belonging to it.*

Definition 2.15 ([22]). Let $S(X, E)$ and $S(Y, K)$ be the families of all soft sets over X and Y, respectively.

(i) The mapping $f_{\beta\psi} : S(X, E) \to S(Y, K)$ is called a soft mapping from, where $\beta : X \to Y$ and $\psi : E \to K$ are two mappings.

(ii) Let $F_E \in S(X, E)$. Then, the image of F_E under the soft mapping f_{β_ψ} is a soft set over Y, denoted by $f_{\beta\psi}(F_E)$, and defined by:

$$f_{\beta\psi}(F_E)(k) = \begin{cases} \beta(\bigcup_{e \in \psi^{-1}(k) \cap E} F_E(e)), & \text{if } \psi^{-1}(k) \cap E \neq \varnothing; \\ \varnothing, & \text{otherwise.} \end{cases}$$

(iii) Let $G_K \in S(Y, K)$. Then, the pre-image of G_K under the soft mapping $f_{\beta\psi}$ is a soft set over X, denoted by $f^{-1}_{\beta\psi}(G_K)$, and defined by:

$$f^{-1}_{\beta\psi}(G_K)(e) = \begin{cases} \beta^{-1}(G_K(\psi(e))), & \text{if } \psi(e) \in K; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Definition 2.16 ([29]). Let τ be a collection of soft sets over a universal X with a fixed set of parameters E. Then, τ is called a soft topology on X if it satisfies the following conditions:

(i) $\Phi, X_E \in \tau$,

(ii) The union of any number of soft sets in τ belongs to τ,

(iii) The intersection of any two soft sets in τ belongs to τ.

The triple (X, τ, E) is called a soft topological space over$X$. Every element of τ is called an open soft set in X and its complement is called a closed soft set in X.

Definition 2.17 ([32])**.** Let (X, τ, E) be a soft topological space and let $F_E \in S(X, E)$. Then, F_E is called a soft neighborhood of the soft point x_e, if there exists $G_E \in \tau$ such that $x_e \widetilde{\in} G_E \sqsubseteq F_E$. The family of all neighborhoods of x_e, denoted by N_{x_e}.

Definition 2.18 ([29])**.** Let (X, τ, E) be a soft topological space and $F_E \in S(X, E)$. Then, the soft closure of F_E, denoted by $scl(F_E)$, defined by:

$$scl(F_E) = \sqcap\{G_E : F_E \sqsubseteq G_E \text{ and } G_E \text{ is a closed soft set in } X\}.$$

Definition 2.19 ([13])**.** Let (X, τ, E) be a soft topological space and $F_E \in S(X, E)$. Then, the soft interior of F_E, denoted by $sint(F_E)$, defined by:

$$sint(F_E) = \sqcup\{G_E : G_E \sqsubseteq F_E \text{ and } G_E \in \tau\}.$$

Definition 2.20 ([32])**.** Let $f_{\beta\psi} : S(X, E) \to S(Y, K)$, where $\beta : X \to Y$ and $\psi : E \to K$. Let (X, τ, E) and (Y, σ, K) be two soft topological spaces. Then, $f_{\beta\psi}$ is called a soft continuous if $f^{-1}_{\beta\psi}(H_K) \in \tau$ whenever $H_K \in \sigma$.

3. Soft double sets

Definition 3.1. Let X be an initial universe and E be a set of parameters. Let $D(X)$ denote the family of all double sets over the universal X.A soft double set $\widetilde{F}_A$ over the universal X is a mapping from the parameter set E to $D(X)$ with support A i.e., $\widetilde{F}_A : E \longrightarrow D(X)$. In other words a soft double set over the universal X is a parametrized family of double subsets of X, where $\widetilde{F}_A(e) \neq \underline{\varnothing}$ if $e \in A \subseteq E$ and $\widetilde{F}_A(e) = \underline{\varnothing}$ if $e \notin A$.
Note that a soft double set can be written in the following form, $\widetilde{F}_A = \{(e, \widetilde{F}_A(e)) : e \in A \subseteq E, \widetilde{F}_A : E \longrightarrow D(X)\}$.

The family of all soft double sets over X denoted by $SD(X)_E$.
In this paper, we use the notation $\widetilde{F}_E$ for any soft double subset where, $\widetilde{F}_E(e) \neq \varnothing$, $\forall e \in A$ *and* $\widetilde{F}_E(e) = \varnothing$, $\forall e \notin A$.

Definition 3.2. The soft double set $\widetilde{F}_E \in SD(X)_E$ is called a null soft double set, denoted by $\widetilde{\Phi}$, where $\widetilde{F}_E(e) = \underline{\varnothing}$, $\forall e \in E$.

Definition 3.3. The soft double set $\widetilde{F}_E \in SD(X)_E$ is called an absolute soft double set, denoted by $\widetilde{X}$, where $\widetilde{F}_E(e) = \underline{X}$, $\forall e \in E$.

Definition 3.4. Let $\widetilde{F}_E, \widetilde{G}_E \in SD(X)_E$. *Then,* $\widetilde{F}_E$ is a soft double subset of $\widetilde{G}_E$, denoted by $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E$, if $\widetilde{F}_E(e) \subseteq \widetilde{G}_E(e)$, $\forall e \in E$.

Definition 3.5. Let $\widetilde{F}_E, \widetilde{G}_E \in SD(X)_E$. Then, $\widetilde{F}_E$ is equal to $\widetilde{G}_E$, denoted by $\widetilde{F}_E = \widetilde{G}_E$, if $\widetilde{F}_E(e) = \widetilde{G}_E(e)$ $\forall e \in E$.

Definition 3.6. Let $\widetilde{F}_E, \widetilde{G}_E \in SD(X)_E$. Then, the union of $\widetilde{F}_E$ and $\widetilde{G}_E$ is a soft double set $\widetilde{H}_E$ defined by

$$\widetilde{H}_E(e) = (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E)(e) = \widetilde{F}_E(e) \underline{\bigcup} \widetilde{G}_E(e),\ \forall e \in E.$$

We write $\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E = \widetilde{H}_E$.

Definition 3.7. Let $\widetilde{F}_E, \widetilde{G}_E \in SD(X)_E$. Then, the intersection of $\widetilde{F}_E$ and $\widetilde{G}_E$ is a soft double set $\widetilde{H}_E$ defined by:

$$\widetilde{H}_E(e) = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E)(e) = \widetilde{F}_E(e) \underline{\bigcap} \widetilde{G}_E(e),\ \forall e \in E.$$

We write $\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E = \widetilde{H}_E$.

Definition 3.8. Let $\widetilde{F}_E, \widetilde{G}_E \in SD(X)_E$. Then, the difference of $\widetilde{F}_E$ and $\widetilde{G}_E$ is a soft double set $\widetilde{H}_E$ defined by

$$\widetilde{H}_E(e) = \widetilde{F}_E(e) \setminus \widetilde{G}_E(e),\ \forall e \in E.$$

We write $\widetilde{H}_E = \widetilde{F}_E \setminus \widetilde{G}_E$.

Definition 3.9. Let $\widetilde{F}_E \in SD(X)_E$. Then, the complement of $\widetilde{F}_E$, denoted by $\widetilde{F}^c_E$, defined by

$$\widetilde{F}^c_E(e) = \underline{X} \setminus \widetilde{F}_E(e),\ \forall e \in E.$$

Example 3.10. Let $X = \{h_1, h_2, h_3\}, E = \{e_1, e_2, e_3, e_4\}$ and
$\widetilde{F}_E = \{(e_1, (\{h_2\}, \{h_1, h_2\})), (e_2, (\{h_1\}, \{h_1, h_2\})), (e_3, \underline{\varnothing}), (e_4, \underline{\varnothing})\}$. Then

$\widetilde{F}^c_E(e_1) = (X, X) \setminus (\{h_2\}, \{h_1, h_2\}) = (\{h_3\}, \{h_1, h_3\}),$
$\widetilde{F}^c_E(e_2) = (X, X) \setminus (\{h_1\}, \{h_1, h_2\}) = (\{h_3\}, \{h_2, h_3\}),$
$\widetilde{F}^c_E(e_3) = (X, X) \setminus (\varnothing, \varnothing) = \underline{X},$
$\widetilde{F}^c_E(e_4) = (X, X) \setminus (\varnothing, \varnothing) = \underline{X}.$

Thus, $\widetilde{F}^c_E = \{(e_1, (\{h_3\}, \{h_1, h_3\})), (e_2, (\{h_3\}, \{h_2, h_3, (e_3, \underline{X}), (e_4, \underline{X})\}))\}$.

Proposition 3.11. *Let $\widetilde{F}_E, \widetilde{G}_E, \widetilde{H}_E \in SD(X)_E$. Then, $(SD(X)_E, \bigcup, \bigcap, ^c)$ is a Morgan, i.e., it satisfies the following axioms:*

(1) $\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E = \widetilde{G}_E \widetilde{\bigcup} \widetilde{F}_E,$
(2) $\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E = \widetilde{G}_E \widetilde{\bigcap} \widetilde{F}_E,$
(3) $(\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E) \widetilde{\bigcup} \widetilde{H}_E = \widetilde{F}_E \widetilde{\bigcup} (\widetilde{G}_E \widetilde{\bigcup} \widetilde{H}_E),$
(4) $(\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \widetilde{\bigcap} \widetilde{H}_E = \widetilde{F}_E \widetilde{\bigcap} (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E),$
(5) $\widetilde{F}_E \widetilde{\bigcup} \widetilde{\Phi} = \widetilde{F}_E,$
(6) $\widetilde{F}_E \widetilde{\bigcup} \widetilde{X} = \widetilde{X},$
(7) $\widetilde{F}_E \widetilde{\bigcap} \widetilde{\Phi} = \widetilde{\Phi},$
(8) $\widetilde{F}_E \widetilde{\bigcap} \widetilde{X} = \widetilde{F}_E,$
(9) $\widetilde{F}_E \widetilde{\bigcup} \widetilde{F}_E = \widetilde{F}_E,$
(10) $\widetilde{F}_E \widetilde{\bigcap} \widetilde{F}_E = \widetilde{F}_E,$
(11) $\widetilde{F}_E \widetilde{\bigcup} (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E) = (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E) \widetilde{\bigcap} (\widetilde{F}_E \widetilde{\bigcup} \widetilde{H}_E),$
(12) $\widetilde{F}_E \widetilde{\bigcap} (\widetilde{G}_E \widetilde{\bigcup} \widetilde{H}_E) = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \widetilde{\bigcup} (\widetilde{F}_E \widetilde{\bigcap} \widetilde{H}_E),$
(13) $(\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E)^c = \widetilde{F}^c_E \widetilde{\bigcap} \widetilde{G}^c_E,$
(14) $(\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E)^c = \widetilde{F}^c_E \widetilde{\bigcup} \widetilde{G}^c_E.$

Proof. As a sample, we prove the properties $(1), (4)$ and (11), the proofs of the remaining cases are similar.

(1) Let $e \in E$. Then

$$(\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E)(e) = \widetilde{F}_E(e) \underline{\bigcup} \widetilde{G}_E(e) = \widetilde{G}_E(e) \underline{\bigcup} \widetilde{F}_E(e) = (\widetilde{G}_E \widetilde{\bigcup} \widetilde{F}_E)(e).$$

Thus $\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E = \widetilde{G}_E \widetilde{\bigcup} \widetilde{F}_E$.

(4) Let $e \in E$. Then

$$\begin{aligned}[(\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \widetilde{\bigcap} \widetilde{H}_E](e) &= (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E)(e) \underline{\bigcap} \widetilde{H}_E(e) \\ &= \widetilde{F}_E(e) \underline{\bigcap} \widetilde{G}_E(e) \underline{\bigcap} \widetilde{H}_E(e) \\ &= \widetilde{F}_E(e) \underline{\bigcap} (\widetilde{G}_E(e) \underline{\bigcap} \widetilde{H}_E(e)) \\ &= \widetilde{F}_E(e) \underline{\bigcap} (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E)(e) \\ &= [\widetilde{F}_E \widetilde{\bigcap} (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E)](e).\end{aligned}$$

Thus $(\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \widetilde{\bigcap} \widetilde{H}_E = \widetilde{F}_E \widetilde{\bigcap} (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E)$.

(11) Let $e \in E$. Then

$$\begin{aligned}[\widetilde{F}_E \widetilde{\bigcup} (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E)](e) &= \widetilde{F}_E(e) \underline{\bigcup} (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E)(e) \\ &= \widetilde{F}_E(e) \underline{\bigcup} (\widetilde{G}_E(e) \underline{\bigcap} \widetilde{H}_E(e)) \\ &= (\widetilde{F}_E(e) \underline{\bigcup} \widetilde{G}_E(e)) \underline{\bigcap} (\widetilde{F}_E(e) \underline{\bigcup} \widetilde{H}_E(e)) \\ &= (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E)(e) \underline{\bigcap} (\widetilde{F}_E \widetilde{\bigcup} \widetilde{H}_E)(e) \\ &= [(\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E) \widetilde{\bigcap} (\widetilde{F}_E \widetilde{\bigcup} \widetilde{H}_E)](e).\end{aligned}$$

tHUS $\widetilde{F}_E \widetilde{\bigcup} (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E) = (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E) \widetilde{\bigcap} (\widetilde{F}_E \widetilde{\bigcup} \widetilde{H}_E)$. □

Proposition 3.12. *Let* $\widetilde{F}_E, \widetilde{G}_E \in SD(X)_E$. *Then,*

$$\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E \Rightarrow \widetilde{G}^c_E \widetilde{\subseteq} \widetilde{F}^c_E.$$

Proof. Straightforward. □

Remark 3.13. If $\widetilde{F}_E \in SD(X)_E$, then

(1) $\widetilde{F}_E \widetilde{\bigcup} \widetilde{F}^c_E \neq \widetilde{X}$, in general,

(2) $\widetilde{F}_E \widetilde{\bigcap} \widetilde{F}^c_E \neq \widetilde{\Phi}$, in general.

Example 3.14. From Example 3.10, let $\widetilde{G}_E = \{(e_1, (\varnothing, \{h_3\})), (e_2, (\varnothing, \{h_2\})), (e_3, \underline{\varnothing}), (e_4, \underline{\varnothing})\}$.
Then $\widetilde{G}^c_E = \{(e_1, (\{h_1, h_2\}, X)), (e_2, (\{h_1, h_3\}, X)), (e_3, \underline{X}), (e_4, \underline{X})\}$.
Now, $\widetilde{G}_E \widetilde{\bigcup} \widetilde{G}^c_E = \{(e_1, (\{h_1, h_2\}, X)), (e_2, (\{h_1, h_3\}, X)), (e_3, \underline{X}), (e_4, \underline{X})\} \neq \widetilde{X}$ and
$\widetilde{G}_E \widetilde{\bigcap} \widetilde{G}^c_E = \{(e_1, (\varnothing, \{h_3\})), (e_2, (\varnothing, \{h_2\})), (e_3, \underline{\varnothing}), (e_4, \underline{\varnothing})\} \neq \widetilde{\Phi}$.

4. Soft double points and soft double mappings

Definition 4.1. Let $\widetilde{F}_E \in SD(X)_E$. Then, $\widetilde{F}_E$ is called a soft double point over X if there exist $e \in E, x \in X$ and $t \in \{0.5, 1\}$ such that

$$\widetilde{F}_E(\alpha) = \begin{cases} \underline{x}_t, & \text{if } \alpha = e; \\ \underline{\varnothing}, & \text{if } \alpha \in E - \{e\}. \end{cases}$$

and we will denote $\widetilde{F}_E$ by $\widetilde{x}^e_t$. The family of all soft double points over X will be denoted by $SDP(X)_E$.

Definition 4.2. Let $\widetilde{F}_E$ be a soft double set over X. Then, a soft double point $\widetilde{x}_t^e$ belongs to $\widetilde{F}_E$, denoted by $\widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E$, if $\underline{x}_t \underline{\in} \widetilde{F}_E(e)$.

Theorem 4.3. *The soft double set* $\widetilde{F}_E = \widetilde{\bigcup}\{\widetilde{x}_t^e : \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E\}$.

Proof. $\widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E \Leftrightarrow \underline{x}_t \underline{\in} \widetilde{F}_E(e)$.
On one hand,
$\widetilde{\bigcup}\{\widetilde{x}_t^e : \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E\}(e) = \underline{\bigcup}\{\widetilde{x}_t^e(e) : \underline{x}_t \underline{\in} \widetilde{F}_E(e)\} = \underline{\bigcup}\{\underline{x}_t : \underline{x}_t \underline{\in} \widetilde{F}_E(e)\} = \widetilde{F}_E(e)$ [20].
Thus $\widetilde{F}_E = \widetilde{\bigcup}\{\widetilde{x}_t^e : \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E\}$. □

Example 4.4. Let $X = \{h_1, h_2, h_3, h_4, h_5\}$ *and* $E = \{e_1, e_2, e_3\}$. We define $\widetilde{F}_E(e_1) = (\{h_1, h_2\}, \{h_1, h_2, h_3\}), \widetilde{F}_E(e_2) = (\{h_3\}, \{h_1, h_3\}), \widetilde{F}_E(e_3) = (\{h_5\}, X)$.
Then, $SDP(X)_E = \{(h_i)_t^{e_j} : 1 \le i \le 5, 1 \le j \le 3 \text{ and } t \in \{0.5, 1\}\}$.
All points belong to $\widetilde{F}_E$ are:

$(h_1)_1^{e_1}, (h_1)_{0.5}^{e_1}, (h_2)_1^{e_1}, (h_2)_{0.5}^{e_1}, (h_3)_{0.5}^{e_1}$,
$(h_3)_1^{e_2}, (h_3)_{0.5}^{e_2}, (h_1)_{0.5}^{e_2}$,
$(h_5)_1^{e_3}, (h_1)_{0.5}^{e_3}, (h_2)_{0.5}^{e_3}, (h_3)_{0.5}^{e_3}, (h_4)_{0.5}^{e_3}, (h_5)_{0.5}^{e_3}$.

Then, $\widetilde{F}_E = \{(e_1, (\{h_1, h_2\}, \{h_1, h_2, h_3\})), (e_2, (\{h_3\}, \{h_1, h_3\})), (e_3, (\{h_5\}, X))\}$.

Theorem 4.5. *Let* $\widetilde{F}_E, \widetilde{G}_E \in SD(X)_E$.

(1) $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E \Leftrightarrow (\widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E \Rightarrow \widetilde{x}_t^e \widetilde{\in} \widetilde{G}_E)$.
(2) $\widetilde{x}_t^e \widetilde{\in} (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E) \Leftrightarrow \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E$ *or* $\widetilde{x}_t^e \widetilde{\in} \widetilde{G}_E$.
(3) $\widetilde{x}_t^e \widetilde{\in} (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \Leftrightarrow \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E$ *and* $\widetilde{x}_t^e \widetilde{\in} \widetilde{G}_E$.
(4) *If* $\{(\widetilde{F}_E)_s : s \in S\} \subseteq SD(X)_E$, *then*
(i) $\widetilde{x}_t^e \widetilde{\in} \widetilde{\bigcup}_{s \in S} (\widetilde{F}_E)_s \Leftrightarrow (\exists s \in S)(\widetilde{x}_t^e \widetilde{\in} (\widetilde{F}_E)_s)$,
(ii) $\widetilde{x}_t^e \widetilde{\in} \widetilde{\bigcap}_{s \in S} (\widetilde{F}_E)_s \Leftrightarrow (\forall s \in S)(\widetilde{x}_t^e \widetilde{\in} (\widetilde{F}_E)_s)$.

Proof. As a sample, we prove the $(1), (2)$ and $(4(\text{ii}))$.

(1) Suppose that $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E$ and let $\widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E$. Then, $\underline{x}_t \underline{\in} \widetilde{F}_E(e)$ implies $\underline{x}_t \underline{\in} \widetilde{G}_E(e)$. Thus, $\widetilde{x}_t^e \widetilde{\in} \widetilde{G}_E$.

Conversely, suppose $\widetilde{F}_E(e) \nsubseteq \widetilde{G}_E(e)$ for some $e \in E$. Then $\exists \underline{x}_t \in DP(X)$ such that $\underline{x}_t \underline{\in} \widetilde{F}_E(e)$ and $\underline{x}_t \underline{\notin} \widetilde{G}_E(e)$. It follows that $\widetilde{x}_t^e \widetilde{\notin} \widetilde{F}_E$ which a contradiction. So the result holds.

(2) $\underline{x}_t \underline{\in} (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E)(e) \Leftrightarrow \underline{x}_t \underline{\in} (\widetilde{F}_E(e) \underline{\bigcup} \widetilde{G}_E(e))$ [by Definition 3.6]
$\Leftrightarrow \underline{x}_t \underline{\in} \widetilde{F}_E(e)$ or $\underline{x}_t \underline{\in} \widetilde{G}_E(e)$.
Then, $\widetilde{x}_t^e \widetilde{\in} (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E) \Leftrightarrow \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E$ *or* $\widetilde{x}_t^e \widetilde{\in} \widetilde{G}_E$.

(4(ii)) $\underline{x}_t \underline{\in} (\widetilde{\bigcap}_{s \in S} (\widetilde{F}_E)_s)(e) \Leftrightarrow \underline{x}_t \underline{\in} \underline{\bigcap}_{s \in S} (\widetilde{F}_E)_s(e)$ [by Definition 3.7].
$\Leftrightarrow (\forall s \in S)(\underline{x}_t \underline{\in} (\widetilde{F}_E)_s)(e)$.
Then, $\widetilde{x}_t^e \widetilde{\in} \widetilde{\bigcap}_{s \in S} (\widetilde{F}_E)_s \Leftrightarrow (\forall s \in S)(\widetilde{x}_t^e \widetilde{\in} (\widetilde{F}_E)_s)$. □

Remark 4.6. Here there is a deviation, $\widetilde{x}_t^e \widetilde{\notin} \widetilde{F}_E \Leftrightarrow \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E^c$ (is not true in general).

Example 4.7. From Example 4.4,
$\widetilde{F}_E^c = \{(e_1, (\{h_4, h_5\}, \{h_3, h_4, h_5\})), (e_2, (\{h_2, h_4, h_5\}, \{h_1, h_2, h_4, h_5\})), (e_3, (\varnothing, \{h_1, h_2, h_3, h_4\}))\}$.

It is clear that $(h_3)_1^{e_1} \widetilde{\notin} \widetilde{F}_E$ and $(h_3)_1^{e_1} \widetilde{\notin} \widetilde{F}_E^c$. Also $(h_3)_{0.5}^{e_1} \widetilde{\in} \widetilde{F}_E$ and $(h_3)_{0.5}^{e_1} \widetilde{\in} \widetilde{F}_E^c$.

Definition 4.8. Two soft double sets $\widetilde{F}_E$ and $\widetilde{G}_E$ are said to be quasi- coincident, denoted by $\widetilde{F}_E\ q\ \widetilde{G}_E$ if $\widetilde{F}_E(e)\ q\ \widetilde{G}_E(e)$, for some $e \in E$. If $\widetilde{F}_E$ is not quasi- coincident with $\widetilde{G}_E$, we write $\widetilde{F}_E\ \not{q}\ \widetilde{G}_E$ or $\widetilde{F}_E(e)\ \not{q}\ \widetilde{G}_E(e)$, $\forall e \in E$.

Proposition 4.9. *Let $\widetilde{F}_E, \widetilde{G}_E, \widetilde{H}_E \in SD(X)$ and $\widetilde{x}_t^e \in SDP(X)_E$.*

(1) $\widetilde{F}_E\ q\ \widetilde{G}_E \Rightarrow \widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E \neq \widetilde{\Phi}$.
(2) $\widetilde{F}_E\ q\ \widetilde{G}_E \Leftrightarrow \widetilde{x}_t^e\ q\ \widetilde{G}_E,\ for\ some\ \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E$.
(3) $\widetilde{F}_E\ \not{q}\ \widetilde{G}_E \Leftrightarrow \widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E^c$.
(4) $\widetilde{x}_t^e\ \not{q}\ \widetilde{F}_E \Leftrightarrow \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E^c$.
(5) $\widetilde{F}_E\ \not{q}\ \widetilde{F}_E^c$.
(6) $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E \Leftrightarrow (\widetilde{x}_t^e\ q\ \widetilde{F}_E \Rightarrow \widetilde{x}_t^e\ q\ \widetilde{G}_E)$.
(7) $\widetilde{F}_E\ \not{q}\ \widetilde{G}_E, \widetilde{H}_E \widetilde{\subseteq} \widetilde{G}_E \Rightarrow \widetilde{F}_E\ \not{q}\ \widetilde{H}_E$.
(8) $\widetilde{x}_t^e\ \not{q}\ (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E) \Leftrightarrow \widetilde{x}_t^e\ \not{q}\ \widetilde{F}_E\ and\ \widetilde{x}_t^e\ \not{q}\ \widetilde{G}_E$.
(9) $\widetilde{x}_t^e\ \not{q}\ (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \Leftrightarrow \widetilde{x}_t^e\ \not{q}\ \widetilde{F}_E\ or\ \widetilde{x}_t^e\ \not{q}\ \widetilde{G}_E$.

Proof. (1) $\widetilde{F}_E\ q\ \widetilde{G}_E \Leftrightarrow \widetilde{F}_E(e)\ q\ \widetilde{G}_E(e)$ for some $e \in E$
$\Rightarrow \widetilde{F}_E(e) \underline{\bigcap} \widetilde{G}_E(e) \neq \widetilde{\Phi}(e)$
$\Rightarrow \widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E \neq \widetilde{\Phi}$ [19].

(2) $\widetilde{F}_E\ q\ \widetilde{G}_E \Leftrightarrow \widetilde{F}_E(e)\ q\ \widetilde{G}_E(e)$ for some $e \in E$
$\Leftrightarrow \underline{x}_t\ q\ \widetilde{G}_E(e)$ for some $\underline{x}_t \underline{\in} \widetilde{F}_E(e)$ [19]
$\Leftrightarrow \widetilde{x}_t^e(e)\ q\ \widetilde{G}_E(e)$ for some $\widetilde{x}_t^e(e) \underline{\in} \widetilde{F}_E(e)$
$\Leftrightarrow \widetilde{x}_t^e\ q\ \widetilde{G}_E$ for some $\widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E$.

(3) $\widetilde{F}_E\ \not{q}\ \widetilde{G}_E \Leftrightarrow \widetilde{F}_E(e)\ \not{q}\ \widetilde{G}_E(e)\ \forall e \in E$
$\Leftrightarrow \widetilde{F}_E(e) \subseteq (\widetilde{G}_E(e))^c = \widetilde{G}_E^c(e)$ [19]
$\Leftrightarrow \widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E^c$. (4) $\widetilde{x}_t^e\ \not{q}\ \widetilde{F}_E \Leftrightarrow \widetilde{x}_t^e(e)\ \not{q}\ \widetilde{F}_E(e)\ \forall e \in E$
$\Leftrightarrow \underline{x}_t \underline{\in} (\widetilde{F}_E(e))^c = \widetilde{F}_E^c(e)$ [19]
$\Leftrightarrow \widetilde{x}_t^e(e) \underline{\in} \widetilde{F}_E^c(e)$
$\Leftrightarrow \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E^c$.

(5), (6) and (7) are obvious.

(8) $\widetilde{x}_t^e\ \not{q}\ (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E) \Leftrightarrow \widetilde{x}_t^e \widetilde{\in} (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E)^c$ by (4)
$\Leftrightarrow \widetilde{x}_t^e \widetilde{\in} (\widetilde{F}_E^c \widetilde{\bigcap} \widetilde{G}_E^c)$ [by proposition 3.11]
$\Leftrightarrow \widetilde{x}_t^e \widetilde{\in} \widetilde{F}_E^c\ and\ \widetilde{x}_t^e \widetilde{\in} \widetilde{G}_E^c$
$\Leftrightarrow \widetilde{x}_t^e\ \not{q}\ \widetilde{F}_E\ and\ \widetilde{x}_t^e\ \not{q}\ \widetilde{G}_E$.

(9) It is similar to (8). □

Definition 4.10. Let $SD(X)_E$ and $SD(Y)_K$ be the families of all soft double sets over X and Y, respectively.

(i) The mapping $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ is called a soft double mapping, where $\beta : X \to Y$ and $\psi : E \to K$ are two mappings.

(ii) Let $\widetilde{F}_E \widetilde{\in} SD(X)_E$. Then, the image of $\widetilde{F}_E$ under the soft double mapping $f_{\beta\psi}$is the soft double set over Y, denoted by $f_{\beta\psi}(\widetilde{F}_E)$ and defined by

$$f_{\beta\psi}(\widetilde{F}_E)(k) = \begin{cases} \beta(\underline{\bigcup}_{e \in \psi^{-1}(k) \cap E} \widetilde{F}_E(e)), & \text{if } \psi^{-1}(k) \cap E \neq \varnothing, \\ \underline{\varnothing}, & \text{otherwise.} \end{cases}$$

(iii) Let $\widetilde{G}_K \in SD(Y)_K$. Then, the pre-image of $\widetilde{G}_K$ under the soft double mapping $f_{\beta\psi}$ is the soft double set over X, denoted by $f^{-1}_{\beta\psi}(\widetilde{G}_K)$ and defined by

$$f^{-1}_{\beta\psi}(\widetilde{G}_K)(e) = \beta^{-1}(\widetilde{G}_K(\psi(e))).$$

Example 4.11. Let $X = \{x, y\}, Y = \{p, q\}, E = \{e_1, e_2, e_3\}, K = \{e'_1, e'_2\}$,
$\beta : X \to Y : \beta(x) = p, \beta(y) = q$,
$\psi : E \to K : \psi(e_1) = \psi(e_2) = e'_1, \psi(e_3) = e'_2$.
Let $\widetilde{F}_E$ be a soft double set defined by

$$\widetilde{F}_E(e_1) = (\{x\}, \{x, y\}), \widetilde{F}_E(e_2) = (\{x\}, \{x\}), \widetilde{F}_E(e_3) = \underline{\varnothing}.$$

Then

$$\begin{aligned} f_{\beta\psi}(\widetilde{F}_E)(e'_1) &= \beta(\underline{\bigcup}_{e\in\psi^{-1}(e'_1)\cap E}\widetilde{F}_E(e)) \\ &= \beta(\underline{\bigcup}_{e\in\{e_1,e_2\}}\widetilde{F}_E(e)) \\ &= \beta(\widetilde{F}_E(e_1)\underline{\bigcup}\widetilde{F}_E(e_2)) \\ &= \beta((\{x\}, \{x, y\}) \\ &= (\beta(\{x\}), \beta(\{x, y\})) \\ &= (\{p\}, \{p, q\}) \end{aligned}$$

and

$$\begin{aligned} f_{\beta\psi}(\widetilde{F}_E)(e'_2) &= \beta(\underline{\bigcup}_{e\in\psi^{-1}(e'_2)\cap E}\widetilde{F}_E(e)) \\ &= \beta(\underline{\bigcup}_{e\in\{e_3\}}\widetilde{F}_E(e)) \\ &= \beta(\widetilde{F}_E(e_3)) = \underline{\varnothing}. \end{aligned}$$

Thus, $f_{\beta\psi}(\widetilde{F}_E) = \{(e'_1, (\{p\}, \{p, q\})), (e'_2, \underline{\varnothing})\}$.

To find $f^{-1}_{\beta\psi}(\widetilde{G}_K)$ let, $\widetilde{G}_K(e'_1) = (\{q\}, \{q\})$ and $\widetilde{G}_K(e'_2) = (\{p\}, \{p, q\})$.
Then

$$\begin{aligned} f^{-1}_{\beta\psi}(\widetilde{G}_K)(e_1) &= \beta^{-1}(\widetilde{G}_K(\psi(e_1))) \\ &= \beta^{-1}(\widetilde{G}_K(e'_1)) \\ &= \beta^{-1}(\{q\}, \{q\}) \\ &= \beta^{-1}(\{q\}), (\beta^{-1}\{q\})) \\ &= (\{y\}, \{y\}), \\ f^{-1}_{\beta\psi}(\widetilde{G}_K)(e_2) &= \beta^{-1}(\widetilde{G}_K(\psi(e_2))) \\ &= \beta^{-1}(\widetilde{G}_K(e'_1) \\ &= \beta^{-1}(\{q\}, \{q\}) \\ &= \beta^{-1}(\{q\}), (\beta^{-1}\{q\}) \\ &= (\{y\}, \{y\}) \end{aligned}$$

and

$$\begin{aligned} f^{-1}_{\beta\psi}(\widetilde{G}_K)(e_3) &= \beta^{-1}(\widetilde{G}_K(\psi(e_3))) \\ &= \beta^{-1}(\widetilde{G}_K(e'_2)) \\ &= \beta^{-1}(\{p\}, \{p, q\}) \\ &= (\{x\}, \{x, y\}). \end{aligned}$$

So, $f^{-1}_{\beta\psi}(\widetilde{G}_K) = \{(e_1, (\{y\}, \{y\})), (e_2, (\{y\}, \{y\})), (e_3, (\{x\}, \{x, y\}))\}$.

Proposition 4.12. *Let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K, \widetilde{F}_E, \widetilde{G}_E \in SD(X)_E, (\widetilde{F}_E)_i \subseteq SD(X)_E, i \in I$ and $\widetilde{H}_K, \widetilde{L}_K \in SD(Y)_K, (\widetilde{H}_K)_i \subseteq SD(Y)_K, i \in I$, where I is an index set.*

(1) *If* $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E$*, then* $f_{\beta\psi}(\widetilde{F}_E) \widetilde{\subseteq} f_{\beta\psi}(\widetilde{G}_E)$.
(2) *If* $\widetilde{H}_K \widetilde{\subseteq} \widetilde{L}_K$*, then* $f^{-1}_{\beta\psi}(\widetilde{H}_K) \widetilde{\subseteq} f^{-1}_{\beta\psi}(\widetilde{L}_K)$.
(3) $\widetilde{F}_E \widetilde{\subseteq} f^{-1}_{\beta\psi}(f_{\beta\psi}(\widetilde{F}_E))$*, the equality holds if* $f_{\beta\psi}$ *is an injective.*
(4) $f_{\beta\psi}(f^{-1}_{\beta\psi}(\widetilde{H}_K)) \widetilde{\subseteq} \widetilde{H}_K$*, the equality holds if* $f_{\beta\psi}$ *is a surjective.*
(5) $f_{\beta\psi}(\widetilde{\bigcup}_{i\in I}(\widetilde{F}_E)_i) = \widetilde{\bigcup}_{i\in I}(f_{\beta\psi}(\widetilde{F}_E)_i)$.
(6) $f_{\beta\psi}(\widetilde{\bigcap}_{i\in I}(\widetilde{F}_E)_i) \widetilde{\subseteq} \widetilde{\bigcap}_{i\in I}(f_{\beta\psi}(\widetilde{F}_E)_i)$.
(7) $f^{-1}_{\beta\psi}(\widetilde{\bigcup}_{i\in I}(\widetilde{H}_K)_i) = \widetilde{\bigcup}_{i\in I}(f^{-1}_{\beta\psi}(\widetilde{H}_K)_i)$.
(8) $f^{-1}_{\beta\psi}(\widetilde{\bigcap}_{i\in I}(\widetilde{H}_K)_i) = \widetilde{\bigcap}_{i\in I}(f^{-1}_{\beta\psi}(\widetilde{H}_K)_i)$.
(9) $f^{-1}_{\beta\psi}(\widetilde{Y}) = \widetilde{X}$.
(10) $f^{-1}_{\beta\psi}(\widetilde{H}^c_K) = (f^{-1}_{\beta\psi}(\widetilde{H}_K))^c$.
(11) $f^{-1}_{\beta\psi}(\widetilde{\Phi}_Y) = \widetilde{\Phi}_X$.
(12) $f_{\beta\psi}(\widetilde{X}) = \widetilde{Y}$ *if* $f_{\beta\psi}$ *is a surjective.*
(13) $f_{\beta\psi}(\widetilde{\Phi}_X) = \widetilde{\Phi}_Y$.

Proof. (1) $f_{\beta\psi}(\widetilde{F}_E)(k) = \beta(\underline{\bigcup}_{e\in\psi^{-1}(k)\cap E}\widetilde{F}_E(e))$
$\subseteq \beta(\underline{\bigcup}_{e\in\psi^{-1}(k)\cap E}\widetilde{G}_E(e)),\ (\widetilde{F}_E(e) \subseteq \widetilde{G}_E(e))$
$\subseteq f_{\beta\psi}(\widetilde{G}_E)(k)$.
Then, $f_{\beta\psi}(\widetilde{F}_E) \widetilde{\subseteq} f_{\beta\psi}(\widetilde{G}_E)$.

(2) Since $\widetilde{H}_K \widetilde{\subseteq} \widetilde{L}_K$, $\widetilde{H}_K(e\text{`}) \subseteq \widetilde{L}_K(e\text{`})\ \forall e\text{`} \in K$. Then $\widetilde{H}_K(\psi(e)) \subseteq \widetilde{L}_K(\psi(e))$, *where* $\psi(e) = e\text{`}$. Thus $\beta^{-1}(\widetilde{H}_K(\psi(e))) \subseteq \beta^{-1}(\widetilde{L}_K(\psi(e)))$. So $f^{-1}_{\beta\psi}(\widetilde{H}_K)(e) \subseteq f^{-1}_{\beta\psi}(\widetilde{L}_K)(e)$, $\forall e \in E$. Hence, $f^{-1}_{\beta\psi}(\widetilde{H}_K) \widetilde{\subseteq} f^{-1}_{\beta\psi}(\widetilde{L}_K)$.

(3) $f^{-1}_{\beta\psi}(f_{\beta\psi}(\widetilde{F}_E))(e) = \beta^{-1}[f_{\beta\psi}(\widetilde{F}_E)(\psi(e))]$
$= \beta^{-1}[\beta(\underline{\bigcup}_{e\in\psi^{-1}(\psi(e))\cap E}\widetilde{F}_E(e))]$
$= \beta^{-1}\beta(\widetilde{F}_E(e))$
$\widetilde{\supseteq} \widetilde{F}_E(e)$.
Then, $f^{-1}_{\beta\psi}(f_{\beta\psi}(\widetilde{F}_E))(e) \widetilde{\supseteq} \widetilde{F}_E(e)$. Thus, $f^{-1}_{\beta\psi}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\supseteq} \widetilde{F}_E$.

Suppose $f_{\beta\psi}$ is one-to-one. Then β is one-to-one and ψ is one-to-one. So, $f^{-1}_{\beta\psi}(f_{\beta\psi}(\widetilde{F}_E)) = \widetilde{F}_E$.

(4) It is similar to (3).

(5) $f_{\beta\psi}(\widetilde{\bigcup}_{i\in I}(\widetilde{F}_E)_i)(k) = \beta(\underline{\bigcup}_{e\in\psi^{-1}(k)\cap E}(\widetilde{\bigcup}_{i\in I}(\widetilde{F}_E)_i(e)))$
$= \widetilde{\bigcup}_{i\in I}(\beta(\underline{\bigcup}_{e\in\psi^{-1}(k)\cap E}(\widetilde{F}_E)_i(e))$
$= \widetilde{\bigcup}_{i\in I}(f_{\beta\psi}(\widetilde{F}_E)_i)$.
Then, $f_{\beta\psi}(\widetilde{\bigcup}_{i\in I}(\widetilde{F}_E)_i) = \widetilde{\bigcup}_{i\in I}(f_{\beta\psi}(\widetilde{F}_E)_i)$.

(6), (7) and (8) are similar to (5).

(9) It is obvious.

(10) $f^{-1}_{\beta\psi}(\widetilde{H}^c_K)(e) = \beta^{-1}(\widetilde{H}^c_K(\psi(e))$
$= \beta^{-1}(\underline{Y} \setminus \widetilde{H}_K(\psi(e)))$

$= \underline{X} \setminus \beta^{-1}(\widetilde{H}_K(\psi(e)).$
And $(f^{-1}_{\beta\psi}(\widetilde{H}_K)^c(e) = \underline{X} \setminus f^{-1}_{\beta\psi}(\widetilde{H}_K(e)$
$= \underline{X} \setminus \beta^{-1}(\widetilde{H}_K(\psi(e)).$
Then, the result holds.
(11), (12) and (13) are obvious. □

5. Soft double topology

Definition 5.1. Let $\widetilde{\tau}$ be a collection of soft double sets over X, i. e., $\widetilde{\tau} \subseteq SD(X)_E$. Then, $\widetilde{\tau}$ is said to be a soft double topology over X, if it satisfies the following conditions:
(i) $\widetilde{\Phi}, \widetilde{X} \in \widetilde{\tau}$,
(ii) The union of any number of soft double sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$,
(iii) The intersection of any two soft double sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.
The triple $(X, \widetilde{\tau}, E)$ is called a soft double topological space. Every member of $\widetilde{\tau}$ is called an open soft double set and its complement is called a closed soft double set. The family of all closed soft double sets we denoted by $\widetilde{\tau}^c$.

Example 5.2. Let $X = \{h_1, h_2, h_3\}, E = \{e_1, e_2\}$ and let
$$\widetilde{\tau} = \{\widetilde{\Phi}, \widetilde{X}, \widetilde{F}^1_E, \widetilde{F}^2_E, \widetilde{F}^3_E, \widetilde{F}^4_E, \widetilde{F}^5_E, \widetilde{F}^6_E, \widetilde{F}^7_E, \widetilde{F}^8_E\},$$
where
$\widetilde{F}^1_E(e_1) = (\{h_1, h_2\}, X), \widetilde{F}^1_E(e_2) = \underline{\varnothing},$
$\widetilde{F}^2_E(e_1) = \underline{\varnothing}, \widetilde{F}^2_E(e_2) = (\{h_1, h_3\}, X),$
$\widetilde{F}^3_E(e_1) = (\{h_1\}, X), \widetilde{F}^3_E(e_2) = \underline{\varnothing},$
$\widetilde{F}^4_E(e_1) = \underline{\varnothing}, \widetilde{F}^4_E(e_2) = (\{h_3\}, X),$
$\widetilde{F}^5_E(e_1) = (\{h_1\}, X), \widetilde{F}^5_E(e_2) = (\{h_3\}, X),$
$\widetilde{F}^6_E(e_1) = (\{h_1, h_2\}, X), \widetilde{F}^6_E(e_2) = (\{h_3\}, X),$
$\widetilde{F}^7_E(e_1) = (\{h_1\}, X), \widetilde{F}^7_E(e_2) = (\{h_1, h_3\}, X),$
$\widetilde{F}^8_E(e_1) = (\{h_1, h_2\}, X), \widetilde{F}^8_E(e_2) = (\{h_1, h_3\}, X).$
Then, $\widetilde{\tau}$ is a soft double topology over X.
Also,
$\widetilde{F}^{1c}_E(e_1) = (\varnothing, \{h_3\}), \widetilde{F}^{1c}_E(e_2) = \underline{X},$
$\widetilde{F}^{2c}_E(e_1) = \underline{X}, \widetilde{F}^{2c}_E(e_2) = (\varnothing, \{h_2\}),$
$\widetilde{F}^{3c}_E(e_1) = (\varnothing, \{h_2, h_3\}), \widetilde{F}^{3c}_E(e_2) = \underline{X},$
$\widetilde{F}^{4c}_E(e_1) = \underline{X}, \widetilde{F}^{4c}_E(e_2) = (\varnothing, \{h_1, h_2\}),$
$\widetilde{F}^{5c}_E(e_1) = (\varnothing, \{h_2, h_3\}), \widetilde{F}^{5c}_E(e_1) = (\varnothing, \{h_1, h_2\}),$
$\widetilde{F}^{6c}_E(e_1) = (\varnothing, \{h_3\}), \widetilde{F}^{6c}_E(e_1) = (\varnothing, \{h_1, h_2\}),$
$\widetilde{F}^{7c}_E(e_1) = (\varnothing, \{h_2, h_3\}), \widetilde{F}^{7c}_E(e_1) = (\varnothing, \{h_2\}),$
$\widetilde{F}^{8c}_E(e_1) = (\varnothing, \{h_3\}), \widetilde{F}^{8c}_E(e_1) = (\varnothing, \{h_2\}).$
Thus, $\widetilde{\tau}^c = \{\widetilde{\varnothing}, \widetilde{X}, \widetilde{F}^{1c}_E, \widetilde{F}^{2c}_E, \widetilde{F}^{3c}_E, \widetilde{F}^{4c}_E, \widetilde{F}^{5c}_E, \widetilde{F}^{6c}_E, \widetilde{F}^{7c}_E, \widetilde{F}^{8c}_E\}.$

Proposition 5.3. *Let $(X, \widetilde{\tau}, E)$ be a soft double topological space. Then*
(1) $\tau_1 = \{A_1 \subseteq X : \exists \widetilde{F}_E \in \widetilde{\tau}, A_2 \subseteq X$ *and* $\widetilde{F}_E(e) = (A_1, A_2)\ \forall e \in E\}$ *is an ordinary topology on* X,

(2) $\tau_2 = \{A_2 \subseteq X : \exists \widetilde{F}_E \in \widetilde{\tau}, A_1 \subseteq X$ *and* $\widetilde{F}_E(e) = (A_1, A_2)\ \forall e \in E\}$ *is an ordinary topology on* X,

(3) $\tau_3 = \{A \subseteq X : \exists \widetilde{F}_E \in \widetilde{\tau}$ *and* $\widetilde{F}_E(e) = (A, A)\ \forall e \in E\}$ *is an ordinary topology on* X.

Proof. As a sample, we prove the property (1), the proofs of the remaining cases are similar.

(i) Clearly, $\varnothing, X \in \tau_1$, since $\widetilde{\Phi} \in \widetilde{\tau}$ and $\widetilde{\Phi}(e) = (\varnothing, \varnothing)$. Also $\widetilde{X} \in \widetilde{\tau}$ and $\widetilde{X}(e) = (X, X)$.

(ii) Let $A_1, B_1 \in \tau_1$. Then, $\exists \widetilde{F}_E, \widetilde{G}_E \widetilde{\in} \widetilde{\tau}$ and $A_2, B_2 \subseteq X$ such that

$$\widetilde{F}_E(e) = (A_1, A_2), \widetilde{G}_E(e) = (B_1, B_2).$$

Thus

$$\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E \widetilde{\in} \widetilde{\tau}$$

and

$$(\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E)(e) = \widetilde{F}_E(e) \underline{\bigcap} \widetilde{G}_E(e) = (A_1 \cap B_1, A_2 \cap B_2).$$

So, $A_1 \cap B_1 \in \tau_1$.

(iii) Let $(A_1)_s \in \tau_1\ \forall s \in S$. Then, $\exists (A_2)_s \subseteq X$ and $(\widetilde{F}_E)_s \in \widetilde{\tau}$ such that

$$(\widetilde{F}_E)_s(e) = ((A_1)_s, (A_2)_s)\ \forall e \in E.$$

Thus, $\underline{\bigcup}_{s \in S} (\widetilde{F}_E)_s(e) = (\widetilde{\bigcup}_{s \in S} (\widetilde{F}_E)_s)(e) = (\bigcup_{s \in S} (A_1)_s, \bigcup_{s \in S} (A_2)_s)$. So $\widetilde{\bigcup}_{s \in S} (\widetilde{F}_E)_s \widetilde{\in} \widetilde{\tau}$. Hence, $\bigcup_{s \in S} (A_1)_s \in \tau_1$. Therefore, τ_1 is an ordinary topology on X. □

Proposition 5.4. *Let* $(X, \widetilde{\tau}, E)$ *be a soft double topological space. Then,* $\tau_e = \{\widetilde{F}_E(e) : \widetilde{F}_E \in \widetilde{\tau}\}$ *is a double topology on* X *for all* $e \in E$.

Proof. Straightforward. □

The following example shows that the converse of proposition 5.4 is not true in general.

Example 5.5. Let $X = \{h_1, h_2, h_3\}, E = \{e_1, e_2\}$ and let $\widetilde{\tau} = \{\widetilde{\Phi}, \widetilde{X}, \widetilde{F}^1_E, \widetilde{F}^2_E, \widetilde{F}^3_E, \widetilde{F}^4_E\}$, where

$\widetilde{F}^1_E(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}^1_E(e_2) = (\{h_1\}, \{h_1\})$,
$\widetilde{F}^2_E(e_1) = (\{h_2, h_3\}, \{h_2, h_3\}), \widetilde{F}^2_E(e_2) = (\{h_1, h_2\}, \{h_1, h_2\})$,
$\widetilde{F}^3_E(e_1) = (\{h_1, h_2\}, \{h_1, h_2\}), \widetilde{F}^3_E(e_2) = (\{h_1, h_2\}, \{h_1, h_2\})$,
$\widetilde{F}^4_E(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}^4_E(e_2) = (\{h_1, h_3\}, \{h_1, h_3\})$.

Then, $\widetilde{\tau}_{e_1} = \{\underline{\varnothing}, \underline{X}, (\{h_2\}, \{h_2\}), (\{h_1, h_2\}, \{h_1, h_2\}), (\{h_2, h_3\}, \{h_2, h_3\})\}$ and $\widetilde{\tau}_{e_2} = \{\underline{\varnothing}, \underline{X}, (\{h_1\}, \{h_1\}), (\{h_1, h_2\}, \{h_1, h_2\}), (\{h_1, h_3\}, \{h_1, h_3\})\}$ are double topologies on X. However $\widetilde{\tau}$ is not a soft double topology on X, because $\widetilde{F}^2_E \widetilde{\bigcup} \widetilde{F}^3_E \notin \widetilde{\tau}$.

Definition 5.6. Let $(X, \widetilde{\tau}, E)$ be a soft double topological space and let $\widetilde{F}_E \in SD(X)_E$. Then, $\widetilde{F}_E$ is called a quasi-neighborhood of a soft double point $\widetilde{x}^e_t$, if there exists $\widetilde{G}_E \in \widetilde{\tau}$ such that $\widetilde{x}^e_t q \widetilde{G}_E \widetilde{\subseteq} \widetilde{F}_E$. The family of all quasi-neighborhoods of $\widetilde{x}^e_t$ denoted by $N^q_{(\widetilde{x}^e_t)_E}$.

Definition 5.7. Let $(X, \widetilde{\tau}, E)$ be a soft double topological space and let $\widetilde{F}_E \in SD(X)_E$. Then, the soft double closure of $\widetilde{F}_E$, denoted by $cl_{\widetilde{\tau}}(\widetilde{F}_E)$ and defined by:

$$cl_{\widetilde{\tau}}(\widetilde{F}_E) = \widetilde{\bigcap}\{\widetilde{G}_E \in \widetilde{\tau}^c : \widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E\}.$$

Proposition 5.8. *Let $(X, \widetilde{\tau}, E)$ be a soft double topological space and let $\widetilde{F}_E, \widetilde{G}_E \widetilde{\in} SD(X)_E$.*

(1) $cl_{\widetilde{\tau}}(\widetilde{F}_E)$ *is the smallest closed soft double set containing* $\widetilde{F}_E$.
(2) $cl_{\widetilde{\tau}}(\widetilde{\Phi}) = \widetilde{\Phi}$ *and* $cl_{\widetilde{\tau}}(\widetilde{X}) = \widetilde{X}$.
(3) $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E \Rightarrow cl_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{G}_E)$.
(4) $cl_{\widetilde{\tau}}(\widetilde{F}_E \widetilde{\cup} \widetilde{G}_E) = cl_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\cup} cl_{\widetilde{\tau}}(\widetilde{G}_E)$.
(5) $cl_{\widetilde{\tau}}(\widetilde{F}_E \widetilde{\cap} \widetilde{G}_E) \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\cap} cl_{\widetilde{\tau}}(\widetilde{G}_E)$.
(6) $\widetilde{F}_E \in \widetilde{\tau}^c \Leftrightarrow \widetilde{F}_E = cl_{\widetilde{\tau}}(\widetilde{F}_E)$.
(7) $cl_{\widetilde{\tau}}(cl_{\widetilde{\tau}}(\widetilde{F}_E)) = cl_{\widetilde{\tau}}(\widetilde{F}_E)$.

Proof. (1) and (2) are obvious.

(3) Since $\widetilde{F}_E \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{F}_E), \widetilde{G}_E \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{G}_E)$ and $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E$, $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{G}_E)$ implies $\widetilde{F}_E \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{G}_E)$. Then $cl_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{G}_E)$, by (1).

(4) Since $\widetilde{F}_E \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{F}_E)$ and $\widetilde{G}_E \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{G}_E)$, $\widetilde{F}_E \widetilde{\cup} \widetilde{G}_E \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\cup} cl_{\widetilde{\tau}}(\widetilde{G}_E)$.
Then $cl_{\widetilde{\tau}}(\widetilde{F}_E \widetilde{\cup} \widetilde{G}_E) \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\cup} cl_{\widetilde{\tau}}(\widetilde{G}_E)$, by (1).
Since $\widetilde{F}_E \widetilde{\subseteq} \widetilde{F}_E \widetilde{\cup} \widetilde{G}_E$ and $\widetilde{G}_E \widetilde{\subseteq} \widetilde{F}_E \widetilde{\cup} \widetilde{G}_E$,
$cl_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{F}_E \widetilde{\cup} \widetilde{G}_E)$ and $cl_{\widetilde{\tau}}(\widetilde{G}_E) \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{F}_E \widetilde{\cup} \widetilde{G}_E)$ by (3).
Thus, $cl_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\cup} cl_{\widetilde{\tau}}(\widetilde{G}_E) \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{F}_E \widetilde{\cup} \widetilde{G}_E)$. So, $cl_{\widetilde{\tau}}(\widetilde{F}_E \widetilde{\cup} \widetilde{G}_E) = cl_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\cup} cl_{\widetilde{\tau}}(\widetilde{G}_E)$.
(5) it is similar to (4).
(6) and (7) are obvious. □

Definition 5.9. Let $(X, \widetilde{\tau}, E)$ be a soft double topological space and let $\widetilde{F}_E \in SD(X)_E$. Then, the soft double interior of $\widetilde{F}_E$, denoted by $int_{\widetilde{\tau}}(\widetilde{F}_E)$ and defined by:

$$int_{\widetilde{\tau}}(\widetilde{F}_E) = \widetilde{\bigcup}\{\widetilde{G}_E : \widetilde{G}_E \widetilde{\subseteq} \widetilde{F}_E \text{ and } \widetilde{G}_E \in \widetilde{\tau}\}.$$

Proposition 5.10. *Let $(X, \widetilde{\tau}, E)$ be a soft double topological space and let $\widetilde{F}_E, \widetilde{G}_E \widetilde{\in} SD(X)_E$.*

(1) $int_{\widetilde{\tau}}(\widetilde{F}_E)$ *is the largest open soft double subset of* $\widetilde{F}_E$.
(2) $int_{\widetilde{\tau}}(\widetilde{\Phi}) = \widetilde{\Phi}$ *and* $int_{\widetilde{\tau}}(\widetilde{X}) = \widetilde{X}$.
(3) $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E \Rightarrow int_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\subseteq} int_{\widetilde{\tau}}(\widetilde{G}_E)$.
(4) $int_{\widetilde{\tau}}(\widetilde{F}_E \widetilde{\cap} \widetilde{G}_E) = int_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\cap} int_{\widetilde{\tau}}(\widetilde{G}_E)$.
(5) $int_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\cup} int_{\widetilde{\tau}}(\widetilde{G}_E) \widetilde{\subseteq} int_{\widetilde{\tau}}(\widetilde{F}_E \widetilde{\cup} \widetilde{G}_E)$.
(6) $\widetilde{F}_E \widetilde{\in} \widetilde{\tau} \Leftrightarrow \widetilde{F}_E = int_{\widetilde{\tau}}(\widetilde{F}_E)$.
(7) $int_{\widetilde{\tau}}(int_{\widetilde{\tau}}(\widetilde{F}_E)) = int_{\widetilde{\tau}}(\widetilde{F}_E)$.

Proof. (1) and (2) are obvious.

(3) Since $int_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\subseteq} \widetilde{F}_E$ and $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E$, $int_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\subseteq} \widetilde{G}_E$. Then, $int_{\widetilde{\tau}}(\widetilde{F}_E) \widetilde{\subseteq} int_{\widetilde{\tau}}(\widetilde{G}_E)$ by (1).

(4) Since $int_{\widetilde{\tau}}(\widetilde{F}_E)\widetilde{\subseteq}\widetilde{F}_E$ and $int_{\widetilde{\tau}}(\widetilde{G}_E)\widetilde{\subseteq}\widetilde{G}_E$, $int_{\widetilde{\tau}}(\widetilde{F}_E)\widetilde{\cap}int_{\widetilde{\tau}}(\widetilde{G}_E)\widetilde{\subseteq}\widetilde{F}_E\widetilde{\cap}\widetilde{G}_E$.
Then $int_{\widetilde{\tau}}(\widetilde{F}_E)\widetilde{\cap}int_{\widetilde{\tau}}(\widetilde{G}_E)\widetilde{\subseteq}int_{\widetilde{\tau}}(\widetilde{F}_E\widetilde{\cap}\widetilde{G}_E)$, by (1).
Since $\widetilde{F}_E\widetilde{\cap}\widetilde{G}_E\widetilde{\subseteq}\widetilde{F}_E$ and $\widetilde{F}_E\widetilde{\cap}\widetilde{G}_E\widetilde{\subseteq}\widetilde{G}_E$,

$int_{\widetilde{\tau}}(\widetilde{F}_E\widetilde{\cap}\widetilde{G}_E)\widetilde{\subseteq}int_{\widetilde{\tau}}(\widetilde{F}_E)$ and $int_{\widetilde{\tau}}(\widetilde{F}_E\widetilde{\cap}\widetilde{G}_E)\widetilde{\subseteq}int_{\widetilde{\tau}}(\widetilde{G}_E)$ by (3)

Thus, $int_{\widetilde{\tau}}(\widetilde{F}_E\widetilde{\cap}\widetilde{G}_E)\widetilde{\subseteq}int_{\widetilde{\tau}}(\widetilde{F}_E)\widetilde{\cap}int_{\widetilde{\tau}}(\widetilde{G}_E)$. So, $int_{\widetilde{\tau}}(\widetilde{F}_E\widetilde{\cap}\widetilde{G}_E) = int_{\widetilde{\tau}}(\widetilde{F}_E)\widetilde{\cap}int_{\widetilde{\tau}}(\widetilde{G}_E)$.
(5) it is similar to (4).
(6) and (7) are obvious. □

Proposition 5.11. *Let $\widetilde{F}_E \in SD(X)$. Then $int_{\widetilde{\tau}}(\widetilde{F}_E) = [cl_{\widetilde{\tau}}(\widetilde{F}^c_E)]^c$.*

Proof. Since $\widetilde{F}^c_E\widetilde{\subseteq}cl_{\widetilde{\tau}}(\widetilde{F}^c_E)$, $[cl_{\widetilde{\tau}}(\widetilde{F}^c_E)]^c\widetilde{\subseteq}\widetilde{F}_E$, by Proposition 3.12.
But, $cl_{\widetilde{\tau}}(\widetilde{F}^c_E)$ is a closed soft double set. Then $[cl_{\widetilde{\tau}}(\widetilde{F}^c_E)]^c$ is an open soft double set. Thus $[cl_{\widetilde{\tau}}(\widetilde{F}^c_E)]^c\widetilde{\subseteq}int_{\widetilde{\tau}}(\widetilde{F}_E)$, by Proposition 5.10.
Also since $int_{\widetilde{\tau}}(\widetilde{F}_E)\widetilde{\subseteq}\widetilde{F}_E$, $\widetilde{F}^c_E\widetilde{\subseteq}[int_{\widetilde{\tau}}(\widetilde{F}_E)]^c$, by Proposition 3.12. So, by Proposition 5.8, $cl_{\widetilde{\tau}}(\widetilde{F}^c_E)\widetilde{\subseteq}[int_{\widetilde{\tau}}(\widetilde{F}_E)]^c$. Hence, $int_{\widetilde{\tau}}(\widetilde{F}_E)\widetilde{\subseteq}[cl_{\widetilde{\tau}}(\widetilde{F}^c_E)]^c$. Therefore, $int_{\widetilde{\tau}}(\widetilde{F}_E) = [cl_{\widetilde{\tau}}(\widetilde{F}^c_E)]^c$. □

Proposition 5.12. *Let $\widetilde{F}_E \in SD(X)$ and $\widetilde{x}^e_t \in PSD(X)$. Then*

$$\widetilde{x}^e_t \; q \; cl_{\widetilde{\tau}}(\widetilde{F}_E) \Leftrightarrow \; \forall \widetilde{G}_E \in \widetilde{\tau}, \widetilde{x}^e_t\widetilde{\in}\widetilde{G}_E, \widetilde{G}_E \; q \; \widetilde{F}_E.$$

Proof. Suppose $\widetilde{x}^e_t \; q \; cl_{\widetilde{\tau}}(\widetilde{F}_E)$ and let $\exists \; \widetilde{G}_E \in \widetilde{\tau}$ such that $\widetilde{x}^e_t\widetilde{\in}\widetilde{G}_E$ and $\widetilde{G}_E \not{q} \; \widetilde{F}_E$. Then $\widetilde{F}_E\widetilde{\subseteq}\widetilde{G}^c_E$. Thus $cl_{\widetilde{\tau}}(\widetilde{F}_E)\widetilde{\subseteq}\widetilde{G}^c_E$. Since $\widetilde{x}^e_t \not{q} \; \widetilde{G}^c_E$, $\widetilde{x}^e_t \not{q} \; cl_{\widetilde{\tau}}(\widetilde{F}_E)$. This is a contradiction. Thus the necessary condition holds.

Conversely, let $\widetilde{x}^e_t \not{q} \; cl_{\widetilde{\tau}}(\widetilde{F}_E)$. Then $\widetilde{x}^e_t\widetilde{\in}(cl_{\widetilde{\tau}}(\widetilde{F}_E))^c \in \widetilde{\tau}$ and $\widetilde{F}_E \not{q} \; (cl_{\widetilde{\tau}}(\widetilde{F}_E))^c$. This is a contradiction. Hence the sufficient condition holds. □

Definition 5.13. Let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$, where $\beta : X \to Y$ and $\psi : E \to K$. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\sigma}, K)$ be two soft double topological spaces. Then, $f_{\beta\psi}$ is called a soft double continuous mapping denoted by SD continuous if $f^{-1}_{\beta\psi}(\widetilde{H}_K) \in \widetilde{\tau}$, whenever $\widetilde{H}_K \in \widetilde{\sigma}$.

Proposition 5.14. *Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\sigma}, K)$ be two soft double topological spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a mapping, $\widetilde{F}_E \in SD(X)_E$ and $\widetilde{H}_K \in SD(Y)_K$. Then, the following conditions are equivalent:*

(1) *$f_{\beta\psi}$ is an SD-continuous,*
(2) *$f^{-1}_{\beta\psi}(\widetilde{H}_K) \in \widetilde{\tau}^c \; \forall \widetilde{H}_K \in \widetilde{\sigma}^c$,*
(3) *$f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))\widetilde{\subseteq}cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E)) \; \forall \widetilde{F}_E \in SD(X)_E$,*
(4) *$cl_{\widetilde{\tau}}(f^{-1}_{\beta\psi}(\widetilde{H}_K))\widetilde{\subseteq}f^{-1}_{\beta\psi}(cl_{\widetilde{\sigma}}(\widetilde{H}_K)) \; \forall \widetilde{H}_K \in SD(Y)_K$,*
(5) *$f^{-1}_{\beta\psi}(int_{\widetilde{\sigma}}(\widetilde{H}_K))\widetilde{\subseteq}int_{\widetilde{\tau}}(f^{-1}_{\beta\psi}(\widetilde{H}_K)) \; \forall \widetilde{H}_K \in SD(Y)_K$.*

Proof. (1)⇒(2): Let $\widetilde{H}_K \in \widetilde{\sigma}^c$. Then $\widetilde{H}^c_K \in \widetilde{\sigma}$. It follows that, $f^{-1}_{\beta\psi}(\widetilde{H}^c_K) = (f^{-1}_{\beta\psi}(\widetilde{H}_K))^c$ is $\widetilde{\tau}$ open. Thus, $f^{-1}_{\beta\psi}(\widetilde{H}_K)$ is $\widetilde{\tau}$ closed. So (2) holds.

(2) ⇒ (3): Since $cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E)) \in \widetilde{\sigma}^c$, by(2), $f^{-1}_{\beta\psi}(cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E))) \in \widetilde{\tau}^c$.
Now, since $\widetilde{F}_E\widetilde{\subseteq}f^{-1}_{\beta\psi}(f_{\beta\psi}(\widetilde{F}_E))\widetilde{\subseteq}f^{-1}_{\beta\psi}(cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E)))$ [by proposition 4.12],
$\widetilde{F}_E\widetilde{\subseteq}f^{-1}_{\beta\psi}(cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E)))$. Thus $cl_{\widetilde{\tau}}(\widetilde{F}_E)\widetilde{\subseteq}f^{-1}_{\beta\psi}(cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E)))$, by proposition 5.8.

So $f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))\widetilde{\subseteq}f_{\beta\psi}(f_{\beta\psi}^{-1}(cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E))))\widetilde{\subseteq}cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E))$, by proposition 4.12. Hence, $f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))\widetilde{\subseteq}cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E))$.

(3) $\Rightarrow$ (4): Let $\widetilde{H}_K \in SD(Y)_K$. Then $f_{\beta\psi}^{-1}(\widetilde{H}_K) \in SD(X)_E$ Thus $f_{\beta\psi}(cl_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K)))\widetilde{\subseteq}cl_{\widetilde{\sigma}}(f_{\beta\psi}(f_{\beta\psi}^{-1}(\widetilde{H}_K)))$, by(3). But $f_{\beta\psi}(f_{\beta\psi}^{-1}(\widetilde{H}_K))\widetilde{\subseteq}\widetilde{H}_K$. So $f_{\beta\psi}^{-1}(f_{\beta\psi}(cl_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K))))\widetilde{\subseteq}f_{\beta\psi}^{-1}(cl_{\widetilde{\sigma}}(\widetilde{H}_K))$, by proposition 5.8.
Hence, $cl_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K))\widetilde{\subseteq}f_{\beta\psi}^{-1}(cl_{\widetilde{\sigma}}(\widetilde{H}_K))$, by proposition 4.12

(4) $\Rightarrow$ (5): Since $cl_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K))\widetilde{\subseteq}f_{\beta\psi}^{-1}(cl_{\widetilde{\sigma}}(\widetilde{H}_K))$, $(f_{\beta\psi}^{-1}(cl_{\widetilde{\sigma}}(\widetilde{H}_K))^c\widetilde{\subseteq}(cl_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K)))^c$. But, $(cl_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K)))^c = int_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K)^c)$. Then $(f_{\beta\psi}^{-1}(cl_{\widetilde{\sigma}}(\widetilde{H}_K)))^c\widetilde{\subseteq}int_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K)^c)$. Thus $f_{\beta\psi}^{-1}(int_{\widetilde{\sigma}}(\widetilde{H}_K)^c)\widetilde{\subseteq}int_{\widetilde{\tau}}((f_{\beta\psi}^{-1}(\widetilde{H}_K)^c)$. Take $(\widetilde{H}_K)^c = \widetilde{L}_K \in SD(Y)_K$.
So, $f_{\beta\psi}^{-1}(int_{\widetilde{\sigma}}(\widetilde{H}_K))\widetilde{\subseteq}int_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K))$.

(5) $\Rightarrow$ (1): Let $\widetilde{H}_K \in \widetilde{\sigma}$. Then $\widetilde{H}_K = int_{\widetilde{\sigma}}(\widetilde{H}_K)$. By (5), we have $f_{\beta\psi}^{-1}(int_{\widetilde{\sigma}}(\widetilde{H}_K))\widetilde{\subseteq}int_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K))$. It follows that $f_{\beta\psi}^{-1}(\widetilde{H}_K)\widetilde{\subseteq}int_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K))$. But, $int_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K))\widetilde{\subseteq}f_{\beta\psi}^{-1}(\widetilde{H}_K)$. Then $int_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K)) = f_{\beta\psi}^{-1}(\widetilde{H}_K)$. Thus, $f_{\beta\psi}^{-1}(\widetilde{H}_K) \in \widetilde{\tau}$. So $f_{\beta\psi}$ is an SD-continuous mapping. □

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