On fuzzy multiset regular grammars

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ABSTRACT. The main goal of this paper is to introduce and study fuzzy multiset regular grammar to show that fuzzy multiset finite automata and fuzzy multiset regular grammars are equivalent. We also study fuzzy multiset linear grammars and fuzzy multiset regular grammars in normal form. Furthermore, we show the equivalence of fuzzy multiset regular grammars, fuzzy multiset left linear grammars, fuzzy multiset right linear grammars, fuzzy multiset grammars in normal form.

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1. Introduction

Finite automata are conceptual machines capable of recognizing whether a string (i.e., a sequence of characters) belongs to some formal language or not. In addition, finite automata have found many applications that are discussed in the literature (e.g., see [6, 14, 15, 26]). Fuzzy finite automata were introduced and studied by Wee [23] and Santos [16] in the 1960s. These conceptual machines incorporated vagueness [9] as realized by fuzzy sets [27] in the machinery of finite automata. In general, fuzzy finite automata have found many applications that include pattern recognition, the theory of databases, and machine learning systems (cf., [8, 10, 12, 24]). In particular, Wee [24] proposed a pattern recognition and nonsupervised learning scheme in automatic control. Moreover, fuzzy automata have been proposed as a model of machine learning systems in design and computation.

A multiset (or bag) is a collection of elements in which elements may occur more than once. A multiset is a natural generalization of the notion of a set and they have been described, among others, by Syropoulos [17]. Yager [25] was the first author who discussed fuzzy multisets. In real life, there are many situations where we deal with collections of objects in which repeated elements matter. From a practical
point of view, multisets are very useful structure arising in many areas of mathematics and computer science. Following the advent of multiset theory, the concept of multiset finite automata has been introduced in [5, 7] and established their connection with membrane computing [4] whereas fuzzy multisets have been used in different models of fuzzy computation [18]. In recent years, multiset processing has appeared frequently in various fields of mathematics, computer science, biology and biochemistry (cf., [1, 2, 11, 13, 20]). Multiset “languages” are not exactly formal languages since tokens can occur anywhere in the “string.” We use the term multistring to describe such generalized strings. A multistring can be best described by a “solution” that contains the characters that make it up. The various rules of the “grammar” are actually multiset processing rules. A multiset automaton is device that takes as input a solution and alters it. How the machine can pick elements from the solution is not specified but one can readily provide such a mechanism. A computation is successful if no rule can be applied any more. In this case, we can say that the multistring has been accepted. Otherwise, the multistring has not been accepted. In a sense, a multiset automaton functions just like the substitution operator of the Perl programming language. In particular, one can view each “production rule” as an instance of the $s/B/A/g$ operator that globally substitutes $B$ with $A$.

Following the trend of generalization, Wang, Yin and Gu [22] introduced fuzzy multiset finite automata as a generalization of multiset finite automata. Unfortunately, these authors did not make it clear that their automata process fuzzy “solutions” instead of fuzzy “strings”. At this point it is important to understand what is a fuzzy solution? The obvious answer is that a fuzzy solution is an ordinary solution with fuzzy molecules (see the relevant discussion in [18] and also [19]). Tiwari, Gautam and Dubey [21] proved that a fuzzy multiset language is accepted by a fuzzy multiset finite automaton if and only if it is accepted by a deterministic fuzzy multiset finite automaton.

In what follows, we introduce and study fuzzy multiset regular automata and the corresponding fuzzy multiset regular languages. After presenting some preliminary ideas, we proceed with fuzzy multiset finite automata and show that fuzzy multiset finite automata are equivalent to fuzzy multiset regular grammars. In addition, we prove that fuzzy multiset regular grammars, fuzzy multiset left linear grammars, fuzzy multiset right linear grammars, and fuzzy multiset regular grammars in normal form are equivalent except for an empty string.

2. Preliminaries

In this section, we recollect some concepts and notations associated with multisets and multiset finite automata that are required in what follows. The notions related to multisets and which are used in this paper are fairly standard and can be found in the literature (cf., [5, 17, 21, 22]). Let us start with a formal definition of multisets.

**Definition 2.1.** If $\Sigma$ is a finite set, then $\alpha : \Sigma \rightarrow \mathbb{N}$ characterizes a multiset over $\Sigma$, where $\mathbb{N}$ denotes the set of positive integers including 0. The cardinality of $\alpha$,
denoted by $|\alpha|$, is defined by
\[
|\alpha| = \sum_{x \in \Sigma} \alpha(x).
\]

Typically, simple multisets are specified using the standard set notation (e.g., \{1, 1, 2, 2, 3, 4, 4\}). The set of all multisets over $\Sigma$ is denoted by $\Sigma^\ast$. The null multiset, $0_\Sigma \in \Sigma^\ast$, is the multiset where all elements of $\Sigma$ belong zero times to $0_\Sigma$, that is, $0_\Sigma(x) = 0$ for all $x \in \Sigma$. Note that the null multiset is not the same as the empty multiset! Also, note that here $\Sigma^\ast$ can be identified with $\mathbb{N}^{\mid\Sigma\mid}$.

For any two multisets $\alpha, \beta \in \Sigma^\ast$, the inclusion operation, $\subseteq$, the addition operation, $\oplus$, and the difference operation, $\ominus$, are defined as follows:

- $\alpha \subseteq \beta$ if and only if $\forall x \in \Sigma : \alpha(x) \leq \beta(x)$,
- $\forall x \in \Sigma : (\alpha \oplus \beta)(x) = \alpha(x) + \beta(x)$,
- $\forall x \in \Sigma : (\alpha \ominus \beta)(x) = \max(0, \alpha(x) - \beta(x))$.

It is not hard to see that $\Sigma^\ast$ is a commutative monoid with operation $\oplus$ and identity element the multiset $0_\Sigma$. Furthermore, $\alpha \subset \beta$ if and only if $\alpha \subseteq \beta$, $\alpha \neq \beta$, and

$$A \oplus B = \bigcup_{\substack{\alpha \in A \\ \beta \in B}} \alpha \oplus \beta,$$

for $A, B \subseteq \Sigma^\ast$. We also use the notation $\langle y \rangle$ for a singleton multiset:

$$\langle y \rangle(x) = \begin{cases} 0, & \text{if } y = x \\ 1, & \text{if } y \neq x \end{cases}$$

for all $x \in \Sigma$. For any set $A$, let $\overline{A} = \{\langle a \rangle \mid a \in A\}$. Assume that $\Sigma = \{a, b, c\}$. Then, the multiset $\alpha = \{a, a, b, c, c\}$ is equivalent to $\langle a \rangle \oplus \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle \oplus \langle c \rangle$. The definitions that follow have been introduced in [5].

**Definition 2.2.** A multiset finite automaton (MFA) is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where

- $Q$ and $\Sigma$ are nonempty finite sets called the state-set and input-set, respectively,
- $\delta : Q \times \Sigma^\ast \rightarrow 2^Q$ is a map called transition map,
- $q_0 \in Q$ is called the initial state, and
- $F \subseteq Q$ is called the set of final states.

A configuration of a MFA $M$ is a pair $(p, \alpha)$, where $p$ and $\alpha$ denote current state and current multiset, respectively. The transition in a multiset finite automaton are described with the help of configurations. The transition from configuration $(p, \alpha)$ leads to configuration $(q, \beta)$ if there exists a multiset $\gamma \in \Sigma^\ast$ with $\gamma \subseteq \alpha$, $q \in \delta(p, \gamma)$ and $\beta = \alpha \ominus \gamma$, and is denoted by $(p, \alpha) \rightarrow (q, \beta)$. We shall denote by $\rightarrow^*$, the reflexive and transitive closure of this operation.

**Definition 2.3.** For a given set $\Sigma$, a multiset language $L$ is a subset of $\Sigma^\ast$. In addition, a multiset language $L \subseteq \Sigma^\ast$ is accepted by a MFA $M = (Q, \Sigma, \delta, q_0, F)$ if

$$L(M) = \left\{ \alpha \in \Sigma^\ast : \delta(q_0, \alpha) \cap F \neq \emptyset \right\},$$
for all $\alpha \in \Sigma^\mu$.

**Definition 2.4.** A multiset grammar is a structure $G = (V_N, V_T, S, P)$, where $V_N$ and $V_T$ are finite sets of non-terminal and terminal symbols, with $V_N \cap V_T = \emptyset$, $S \in V_N$ is the starting symbol, and $P \subseteq (V^\mu \cup V_N) \times V^\mu$ is a finite set of production rules, where $V = V_N \cup V_T$.

We have recollected the basic definitions and ideas of the theory of multisets as well as the notions a multiset automaton and of a multiset language. The next step is to generalize these ideas in the fuzzy theoretic framework.

### 3. Fuzzy multiset finite automata

In this section, we study the concepts of fuzzy multiset finite automata and fuzzy multiset languages. Let us start with the following definitions and results, which are borrowed from [21, 22].

**Definition 3.1.** A fuzzy multiset finite automaton (FMFA) is a quintuple $M = (Q, \Sigma, \delta, \sigma, \tau)$, where

- $Q$ and $\Sigma$ are nonempty finite sets called the state-set and input-set, respectively,
- $\delta : Q \times \Sigma^\mu \times Q \rightarrow [0, 1]$ is a map called transition map,
- $\sigma : Q \rightarrow [0, 1]$ is a map called the fuzzy set of initial states, and
- $\tau : Q \rightarrow [0, 1]$ is a map called the fuzzy set of final states.

A configuration of a fuzzy multiset finite automaton $M$ is a pair $(p, \alpha)$, where $p$ and $\alpha$ denote the current state and the current multiset, respectively. Transitions in a fuzzy multiset finite automaton are described with the help of configurations. The transition from configuration $(p, \alpha)$ leads to configuration $(q, \beta)$ with membership degree $k \in [0, 1]$ if there exists a multiset $\gamma \in \Sigma^\mu$ with $\gamma \subseteq \alpha$, $\delta(p, \gamma, q) = k$ and $\beta = \alpha \ominus \gamma$. This transition is written as $(p, \alpha) \overset{k}{\rightarrow} (q, \beta)$.

The expression $\overset{k'}{\rightarrow}^*$ denotes the reflexive and transitive closure of $\overset{k'}{\rightarrow}$. This means that for $(p, \alpha), (q, \beta) \in Q \times \Sigma^\mu$, $(p, \alpha) \overset{k'}{\rightarrow}^* (q, \beta)$, if for some $n \geq 0$, there exist $(n + 1)$ states $q_0, \ldots, q_n$ and $(n + 1)$ multisets $\alpha_0, \ldots, \alpha_n$ such that $q_0 = p$, $q_n = q$, $\alpha_0 = \alpha$, $\alpha_n = \beta$, and $(q_i, \alpha_i) \overset{k_i}{\rightarrow} (q_{i+1}, \alpha_{i+1})$, for all $i = 0, n - 1$, where $k' = k_0 \land k_1 \land \cdots \land k_{n-1}$. Next, we define

$$\mu_M((p, \alpha) \rightarrow^* (q, \beta)) = \bigvee \{ \mu_M((p, \alpha) \rightarrow (r, \alpha \ominus \gamma)) \land \mu_M((r, \alpha \ominus \gamma) \rightarrow (q, \beta)) : r \in Q, \gamma \in \Sigma^\mu \land \gamma \subseteq \alpha \}$$

and

$$\mu_M((p, \alpha) \rightarrow (q, \alpha)) = \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{if } p \neq q. \end{cases}$$
Example 3.2. Let $M = (Q, \Sigma, \delta, \sigma, \tau)$ be a FMFA, where $Q = \{q_1, q_2, q_3, q_4\}$, $\Sigma = \{a, b, c\}$. Also, assume that the fuzzy transition function is defined as follows:

$$
\begin{align*}
\delta(q_1, \langle b \rangle, q_3) &= 0.5 \\
\delta(q_3, \langle b \rangle \uplus \langle c \rangle, q_4) &= 0.3 \\
\delta(q_4, \langle a \rangle, q_2) &= 0.7 \\
\delta(q_1, \langle b \rangle \uplus \langle c \rangle, q_3) &= 0.3 \\
\delta(q_3, \langle b \rangle, q_4) &= 0.6 \\
\delta(q_2, \langle a \rangle, q_4) &= 0.2.
\end{align*}
$$

If $\alpha = \langle a \rangle \uplus \langle a \rangle \uplus \langle b \rangle \uplus \langle b \rangle \uplus \langle c \rangle$ and $\beta = 0_{\Sigma}$, then, the transition steps, $((q_1, \alpha) \to ^{(q_2, \beta)})$ are “expanded” as follows.

1. $(q_1, \langle a \rangle \uplus \langle a \rangle \uplus \langle b \rangle \uplus \langle b \rangle \uplus \langle c \rangle) \xrightarrow{0.5} (q_3, \langle a \rangle \uplus \langle a \rangle \uplus \langle b \rangle \uplus \langle c \rangle) \xrightarrow{0.3} (q_4, \langle a \rangle \uplus \langle a \rangle) \xrightarrow{0.7} (q_2, \langle a \rangle) \xrightarrow{0.3} (q_4, \langle a \rangle \uplus \langle a \rangle \uplus \langle b \rangle \uplus \langle c \rangle) \xrightarrow{0.7} (q_2, \langle a \rangle \uplus \langle b \rangle) \xrightarrow{0.2} (q_4, \langle b \rangle) \xrightarrow{0.6} (q_3, 0_{\Sigma}).$

Thus

$$
\mu_M((q_1, \alpha) \to ^{(q_2, \beta)}) = \bigvee \left\{ 0.5 \land 0.3 \land 0.7 \land 0.2, 0.3 \land 0.7 \land 0.2 \land 0.6 \right\} = \bigvee \{0.2, 0.2\} = 0.2.
$$

Definition 3.3. For a given set $\Sigma$, a fuzzy multiset language is a map $L : \Sigma^\uplus \to [0, 1]$. Let $M = (Q, \Sigma, \delta, \sigma, \tau)$ be a fuzzy multiset finite automaton. Then, the set

$$L(M) = \left\{ \alpha \in \Sigma^\uplus \mid \alpha \text{ is accepted by } M \right\}
$$

is called the fuzzy multiset language of $M$. A fuzzy multiset language $L : \Sigma^\uplus \to [0, 1]$ is accepted by a FMFA $M = (Q, \Sigma, \delta, \sigma, \tau)$ if

$$L(\alpha) = \bigvee \left\{ \sigma(q) \land \mu_M((q, \alpha) \to ^*(p, 0_{\Sigma})) \land \tau(p) : q, p \in Q \right\},
$$

for all $\alpha \in \Sigma^\upplus$. Furthermore, a fuzzy multiset language $L : \Sigma^\upplus \to [0, 1]$ is called regular if there exists a FMFA $M$ such that $L = L(M)$. We shall denote by $L(M)$ a fuzzy multiset language $L$, if $L$ is accepted by a FMFA $M$.

Example 3.4. Assume that $M = (Q, \Sigma, \delta, \sigma, \tau)$ is a FMFA. Also assume that $Q = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8\}$, $\Sigma = \{a, b, c\}$, $\sigma(q_1) = 0.2$, $\tau(q_5) = 0.3$, $\tau(q_8) = 0.6$, and
that \( \delta \) is defined as follows:
\[
\begin{align*}
\delta(q_1, (c), q_2) &= 0.2, & \delta(q_4, (0_\Sigma), q_7) &= 0.2, \\
\delta(q_1, (a), q_3) &= 0.3, & \delta(q_5, (a), q_6) &= 0.3, \\
\delta(q_2, (a), q_3) &= 0.2, & \delta(q_6, (a), q_6) &= 0.4, \\
\delta(q_2, (a) \sqcup (b) \sqcup (c), q_4) &= 0.9, & \delta(q_6, (0_\Sigma), q_8) &= 0.4, \\
\delta(q_3, (b) \sqcup (c), q_4) &= 0.6, & \delta(q_7, (c), q_8) &= 0.8, \\
\delta(q_1, (a) \sqcup (b), q_7) &= 0.5, & \delta(q_7, (a), q_8) &= 0.2, \\
\delta(q_4, (c), q_5) &= 0.3, & \delta(q_8, (c), q_8) &= 0.7.
\end{align*}
\]
Then, if \( \alpha = (2, 1, 2) \) can be written as \( (a) \sqcup (a) \sqcup (b) \sqcup (c) \sqcup (c) \), the transition steps of \( \alpha \) in \( M \) are as follows:

1. \( (q_1, (a) \sqcup (a) \sqcup (b) \sqcup (c) \sqcup (c)) \)  \( \xrightarrow{0.2} \) \( (q_2, (a) \sqcup (a) \sqcup (b) \sqcup (c)) \)  \( \xrightarrow{0.2} \) \( (q_3, (a) \sqcup (b) \sqcup (c)) \)  \( \xrightarrow{0.6} \) \( (q_4, (a)) \)

2. \( (q_1, (a) \sqcup (a) \sqcup (b) \sqcup (c) \sqcup (c)) \)  \( \xrightarrow{0.2} \) \( (q_2, (a) \sqcup (a) \sqcup (b) \sqcup (c)) \)  \( \xrightarrow{0.2} \) \( (q_3, (a) \sqcup (b) \sqcup (c)) \)  \( \xrightarrow{0.5} \) \( (q_7, (c)) \)  \( \xrightarrow{0.8} \) \( (q_6, (0_\Sigma)) \)

3. \( (q_1, (a) \sqcup (a) \sqcup (b) \sqcup (c) \sqcup (c)) \)  \( \xrightarrow{0.2} \) \( (q_2, (a) \sqcup (a) \sqcup (b) \sqcup (c)) \)  \( \xrightarrow{0.9} \) \( (q_4, (a)) \)  \( \xrightarrow{0.2} \) \( (q_7, (a)) \)  \( \xrightarrow{0.2} \) \( (q_8, (0_\Sigma)) \)

4. \( (q_1, (a) \sqcup (a) \sqcup (b) \sqcup (c) \sqcup (c)) \)  \( \xrightarrow{0.3} \) \( (q_3, (a) \sqcup (b) \sqcup (c) \sqcup (c)) \)  \( \xrightarrow{0.6} \) \( (q_4, (a) \sqcup (c)) \)  \( \xrightarrow{0.3} \) \( (q_5, (a)) \)  \( \xrightarrow{0.3} \) \( (q_6, (0_\Sigma)) \)

5. \( (q_1, (a) \sqcup (a) \sqcup (b) \sqcup (c) \sqcup (c)) \)  \( \xrightarrow{0.3} \) \( (q_3, (a) \sqcup (b) \sqcup (c) \sqcup (c)) \)  \( \xrightarrow{0.6} \) \( (q_4, (a) \sqcup (c)) \)  \( \xrightarrow{0.2} \) \( (q_7, (a) \sqcup (c)) \)  \( \xrightarrow{0.2} \) \( (q_8, (c)) \)  \( \xrightarrow{0.7} \) \( (q_8, (0_\Sigma)) \).
Thus

\[(q_1, \langle a \rangle \sqcup \langle a \rangle \sqcup \langle b \rangle \sqcup \langle c \rangle \sqcup \langle c \rangle) \xrightarrow{0.3} (q_3, \langle a \rangle \sqcup \langle b \rangle \sqcup \langle c \rangle \sqcup \langle c \rangle)\]
\[\quad \xrightarrow{0.6} (q_4, \langle a \rangle)\]
\[\quad \xrightarrow{0.2} (q_7, \langle a \rangle)\]
\[\quad \xrightarrow{0.8} (q_6, \langle a \rangle)\]
\[\quad \xrightarrow{0.4} (q_6, \langle 0 \Sigma \rangle).\]

\[(q_1, \langle a \rangle \sqcup \langle a \rangle \sqcup \langle b \rangle \sqcup \langle c \rangle \sqcup \langle c \rangle) \xrightarrow{0.3} (q_3, \langle a \rangle \sqcup \langle b \rangle \sqcup \langle c \rangle \sqcup \langle c \rangle)\]
\[\quad \xrightarrow{0.5} (q_7, \langle c \rangle)\]
\[\quad \xrightarrow{0.8} (q_6, \langle c \rangle)\]
\[\quad \xrightarrow{0.4} (q_8, \langle c \rangle)\]
\[\quad \xrightarrow{0.7} (q_8, \langle 0 \Sigma \rangle).\]

Thus

\[\mu_M(\alpha) = \bigvee \left\{ \sigma(q_1) \land 0.2 \land 0.2 \land 0.6 \land 0.2 \land 0.2 \land \tau(q_8), \right.\]
\[\quad \sigma(q_1) \land 0.2 \land 0.2 \land 0.5 \land 0.8 \land \tau(q_6),\]
\[\quad \sigma(q_1) \land 0.2 \land 0.9 \land 0.2 \land 0.2 \land \tau(q_8),\]
\[\quad \sigma(q_1) \land 0.3 \land 0.6 \land 0.3 \land 0.3 \land \tau(q_6),\]
\[\quad \sigma(q_1) \land 0.3 \land 0.6 \land 0.2 \land 0.2 \land 0.7 \land \tau(q_8),\]
\[\quad \sigma(q_1) \land 0.3 \land 0.6 \land 0.2 \land 0.8 \land 0.4 \land \tau(q_6),\]
\[\quad \sigma(q_1) \land 0.3 \land 0.5 \land 0.8 \land 0.4 \land 0.7 \land \tau(q_8)\}
\[= \bigvee \left\{ 0.2 \land 0.2 \land 0.2 \land 0.6 \land 0.2 \land 0.2 \land 0.6, \right.\]
\[\quad 0.2 \land 0.2 \land 0.2 \land 0.5 \land 0.8 \land 0.3,\]
\[\quad 0.2 \land 0.2 \land 0.9 \land 0.2 \land 0.2 \land 0.6,\]
\[\quad 0.2 \land 0.3 \land 0.6 \land 0.3 \land 0.3 \land 0.3,\]
\[\quad 0.2 \land 0.3 \land 0.6 \land 0.2 \land 0.2 \land 0.7 \land 0.6,\]
\[\quad 0.2 \land 0.3 \land 0.6 \land 0.2 \land 0.8 \land 0.4 \land 0.3,\]
\[\quad 0.2 \land 0.3 \land 0.5 \land 0.8 \land 0.4 \land 0.7 \land 0.6\}
\[= \bigvee \left\{ 0.2, 0.2, 0.2, 0.2, 0.2, 0.2 \right\}
\[= 0.2.\]
Definition 3.5. A language $L \subseteq \Sigma^\omega$ is called a fuzzy multiset automaton language if there exists a fuzzy multiset automaton $M$ such that $L(M) = L$.

Theorem 3.6. If $L_1$ and $L_2$ are fuzzy multiset automata languages over $\Sigma$, then $L_1 \cap L_2$, $L_1 \uplus L_2$ and $L_1^\omega$ are fuzzy multiset automata languages over $\Sigma$.

Proof. The proof is identical to the one given by Wang, Yin and Gu [22] but we include it for reasons of completeness. Assume that $L_1$ and $L_2$ are two fuzzy multiset finite automata languages accepted by the FMFAs $M_1 = (Q_1, \Sigma, \delta_1, \sigma_1, \tau_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, \sigma_2, \tau_2)$, respectively. Let $Q_1 \cap Q_2 = \emptyset$.

1. First of all, we construct a FMFA $M = (Q, \Sigma, \delta, \sigma, \tau)$ such that $Q = Q_1 \cup Q_2$ as follows:

$$\sigma(q) = \begin{cases} \sigma_1(q), & \text{if } q \in Q_1, \\ \sigma_2(q), & \text{if } q \in Q_2 \end{cases} \quad \text{and} \quad \tau(q) = \begin{cases} \tau_1(q), & \text{if } q \in Q_1, \\ \tau_2(q), & \text{if } q \in Q_2 \end{cases}$$

for all $x \in \Sigma^\omega$ and

$$\delta^*(q_1, x, p_1) = \begin{cases} \delta_1(q_1, x, p_1), & \text{if } q_1, p_1 \in Q_1 \\ \delta_2(q_1, x, p_1), & \text{if } q_1, p_1 \in Q_2 \\ 0, & \text{otherwise} \end{cases}$$

Obviously, for all $\omega \in \Sigma^\omega$, we have

$$\mu_M(\omega) = \max_{q,p \in Q} \{ \sigma(q) \land \mu_M((q, \omega) \xrightarrow{\delta} (p, 0_\Sigma)) \land \tau(p) \}$$

$$= \max_{q,p \in Q} \{ \sigma_1(q) \land \mu_{M_1}((q, \omega) \xrightarrow{\delta} (p, 0_\Sigma)) \land \tau_1(p) \} \lor \max_{q,p \in Q} \{ \sigma_2(q) \land \mu_{M_2}((q, \omega) \xrightarrow{\delta} (p, 0_\Sigma)) \land \tau_2(p) \}$$

$$= \mu_{M_1}(\omega) \lor \mu_{M_2}(\omega).$$

Hence $L(M) = L(M_1) \cup L(M_2) = L_1 \cup L_2$ is a FMFA language.

2. Now we construct a FMFA $M = (Q, \Sigma, \delta, \sigma, \tau)$ such that $Q = Q_1 \cap Q_2$:

$$\sigma(q) = \begin{cases} \sigma_1(q), & \text{if } q \in Q_1, \\ 0, & \text{if } q \in Q_2 \end{cases} \quad \text{and} \quad \tau(q) = \begin{cases} 0, & \text{if } q \in Q_1, \\ \tau_2(q), & \text{if } q \in Q_2 \end{cases}$$

Also, for all $x \in \Sigma^\omega \setminus 0_\Sigma$:

$$\delta^*(q_1, x, p_1) = \begin{cases} \delta_1(q_1, x, p_1), & \text{if } q_1, p_1 \in Q_1 \\ \delta_2(q_1, x, p_1), & \text{if } q_1, p_1 \in Q_2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\delta^*(q_1, 0_\Sigma, p_1) = \begin{cases} \delta_1(q_1, 0_\Sigma, p_1), & \text{if } q_1, p_1 \in Q_1 \\ \delta_2(q_1, 0_\Sigma, p_1), & \text{if } q_1, p_1 \in Q_2 \\ \tau_1(q_1) \land \tau_2(p_1), & \text{if } q_1 \in Q_1, p_1 \in Q_2 \\ 0, & \text{otherwise} \end{cases}$$
Obviously, for all $\omega \in \Sigma^\omega$, we have

$$
\mu_M(\omega) = \max_{q,p \in Q} \{ \sigma(q) \land \mu_M((q, \omega) \xrightarrow{\star} (p, 0\Sigma)) \land \tau(p) \} \\
= \max_{\omega_1 \uplus \omega_2 = \omega} \{ \sigma(p) \land \mu_M((q, \omega) \xrightarrow{\star} (q_1, \omega_1)) \land \mu_M((q_1, \omega_1) \xrightarrow{\star} (p_1, \omega_2)) \land \mu_M((p_1, \omega_2) \xrightarrow{\star} (p, 0\Sigma)) \land \tau(p) \} \\
= \max_{\omega_1 \uplus \omega_2 = \omega} \{ \{ \sigma_1(p) \land \mu_{M_1}((q, \omega) \xrightarrow{\star} (q_1, \omega_1)) \land \tau_1(q_1) \land \mu_{M_2}((p_1, \omega_2) \xrightarrow{\star} (p, 0\Sigma)) \land \tau(p) \} \\
= \max_{\omega_1 \uplus \omega_2 = \omega} \{ \{ \{ \sigma_1(p) \land \mu_{M_1}((q, \omega) \xrightarrow{\star} (q_1, 0\Sigma)) \land \tau_1(q_1) \} \land \mu_{M_2}((p_1, \omega_2) \xrightarrow{\star} (p, 0\Sigma)) \land \tau(p) \} \\
= \max \{ \mu_{M_1}(\omega_1) \land \mu_{M_2}(\omega_2) \}.
$$

Hence, $L(M) = L(M_1) \uplus L(M_2) = L_1 \uplus L_2$ is a FMFA language.

(3) We construct a FMFA $M = (Q, \Sigma, \delta, \sigma, \tau)$ such that $Q = Q_1 \cup \{ q_0 \}$ for all $x \in \Sigma^\omega \setminus 0\Sigma$:

$$
\sigma(q) = \begin{cases} 
\sigma_1(q), & \text{if } q \in Q_1, \\
1, & \text{if } q = q_0
\end{cases}
$$

and

$$
\tau(q) = \begin{cases} 
\tau_1(q), & \text{if } q \in Q_1, \\
1, & \text{if } q = q_0
\end{cases}
$$

Also, for all $\omega \in \Sigma^\omega$:

$$
\delta^*(q_1, x, p_1) = \begin{cases} 
\delta_1(q_1, x, p_1), & \text{if } q_1, p_1 \in Q_1, \\
0, & \text{otherwise}
\end{cases}
$$

and

$$
\delta^*(q_1, 0\Sigma, p_1) = \begin{cases} 
\delta_1(q_1, 0\Sigma, p_1), & \text{if } q_1, p_1 \in Q_1, \\
\tau_1(q_1) \land \tau_2(p_1), & \text{if } q_1, p_1 \in Q_1, \\
0, & \text{otherwise}
\end{cases}
$$

Now, (a) if $\omega = 0\Sigma$, then

$$
\mu_M(0\Sigma) = \max_{q,p \in Q} \{ \sigma(q) \land \mu_M((q, 0\Sigma) \xrightarrow{\star} (p, 0\Sigma)) \land \tau(p) \} \\
= \sigma(q_0) \land \mu_M((q_0, 0\Sigma) \xrightarrow{\star} (q_0, 0\Sigma)) \land \tau(q_0) \\
= 1 \\
= \mu_{L_T^\omega}(0\Sigma);$$
Example 3.7. Assume that $M_1 = (Q_1, \Sigma, \delta_1, \sigma_1, \tau_1)$ is a FMFA, where $Q_1 = \{q_1, q_2, q_3\}$, $\Sigma = \{a, b, c\}$, $\sigma_1 = \{(q_1, 0.5)\}$, $\tau_1 = \{(q_2, 0.4), (q_3, 0.6)\}$, and $\delta_1$ is defined as follows:

$$\delta_1(q_1, \langle a \rangle \uplus \langle b \rangle, q_2) = 0.4,$$
$$\delta_1(q_1, \langle c \rangle, q_3) = 0.5,$$

and

$$\delta_1(q_2, \langle a \rangle \uplus \langle c \rangle, q_3) = 0.3.$$ 

Then, we define the language

$$L(M_1) = \{(\langle a \rangle \uplus \langle b \rangle, 0.4), (\langle a \rangle \uplus \langle b \rangle \uplus \langle a \rangle \uplus \langle c \rangle, 0.3), (\langle c \rangle, 0.5)\}.$$ 

Assume that $M_2 = (Q_2, \Sigma, \delta_2, \sigma_2, \tau_2)$ is a FMFA, where $Q_2 = \{p_1, p_2, p_3\}$, $\Sigma = \{a, b, c\}$, $\sigma_2 = \{(p_1, 0.6), (p_2, 0.4)\}$, $\tau_2 = \{(p_2, 0.4), (p_3, 0.5)\}$, and $\delta_2$ is defined as follows:

$$\delta_2(p_1, \langle a \rangle \uplus \langle c \rangle, p_2) = 0.6,$$
$$\delta_2(p_1, \langle c \rangle, p_3) = 1.0.$$ 

Hence, $L(M) = L_1^\omega$ is a FMFA language.

\[\square\]
and
\[ \delta_2(p_2, \langle a \rangle \uplus b), p_3) = 0.4. \]

Then, we find the language
\[ L(M_2) = \{(0, 0.4), (\langle a \rangle \uplus \langle c \rangle, 0.4), (\langle c \rangle, 0.5), (\langle a \rangle \uplus \langle c \rangle \uplus \langle b \rangle, 0.4), (\langle a \rangle \uplus \langle b \rangle, 0.4)\}. \]

From the proof of the above theorem, we have

(1) \( M = (Q_1 \cup Q_2, \Sigma, \delta, \tau) \), where
\[ \begin{align*}
\sigma &= \{(q_1, 0.5), (p_1, 0.6), (p_2, 0.4)\}, \\
\tau &= \{(q_2, 0.4), (q_3, 0.6), (p_2, 0.4), (p_3, 0.5)\},
\end{align*} \]
and the \( \delta \)'s are defined as follows:
\[ \begin{align*}
\delta_1(q_1, \langle a \rangle \uplus \langle b \rangle, q_2) &= 0.4, \\
\delta_1(q_1, \langle c \rangle, q_3) &= 0.5, \\
\delta_1(q_2, \langle a \rangle \uplus \langle c \rangle, q_3) &= 0.3,
\end{align*} \]

and
\[ \begin{align*}
\delta_2(p_1, \langle a \rangle \uplus \langle c \rangle, p_2) &= 0.6, \\
\delta_2(p_1, \langle c \rangle, p_3) &= 1.0, \\
\delta_2(p_2, \langle a \rangle \uplus \langle b \rangle, p_3) &= 0.4.
\end{align*} \]

Now,
\[ L(M) = L(M_1) \cup L(M_2) \]
\[ = \{(\langle a \rangle \uplus \langle b \rangle, 0.4), (\langle a \rangle \uplus \langle b \rangle \uplus \langle a \rangle \uplus \langle c \rangle, 0.3), (\langle c \rangle, 0.5), \\
(0, 0.4), (\langle a \rangle \uplus \langle c \rangle, 0.4), (\langle c \rangle, 0.5), \\
(\langle a \rangle \uplus \langle c \rangle \uplus \langle a \rangle \uplus \langle b \rangle, 0.4), (\langle a \rangle \uplus \langle b \rangle, 0.4)\}. \]

(2) \( M' = (Q_1 \cup Q_2, \Sigma, \delta', \sigma', \tau') \), where \( \sigma' = \{(q_1, 0.5)\}, \tau' = \{(p_2, 0.4), (p_3, 0.5)\}, \)
and \( \delta' \) is defined as follows:
\[ \begin{align*}
\delta'(q_2, 0, p_2) &= 0.6, \\
\delta'(q_2, 0, p_1) &= 0.5, \\
\delta'(q_3, 0, p_2) &= 0.7, \\
\delta'(q_3, 0, p_1) &= 0.8, \\
\delta'(q_1, \langle a \rangle \uplus \langle b \rangle, q_2) &= 0.4, \\
\delta'(q_1, \langle c \rangle, q_3) &= 0.5, \\
\delta'(q_2, \langle a \rangle \uplus \langle c \rangle, q_3) &= 0.3, \\
\delta'(p_1, \langle a \rangle \uplus \langle c \rangle, p_2) &= 0.6, \\
\delta'(p_1, \langle c \rangle, p_3) &= 1.0.
\end{align*} \]
and

\[ \delta'(p_2, (a) \cup (b), p_3) = 0.4. \]

Then,

\[ L(M') = L(M_1) \cup L(M_2) \]

\[ = \{(a) \cup (b), 0.4), ((a) \cup (a) \cup (b) \cup (b), 0.4),\]

\[(a) \cup (a) \cup (b) \cup (b) \cup (c), 0.4), ((a) \cup (a) \cup (b) \cup (c), 0.3),\]

\[((a) \cup (a) \cup (b) \cup (c), 0.3),\]

\[((a) \cup (a) \cup (b) \cup (c), 0.4),\]

\[((c), 0.4), ((a) \cup (b) \cup (c), 0.4),\]

\[((c) \cup (c), 0.5), ((a) \cup (c) \cup (c), 0.4),\]

\[((a) \cup (b) \cup (c) \cup (c), 0.4)\}.

(3) \(M'' = (Q_1 \cup \{q_0\}, \Sigma, \delta'', \sigma'', \tau'')\) is a FMFA, where \(\sigma'' = \{(q_1, 0.5), (q_0, 1)\}\),

\(\tau'' = \{(q_2, 0.4), (q_3, 0.6), (q_0, 1)\}\) and \(\delta''\) is defined as follows:

\[ \delta''(q_2, 0_\Sigma, q_1) = 0.8, \]

\[ \delta''(q_3, 0_\Sigma, q_1) = 0.9, \]

\[ \delta''(q_1, (a) \cup (b), q_2) = 0.4, \]

\[ \delta''(q_1, (c), q_3) = 0.5, \]

and

\[ \delta''(q_2, (a) \cup (c), q_3) = 0.3. \]

Then, we get the following language

\[ L(M'') = L(M_1)^w \]

\[ = \{(0_\Sigma, 1), ((a) \cup (b), 0.4),\]

\[((a) \cup (b) \cup (b) \cup (c), 0.3), ((c), 0.5), \ldots\}.

**Definition 3.8.** An onto function \(f : \Sigma \to \Sigma'^w\) is called a homomorphism if for all \(x, y \in \Sigma, f(x \cup y) = f(x) \cup f(y)\). This homomorphism can be naturally extended to \(f : \Sigma'^w \to \Sigma_1'^w\), where \(f(0_\Sigma) = 0_\Sigma, f(a \cup \omega) = f(a) \cup f(\omega)\) for \(a \in \Sigma\) and \(\omega \in \Sigma'^w\).

The following theorem can be proved by induction on the length of \(b\) \((|b|)\).

**Theorem 3.9.** If \(f : \Sigma \to \Sigma'^w\) is a homomorphism, then for all \(a, b \in \Sigma'^w, f(a \cup b) = f(a) \cup f(b)\).

**Theorem 3.10.** The class of all fuzzy multiset automaton languages is closed under homomorphism and inverse homomorphism.
Proof. Let \( f : \Sigma \to \Sigma^w \) be a homomorphism and \( L \subseteq \Sigma^w \) be a fuzzy multiset finite automaton language. Also, let \( M = (Q, \Sigma, \delta, \sigma, \tau) \) be a fuzzy multiset finite automaton such that \( L(M) = L \). Then, we construct a fuzzy multiset finite automaton \( M' = (Q, \Sigma_1, \delta', \sigma, \tau) \), where \( \delta' : Q \times \Sigma_1 \times Q \to [0,1] \) is defined by \( \delta'(p, \alpha, q) = r \) if and only if there is an \( \omega \in \Sigma^w \) such that \( f(\omega) = \alpha \) and \( \delta'(p, \alpha, q) = r \), where \( r \in [0,1] \). Assume that \( \alpha \in L(M') \). Then, \( \{ \sigma(p) \land \mu_M((p, \alpha) \to (q, 0_{\Sigma_2})) \land \tau(q) \} \neq 0 \), for some \( p, q \in Q \). Also, \( \mu_M'((p, \alpha) \to (q, 0_{\Sigma_2})) \neq 0 \). Thus there is an \( \omega \in \Sigma^w \) such that \( f(\omega) = \alpha \) and then \( \{ \sigma(p) \land \mu_M'((p, \omega) \to (q, 0_{\Sigma_2})) \land \tau(q) \} \neq 0 \). Hence, \( \omega \in L(M) = L \). Therefore, \( \alpha \in f(L) \).

The converse is similar. Assume that \( L \in \Sigma^w \) is a fuzzy multiset finite automaton language and \( M = (Q, \Sigma_1, \delta, \sigma, \tau) \) is a fuzzy multiset finite automaton such that \( L(M) = L \). Then we construct a fuzzy multiset automaton \( M' = (Q, \Sigma, \delta', \sigma, \tau) \), where \( \delta' = \Sigma \times \Sigma^w \), defined by \( \mu_M'((p, \omega) \to (q, 0_{\Sigma_2})) = \mu_M((p, f(\omega)) \to (q, 0_{\Sigma_2})) \), for all \( p, q \in Q, \omega \in \Sigma^w \). Then,

\[
\omega \in L(M') \iff \{ \sigma(p) \land \mu_M'((p, \omega) \to (q, 0_{\Sigma_2})) \land \tau(q) \} \neq 0
\]

\[
\iff \{ \sigma(p) \land \mu_M((p, f(\omega)) \to (q, 0_{\Sigma_2})) \land \tau(q) \} \neq 0
\]

\[
\iff f(\omega) \in L(M) = L
\]

\[
\iff \omega \in f^{-1}(L).
\]

\[\square\]

**Definition 3.11.** Suppose that \( L_1, L_2 \subseteq \Sigma^w \). Then, the quotient of language \( L_1 \) with language \( L_2 \) is the following language:

\[ L_1/L_2 = \{ \omega \in \Sigma^w \mid \exists \nu \in L_2 \text{ such that } \omega \uplus \nu \in L_1 \} \]

**Theorem 3.12.** The class of all fuzzy multiset automata languages is closed under quotient with arbitrary multisets.

**Proof.** Assume that \( L_1 \subseteq \Sigma^w \) is a fuzzy multiset automaton language and that \( L_2 \subseteq \Sigma^w \) is a multiset. Then, there exists a fuzzy multiset automaton \( M = (Q, \Sigma, \delta, \sigma, \tau) \) such that \( L(M) = L_1 \). We construct a fuzzy multiset finite automaton \( M' = (Q, \Sigma, \delta, \sigma, \tau') \), where \( \tau' : Q \to [0,1] \) is defined by

\[
\tau'(q) = \max_{v \in L_2} \left\{ \mu_M((p, v) \to (q', 0_{\Sigma_2})) \land \tau(q') \right\} \neq 0.
\]

Also, \( \omega \in L(M') \) if and only if there exist \( p, q \in Q \) such that

\[
\{ \sigma(p) \land \mu_M((p, \omega) \to (q, 0_{\Sigma_2})) \land \tau'(q) \} \neq 0.
\]

A different way to say the same thing are the following “equations”:

\[
\sigma(p) \neq 0, \quad \mu_M((p, \omega) \to (q, 0_{\Sigma_2})) \neq 0, \quad \text{and} \quad \tau'(q) \neq 0.
\]

Again, these are equivalent to the following for \( p, q, q' \in Q \)

\[
\{ \sigma(p) \land \mu_M((p, \omega) \to (q, 0_{\Sigma_2})) \land \mu_M((p, v) \to (q, 0_{\Sigma_2})) \land \tau(q') \} \neq 0.
\]

Now, for some \( v \in L_2 \) and since \( f'(q) \neq 0 \)

\[
\{ \sigma(p) \land \mu_M((p, \omega \uplus v) \to (q', 0_{\Sigma_2})) \land \tau(q') \} \neq 0.
\]

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that is, for some $p, q' \in Q$ and $v \in L_2$ if and only if $\omega \uplus v \in L(M) = L_1$ for some $v \in L_2$ and only if $\omega \in L_1/L_2$.
The converse part can be proved in a similar way. □

**Theorem 3.13.** The class of all fuzzy multiset finite automaton languages is closed under right quotient by any multiset and reversal of a fuzzy multiset finite language.

**Proof.** The proof is similar to the proof of the previous theorem. □

### 4. FUZZY MULTISET REGULAR GRAMMAR

The concept of regular grammar for both finite state automaton and fuzzy finite state automaton have been introduced in (see also [3, 6, 12, 26]). In this section, we introduce the notion of regular grammar and regular grammar in normal form for fuzzy multiset finite automata and then discuss some properties. The following definition is borrowed from [22].

**Definition 4.1.** A fuzzy multiset grammar is a quintuple $G = (V_N, V_T, S, P)$, where

1. $V_N$ is a finite alphabet of nonterminal symbols,
2. $V_T$ is a finite alphabet of terminal symbols, such that $V_N \cap V_T = \varnothing$,
3. $V$ is the total alphabet ($V = V_N \cup V_T$),
4. $S \in V_N$ is a starting nonterminal symbol,
5. $P$ is a finite set of fuzzy multiset production rules over $V$, that is,

$$P = \left\{ \alpha \rightarrow \beta, \mu_P(\alpha \rightarrow \beta) \mid \alpha \in V^w \uplus V_N, \beta \in V^w, \mu_P(\alpha \rightarrow \beta) \in [0, 1] \right\},$$

where $\mu_P : (V^w \uplus V_N) \times V^w \rightarrow [0, 1]$ is called the fuzzy transition function.

For any $(\alpha \rightarrow \beta, r) \in P$, for convenience, we sometimes abbreviate $(\alpha \rightarrow \beta, r)$ to $\alpha \overset{r}{\rightarrow} \beta$.

**Definition 4.2.** Suppose that $G = (V_N, V_T, S, P)$ is a fuzzy multiset grammar, $B \in V^w$, and $A \in V^w \setminus \{0_N\}$. Then,

1. $B$ immediately derives from $A$ written as $(A \overset{r}{\rightarrow} B)$ if there exist multisets $\alpha, \beta, x$ such that $A = \alpha \uplus x, \beta = \beta \uplus y$ and $\alpha \overset{r}{\rightarrow} \beta \in P$,
2. $B$ derives from $A$ (written $(A \overset{r}{\rightarrow} *B))$ if for some $n \geq 0$ there exist $n + 1$ multisets $v_0, v_1, \ldots, v_n$ such that $v_0 = A, v_n = B$ and $v_i \overset{r_i}{\rightarrow} v_{i+1}$ for $i = 0, 1, \ldots, n-1$, where $r = r_0 \land r_1 \land \cdots \land r_{n-1}$ is called the grade of membership of the derivation of $B$ from $A$ with the derivation chain $A \overset{r_0}{\rightarrow} v_1 \overset{r_1}{\rightarrow} \cdots \overset{r_{n-2}}{\rightarrow} v_{n-2} \overset{r_{n-1}}{\rightarrow} v_{n-1} \overset{r_n}{\rightarrow} B$.

**Definition 4.3.** A fuzzy multiset grammar $G = (V_N, V_T, S, P)$ is said to be regular, if each of its fuzzy multiset productions is either of the form $(\alpha) \overset{r}{\rightarrow} (\alpha) \uplus (\beta)$ or $(\alpha) \overset{r}{\rightarrow} (\alpha)$, where $\alpha, \beta \in V_N, a \in V_T$, and $r \in [0, 1]$.

**Definition 4.4.** A string $w \in V_T^+$ is said to be generated by fuzzy multiset regular grammar $G$ if

$$\max_{q_0 \in V_N} \left\{ S(q_0) \land \mu_G(q_0 \rightarrow *\omega) \right\} \neq 0.$$
**Definition 4.5.** The set of all strings \( \omega \in V_N^* \) that are generated by a fuzzy multiset regular grammar \( G \) is called a fuzzy multiset regular language. We shall denote it by \( L(G) \).

The following three theorems have been proved in [22].

**Theorem 4.6.** For each fuzzy multiset finite automaton \( M = (Q, \Sigma, \delta, \sigma, \tau) \), there exists a fuzzy multiset regular grammar \( G \) such that \( L(G) = L(M) \setminus \{0_\Sigma \} \).

**Theorem 4.7.** For a fuzzy multiset regular grammar \( G = (V_N, V_T, S, P) \) there exists a fuzzy multiset finite automaton \( M \) such that \( L(M) = L(G) \).

**Definition 4.8.** A fuzzy multiset \( G = (V_N, V_T, S, P) \) is said to be in normal form if it has either of the following types of fuzzy multiset production rules:

\[
\langle A \rangle \xrightarrow{r} \langle a \rangle \cup \langle B \rangle \quad \text{or} \quad \langle A \rangle \xrightarrow{r} \langle 0_{V_T} \rangle,
\]

where \( A, B \in V_N, \ a \in V_T, \) and \( r \in [0,1] \). Every fuzzy multiset regular grammar can be reduced to a fuzzy multiset grammar in normal form by changing its fuzzy multiset production \( A \xrightarrow{r} \langle a \rangle \) by two fuzzy multiset productions \( A \xrightarrow{r} \langle a \rangle \cup \langle C \rangle, \ C \notin V_N \) and \( C \xrightarrow{0_{V_T}} \langle 0_{V_T} \rangle \).

**Theorem 4.9.** For every fuzzy multiset grammar \( G = (V_N, V_T, S, P) \) in normal form, there exists a fuzzy multiset automaton \( M \), such that \( L(M) = L(G) \).

**Proof.** Let \( Q = V_N, \ \sigma = S \). Also, for \( \tau : Q \to [0,1] \) it holds that \( \tau(q) = r \) if and only if \( \langle q \rangle \xrightarrow{r} \langle 0_{V_T} \rangle \) is a fuzzy multiset production in \( P \). In addition, for \( \delta : Q \times \Sigma \times Q \to [0,1] \), it holds that \( \mu_M(\langle q, a \rangle \xrightarrow{r} \langle p, a \rangle) \) if and only if \( \langle q \rangle \xrightarrow{r} \langle a \rangle \) \( \cup \langle p \rangle \) is a fuzzy multiset production in \( P \). Then, \( M = (Q, \Sigma, \delta, \sigma, \tau) \) is a fuzzy multiset finite automaton. Also, let \( \omega \in L(G) \) and \( \omega = a_1a_2a_3 \ldots a_{n-1}a_n, \ a_i \in \Sigma \). Then, \[
\max_{q \in V_N} \{ S(q) \land \mu((q, \omega) \to \omega(q_f, 0_\Sigma)) \} \neq 0,
\]

that is, there exists \( q_0 \in V_N \) such that \( S(q_0) \neq 0 \) and \( \mu_M((q_0, \omega) \to \omega(q_f, 0_{V_T})) \), for all \( r \in [0,1] \). Now, \( \langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle \), \( r \in [0,1] \) implies that there exist \( q_1, q_2, \ldots q_n \in V_N \) and \( r_1, r_2, \ldots r_{n-1} \in [0,1] \) such that

\[
\langle q_0 \rangle \xrightarrow{r_1} \langle a_1 \rangle \cup \langle q_1 \rangle \\
\xrightarrow{r_2} \langle a_1 \rangle \cup \langle a_2 \rangle \cup \langle q_2 \rangle \\
\ldots \\
\xrightarrow{r_n} \langle a_1 \rangle \cup \langle a_2 \rangle \cup \ldots \cup \langle a_n \rangle \cup \langle q_n \rangle \\
\xrightarrow{r_{n+1}} \langle a_1 \rangle \cup \langle a_2 \rangle \cup \ldots \cup \langle a_n \rangle \cup \langle 0_{V_T} \rangle \\
= \omega.
\]
Thus for this derivation chain, \( P \) must have the following fuzzy multiset productions:

\[
\langle q_0 \rangle \xrightarrow{r_1} \langle a_1 \rangle \uplus \langle q_1 \rangle,
\langle q_1 \rangle \xrightarrow{r_2} \langle a_2 \rangle \uplus \langle q_2 \rangle,
\vdots
\vdots
\langle q_{n-1} \rangle \xrightarrow{r_n} \langle a_n \rangle \uplus \langle q_n \rangle,
\langle q_n \rangle \xrightarrow{r_{n+1}} 0_{VT}.
\]

Therefore,

\[
\delta(q_0, (a_1), q_1) = r_1,
\delta(q_1, (a_2), q_2) = r_2,
\vdots
\vdots
\delta(q_{n-1}, (a_n), q_n) = r_n,
\tau(q_n) = r_{n+1}.
\]

But then,

\[
\left\{ \mu_M((q_0, a_1) \rightarrow (q_1, a_2)) \land \cdot \cdot \cdot \land \mu_M((q_{n-1}, a_n) \rightarrow (q_n, 0_{\Sigma})) \land \tau(q_n) \right\} \neq 0,
\]

that is, there exist \( q_0, q_n \in Q \), such that

\[
\left\{ \sigma(q_0) \land \mu_M((q_0, \omega) \rightarrow^* (q_n, 0_{\Sigma})) \land \tau(q_n) \right\} \neq 0,
\]

or that

\[
\max_{q \in Q} \left\{ \sigma(q) \land \mu_M((q, \omega) \rightarrow^* (p, 0_{\Sigma})) \right\} \neq 0.
\]

Hence \( \omega \in L(M) \).

The converse can be proved similarly. \( \square \)

**Corollary 4.10.** For every fuzzy multiset grammar \( G \) in normal form, there exists a fuzzy multiset regular grammar \( G_1 \) such that

\[
L(G_1) = L(G) - \{0_{VT}\}.
\]

**Definition 4.11.** A fuzzy multiset grammar \( G = (V_N, V_T, S, P) \) is called a linear grammar, if it has either of the following fuzzy multiset production rules:

- \( (A) \xrightarrow{r} \langle \omega \rangle \)
- \( (A) \xrightarrow{r} \langle \omega_1 \rangle \uplus \langle B \rangle \uplus \langle \omega_2 \rangle \)

where \( A, B \in V_N, \omega_1, \omega_2, \omega \in V_T^*, r \in [0, 1] \). When \( G \) contains production rule of the second case and \( \omega_1 = \{0_{\Sigma}\} \), then \( G \) is called a left linear grammar, while if \( \omega_2 \in \{0_{VT}\} \), then it is called right linear grammar.

**Definition 4.12.** A language \( L \subseteq V_T^* \) is fuzzy multiset linear (left linear, right linear), if there is fuzzy multiset linear (left linear, right linear, respectively) grammar \( G \) such that

\[
L(G) = L.
\]
It is clear that the class of fuzzy multiset regular languages is a subclass of the class of fuzzy linear languages.

**Theorem 4.13.** A fuzzy left linear grammar and a fuzzy right linear grammar both generate the same languages.

**Proof.** Let \( G = (V_N, V_T, S, P) \) be a fuzzy left right linear grammar \( G' = (V_N, V_T, S, P') \) whose fuzzy multiset production rule set \( P' \) is as follows:

1. \( \langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle \) in \( P' \) if and only if \( \langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle \) in \( P \), \( S(q_0) \neq 0 \).
2. \( \langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle \cup \langle A \rangle \) in \( P' \) for \( S(q_0) \neq 0 \) if and only if \( \langle A \rangle \xrightarrow{r} \langle \omega \rangle \) in \( P \); \( S(A) = 0 \).
3. \( \langle A \rangle \xrightarrow{r} \langle \omega \rangle \) and \( \langle A \rangle \xrightarrow{r} \langle \omega \cup q_0 \rangle \) in \( P' \) if and only if \( \langle q_0 \rangle \xrightarrow{r} \langle A \cup \omega \rangle \) in \( P \), \( S(q_0) \neq 0 \).
4. \( \langle A \rangle \xrightarrow{r} \langle \omega \cup B \rangle \) in \( P' \) if and only if \( \langle B \rangle \xrightarrow{r} \langle A \cup \omega \rangle \) in \( P \); \( S(B) = 0 \).

We prove that \( L(G') = L(G) \). Assume that \( \omega \in L(G) \), where \( \omega = \omega_1 \cdots \omega_n \), and \( \omega_i \in V_T^{ \ast \ast } \). Then,

\[
\max_{q \in V_N} \left\{ S(q) \land (\langle q \rangle \xrightarrow{r} \langle \omega \rangle) \right\} \neq 0,
\]

where \( r = r_0 \land r_1 \land \cdots \land r_n \in [0, 1] \). Thus there exists \( q_0 \in V_N \) such that \( S(q_0) \neq 0 \) and \( \langle q \rangle \xrightarrow{r} \langle \omega \rangle \); \( r \neq 0 \) in \( G \). If \( \langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle \) is a fuzzy multiset production rule in \( P \) for some \( r \in [0, 1] \) then \( \langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle \) in \( P' \) and \( \omega \in L(G') \), otherwise these exist \( A_n, A_{n-1}, \ldots, A_2 \in V_N \) and \( r_1, r_2, \ldots, r_n \in (0, 1) \) such that \( \langle q_0 \rangle \xrightarrow{r_1} \langle A_n \rangle \cup (\omega_n) \xrightarrow{r_1} \ldots \rightarrow \langle A_2 \rangle \cup \langle \omega_2 \rangle \) and \( \langle A_2 \rangle \xrightarrow{r_1} \langle \omega_1 \rangle \).

Therefore, \( P' \) should have the following fuzzy multiset production rules:

\[
\langle q_0 \rangle \xrightarrow{r_1} \langle A_n \rangle \cup (\omega_n),
\langle A_n \rangle \xrightarrow{r_1} \langle A_{n-1} \rangle \cup (\omega_{n-1}),
\vdots
\langle A_2 \rangle \xrightarrow{r_1} \langle \omega_1 \rangle \cup (\omega_2),
\]

and

\[
\langle A_2 \rangle \xrightarrow{r_1} \langle \omega_1 \rangle.
\]
Thus, there is a derivation chain for $\omega$ in $G'$ as:

$$
\langle q_0 \rangle \xrightarrow{r_1} \langle \omega_1 \rangle \uplus \langle A_2 \rangle \\
\xrightarrow{r_2} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \langle A_3 \rangle \\
\vdots \\
\xrightarrow{r_{n-2}} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \cdots \uplus \langle \omega_{n-2} \rangle \uplus \langle A_{n-1} \rangle \\
\xrightarrow{r_{n-1}} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \cdots \uplus \langle \omega_{n-1} \rangle \uplus \langle A_n \rangle \\
\xrightarrow{r_n} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \cdots \uplus \langle \omega_{n-1} \rangle \uplus \langle \omega_n \rangle = \omega.
$$

Hence

$$\max_{q \in V_N} \left\{ S(q) \land (\langle q \rangle \xrightarrow{r} \langle \omega \rangle) \right\} \neq 0,$$

that is, $\omega \in L(G')$.

The other part of the proof is similar. 

\begin{proof}
We first construct a fuzzy multiset regular grammar $G_1 = (V'_N, V_T, S, P_1)$ such that $G \sim G_1$. Next, we construct a fuzzy multiset grammar $G' = (V'_N, V_T, S, P')$ in normal form from $G_1$ such that $G' \sim G_1$. There are the two cases which are examined in what follows.

Case 1. Assume that $G = (V_N, V_T, S, P)$ is a fuzzy multiset right linear grammar. Then, any fuzzy multiset production $\langle \alpha \rangle \xrightarrow{r} \langle \omega \rangle \uplus \langle \beta \rangle$ or $\langle \alpha \rangle \xrightarrow{r} \langle \omega \rangle$ of $P$ with $|\omega| \leq 1$, is a fuzzy multiset production rule in $P_1$. The fuzzy multiset production rule $\langle \alpha \rangle \xrightarrow{r} \langle \omega \rangle \uplus \langle \beta \rangle$ is in $P$ with $|\omega| > 1$ and $\omega = a_1 \uplus a_2 \uplus \cdots \uplus a_{n-1} \uplus a_n$. We add in $P_1$ a set of fuzzy multiset production rules

$$
\langle \alpha \rangle \xrightarrow{r} \langle a_1 \rangle \uplus \langle Z_1 \rangle, \\
\langle Z_1 \rangle \xrightarrow{r} \langle a_2 \rangle \uplus \langle Z_2 \rangle, \\
\vdots \\
\langle Z_{n-1} \rangle \xrightarrow{r} \langle a_n \rangle \uplus \langle \beta \rangle
$$

where $Z_1, Z_2, \ldots, Z_{n-1}$ are new variables not in $V_N$. Similarly, for the fuzzy multiset production rules

$$
\langle \alpha \rangle \xrightarrow{r} \langle a_1 \rangle \uplus \langle a_2 \rangle \uplus \cdots \uplus \langle a_m \rangle,
$$

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\end{proof}
$m \geq 2$ and $r \in [0, 1]$, we add a set of fuzzy multiset production rules

\[
\langle \alpha \rangle \xrightarrow{r} \langle a_1 \rangle \cup \langle Y_1 \rangle,
\langle Z_1 \rangle \xrightarrow{r} \langle a_2 \rangle \cup \langle Y_2 \rangle,
\vdots
\langle Y_{m-1} \rangle \xrightarrow{r} \langle a_m \rangle \cup \langle Y_m \rangle,
\langle Y_m \rangle \xrightarrow{r} \langle 0_{VT} \rangle
\]

are in $P_1$, where $Y_1, Y_2, \ldots, Y_{m-1}, Y_m$ are new variables not in $V_N$.

Let $V'_N$ be the set of all such variables in $V_N$ that includes all the new variables introduced in the previous process. Then, $P_1$ of the fuzzy multiset grammar $G_1 = (V'_N, V_T, S, P_1)$ contains the following types of fuzzy multiset production rules:

1. $\langle \alpha \rangle \xrightarrow{r} \langle \omega \rangle \cup \langle \beta \rangle,$
2. $\langle \alpha \rangle \xrightarrow{r} \langle \beta \rangle,$
3. $\langle \alpha \rangle \xrightarrow{r} \langle 0_{V_T} \rangle, \alpha, \beta \in V'_N, 0_{V_T} \in V^\beta_T$ and $\omega \in V_T, r \in [0, 1].$

Next, we prove that $G \sim G_1$. Let $\omega \in L(G)$. Then, $\max\{S(q) \land (\langle q \rangle \rightarrow (\omega)) \neq 0\}$, that is, there exists $q_0 \in V_N$ such that $S(q_0) \neq 0$ and $(\langle q_0 \rangle \rightarrow (\omega)) \neq 0$. If $(q_0) \rightarrow (\omega)$ is a fuzzy multiset production rule in $P$ and $|\omega| = 1$, then clearly $(q_0) \rightarrow (\omega)$ in $P_1$. Now, if $|\omega| > 1$ and $\omega = a_1 \cup a_2 \cup \cdots \cup a_{n-1} \cup a_n$, $a_i \in V_T$, then there exist $q_1, q_2, \ldots, q_{n-1} \in V'_N$ such that $\langle q_0 \rangle \rightarrow (\omega_1) \cup (q_1) \rightarrow (a_1) \cup (a_2) \cup (q_2), \ldots, (q_{n-1} \rightarrow (a_1 \cup a_2 \cup \cdots \cup a_{n-1} \cup a_n) = \omega$, that is, $(q_0) \rightarrow (\omega)$ in $G_1$. Hence $\omega \in L(G_1)$. Again, consider $\omega \in L(G_1)$. Then,

\[
\max_{q \in V'_N} \left\{ S(q) \land (\langle q \rangle \rightarrow (\omega)) \neq 0 \right\},
\]

that is, there exists $q_0 \in V'_N$ such that $S(q_0) \neq 0$ and $(\langle q_0 \rangle \rightarrow (\omega)) \neq 0$ in $G_1$.

Now, $(\langle q_0 \rangle \rightarrow (\omega)) \neq 0$. Therefore, there exists a derivation chain of $\omega$ in $G_1$ as:

\[
\langle q_0 \rangle \xrightarrow{r_1} \langle a_1 \rangle \cup \langle a'_1 \rangle \\
\xrightarrow{r_2} \langle a_1 \rangle \cup \langle a_2 \rangle \cup \langle a'_2 \rangle \\
\xrightarrow{r_3} \ldots \xrightarrow{r_n} \langle a_1 \rangle \cup \langle a_2 \rangle \cup \cdots \cup \langle a_n \rangle \cup \langle a'_n \rangle \\
\xrightarrow{r_{n+1}} \langle \omega_1 \rangle \cup \langle q_1 \rangle \\
\xrightarrow{s_1} \langle \omega_1 \rangle \cup \langle b_1 \rangle \cup \langle \beta'_1 \rangle \\
\xrightarrow{s_2} \langle \omega_1 \rangle \cup \langle b_1 \rangle \cup \langle b_2 \rangle \cup \langle \beta'_2 \rangle \\
\xrightarrow{s_3} \ldots \xrightarrow{s_{m+1}} \langle \omega_1 \rangle \cup \langle b_1 \rangle \cup \langle b_2 \rangle \cup \cdots \cup \langle b_m \rangle \cup \langle \beta'_m \rangle \\
\xrightarrow{s_{m+1}} \langle \omega_1 \rangle \cup \langle \omega_2 \rangle \cup \langle q_2 \rangle
\]
\[
\begin{align*}
\text{Derivation in } P_1, \text{ or that there is a derivation chain of } \omega \text{ in } G \text{ as:} \\
\langle q_0 \rangle \overset{t_1}{\rightarrow} \langle \omega_1 \rangle \cup \langle q_1 \rangle \\
\langle \omega_1 \rangle \overset{s_1}{\rightarrow} \langle \omega_2 \rangle \cup \langle q_2 \rangle \\
\vdots \\
\langle \omega_{r+1} \rangle \overset{t_{r+1}}{\rightarrow} \langle \omega_{n-1} \rangle \cup \langle q_{n-1} \rangle \\
\langle \omega_{n-1} \rangle \cup \langle \omega_n \rangle \cup \langle q_n \rangle \\
\langle \omega_n \rangle \overset{u_n}{\rightarrow} \langle \omega_1 \rangle \cup \langle \omega_2 \rangle \cup \cdots \cup \langle q_{n-1} \rangle \\
\langle q_{n-1} \rangle \cup \langle \omega_n \rangle \cup \langle 0_{V_T} \rangle = \omega,
\end{align*}
\]

Thus the production rules are

\[
\begin{align*}
\langle q_0 \rangle & \overset{t_1}{\rightarrow} \langle a_1 \rangle \cup \langle \alpha'_1 \rangle \\
\langle \alpha'_1 \rangle & \overset{t_2}{\rightarrow} \langle a_2 \rangle \cup \langle \alpha'_2 \rangle \\
& \cdots \\
\langle \alpha'_{r-1} \rangle & \overset{t_r}{\rightarrow} \langle a_r \rangle \cup \langle \alpha'_r \rangle \\
\langle \alpha'_r \rangle & \overset{t_{r+1}}{\rightarrow} \langle a_{r+1} \rangle \cup \langle \alpha'_{r+1} \rangle \\
& \cdots \\
\langle \alpha'_{n-1} \rangle & \overset{t_n}{\rightarrow} \langle a_n \rangle \cup \langle \alpha'_{n} \rangle \\
\langle q_n \rangle & \overset{u_n}{\rightarrow} \langle \omega_n \rangle \cup \langle q_n \rangle \\
\langle q_n \rangle & \overset{u_{n+1}}{\rightarrow} \langle \{0_{V_T}\} \rangle
\end{align*}
\]

are in \( P_1 \), or that there exists \( q_0 \in V_N \) such that \( S(q_0) \neq 0 \) and \( \langle q_0 \rangle \overset{t_1}{\rightarrow} \langle \omega \rangle \neq 0 \) in \( G \). Thus \( \omega \in L(G) \), and hence \( G \sim G_1 \).

Case 2. We note that \( V'_N \) contains the variables in \( V_N \) as well as the new variables introduced in the process of finding \( G_1 \). Fuzzy multiset productions like this one

\[
\langle \alpha \rangle \overset{t}{\rightarrow} \langle \beta \rangle
\]
are called fuzzy multiset chain rules. Now, we describe an algorithm to eliminate all such fuzzy multiset chain rules.

Initially, we construct the set $\cup_{i}(\alpha) = \{\alpha\}$, for $\alpha \in V'_N$, and $\cup_{i+1} = \cup_{i}(\alpha) \cup \{\beta \mid \langle \beta \rangle \overset{r}{\to} \langle Z \rangle \in P_1$ for some $Z \in \cup_{i}(\alpha)$, $r \in [0, 1]\}$. Since $V'_N$ is finite, there exists an integer $K$ such that $\cup_{k+j}(\alpha) = \cup_k(\alpha)$, $j = 1, 2, \ldots$. Next, write $\cup(\alpha)$ instead of $\cup_{k}(\alpha)$ for all $\alpha \in V'_N$. Then, we construct the required fuzzy multiset grammar $G' = (V'_N, V_T, S, P')$, where

1. $\langle \alpha \rangle \overset{s,r}{\to} \langle \alpha \rangle \cup \langle \beta \rangle$, in $P'$ if and only if there is a $Z \in V'_N$ such that $\alpha \in \cup(Z)$ and $\langle Z \rangle \overset{s}{\to} \langle \alpha \rangle \cup \langle \beta \rangle$ in $P_1$,

2. $\langle \alpha \rangle \overset{s,r}{\to} \langle 0_{V_T} \rangle$ in $P'$ if and only if there is $Z \in V'_N$ such that $\alpha \in \cup(Z)$ and $\langle Z \rangle \overset{r}{\to} \langle \{0_{V_T}\} \rangle$ in $P_1$.

Clearly, the fuzzy multiset grammar $G' = (V'_N, V_T, S, P')$ is in the normal form. It is now obvious that $G_1 \sim G'$.

**Corollary 4.15.** Fuzzy multiset right linear grammar is equivalent to fuzzy multiset regular grammar except for $\{0_V\}$.

**Theorem 4.16.** The following are equivalent except for $\{0_V\}$.

1. Fuzzy multiset regular grammar.
2. Fuzzy multiset left linear grammar.
3. Fuzzy multiset right linear grammar.
4. Fuzzy multiset grammar in normal form.

5. Conclusions

In this paper, we have introduced and studied fuzzy multiset grammars. In addition, we proved that fuzzy multiset finite automata, and all other related concepts are equivalent, in the sense that all generate the same languages. We plan to further investigate the properties of these automata but we also hope that members of the scientific community will find useful applications of both fuzzy multiset finite-state automata and fuzzy multiset grammars.

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