Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 12, No. 5, (November 2016), pp. 617–639
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On fuzzy multiset regular grammars

B. K. Sharma, Apostolos Syropoulos, S. P. Tiwari

Received 10 February 2016; Revised 31 March 2016; Accepted 16 May 2016

Abstract. The main goal of this paper is to introduce and study fuzzy multiset regular grammar to show that fuzzy multiset finite automata and fuzzy multiset regular grammars are equivalent. We also study fuzzy multiset linear grammars and fuzzy multiset regular grammars in normal form. Furthermore, we show the equivalence of fuzzy multiset regular grammars, fuzzy multiset left linear grammars, fuzzy multiset right linear grammars, fuzzy multiset grammars in normal form.

2010 AMS Classification: 68Q45, 03D05

Keywords: Fuzzy multiset finite automaton, Fuzzy multiset grammar, Fuzzy multiset linear grammar, Fuzzy multiset regular grammar in normal form.

Corresponding Author: B. K. Sharma (bksharma1@gmail.com)

1. Introduction

Finite automata are conceptual machines capable of recognizing whether a string (i.e., a sequence of characters) belongs to some formal language or not. In addition, finite automata have found many applications that are discussed in the literature (e.g., see [6, 14, 15, 26]). Fuzzy finite automata were introduced and studied by Wee [23] and Santos [16] in the 1960s. These conceptual machines incorporated *vagueness* [9] as realized by fuzzy sets [27] in the machinery of finite automata. In general, fuzzy finite automata have found many applications that include pattern recognition, the theory of databases, and machine learning systems (cf., [8, 10, 12, 24]). In particular, Wee [24] proposed a pattern recognition and nonsupervised learning scheme in automatic control. Moreover, fuzzy automata have been proposed as a model of machine learning systems in design and computation.

A multiset (or bag) is a collection of elements in which elements may occurs more than once. A multiset is a natural generalization of the notion of a set and they have been described, among others, by Syropoulos [17]. Yager [25] was the first author who discussed fuzzy multisets. In real life, there are many situations where we deal with collections of objects in which repeated elements matter. From a practical point of view, multisets are very useful structure arising in many areas of mathematics and computer science. Following the advent of multiset theory, the concept of multiset finite automata has been introduced in [5, 7] and established their connection with membrane computing [4] whereas fuzzy multisets have been used in different models of fuzzy computation [18]. In recent years, multiset processing has appeared frequently in various fields of mathematics, computer science, biology and biochemistry (cf., [1, 2, 11, 13, 20]). Multiset "languages" are not exactly formal languages since tokens can occur anywhere in the "string." We use the term *multistring* to describe such generalized strings. A multistring can be best desrcribed by a "solution" that contains the characters that make it up. The various rules of the "grammar" are actually multiset processing rules. A multiset automaton is device that takes as input a solution and alters it. How the machine can pick elements from the solution is not specified but one can readily provide such a mechanism. A computation is successful if no rule can be applied any more. In this case, we can say that the multistring has been accepted. Otherwise, the multistring has not been accepted. In a sense, a multiset automaton functions just like the substitution operator of the Perl programming language. In particular, one can view each "production rule" as an instance of the `s/B/A/g` operator that *globally* substitutes `B` with `A`.

Following the trend of generalization, Wang, Yin and Gu [22] introduced fuzzy multiset finite automata as a generalization of multiset finite automata. Unfortunately, these authors did not make it clear that their automata process fuzzy "solutions" insted of fuzzy "strings". At this point it is important to understand what is a fuzzy solution? The obvious answer is that a fuzzy solution is an ordinary solution with fuzzy molecules (see the relevant discussion in [18] and also [19]). Tiwari, Gautam and Dubey [21] proved that a fuzzy multiset language is accepted by a fuzzy multiset finite automaton if and only if it is accepted by a deterministic fuzzy multiset finite automaton.

In what follows, we introduce and study fuzzy multiset regular automata and the corresponding fuzzy multiset regular languages. After presenting some preliminary ideas, we proceed with fuzzy multiset finite automata and show that fuzzy multiset finite automata are equivalent to fuzzy multiset regular grammars. In addition, we prove that fuzzy multiset regular grammars, fuzzy multiset left linear grammars, fuzzy multiset right linear grammars, and fuzzy multiset regular grammars in normal form are equivalent except for an empty string.

2. Preliminaries

In this section, we recollect some concepts and notations associated with multisets and multiset finite automata that are required in what follows. The notions related to multisets and which are used in this paper are fairly standard and can be found in the literature (cf., [5, 17, 21, 22]). Let us start with a formal definition of multisets.

Definition 2.1. If Σ is a finite set, then $\alpha : \Sigma \to \mathbb{N}$ characterizes a multiset over Σ, where $\mathbb{N}$ denotes the set of positive integers including 0. The cardinality of α, denoted by $|\alpha|$, is defined by

$$|\alpha| = \sum_{x\in\Sigma} \alpha(x).$$

Typically, simple multisets are specified using the standard set notation (e.g., $\{1, 1, 2, 2, 3, 4, 4\}$).

The set of all multisets over Σ is denoted by $\Sigma^{\uplus}$. The null multiset, $0_\Sigma \in \Sigma^{\uplus}$, is the multiset where all elements of Σ belong zero times to 0_Σ, that is, $0_\Sigma(x) = 0$ for all $x \in \Sigma$. Note that the null multiset is not the same as the empty multiset! Also, note that here $\Sigma^{\uplus}$ can be identified with $\mathbb{N}^{|\Sigma|}$.

For any two multisets $\alpha, \beta \in \Sigma^{\uplus}$, the inclusion operation, $\subseteq$, the addition operation, $\uplus$, and the difference operation, $\ominus$, are define as follows:

- $\alpha \subseteq \beta \Leftrightarrow \forall x \in \Sigma : \alpha(x) \le \beta(x)$,
- $\forall x \in \Sigma : (\alpha \uplus \beta)(x) = \alpha(x) + \beta(x)$,
- $\forall x \in \Sigma : (\alpha \ominus \beta)(x) = max(0, \alpha(x) - \beta(x))$.

It is not hard to see that $\Sigma^{\uplus}$ is a commutative monoid with operation $\uplus$ and identity element the multiset 0_Σ. Furthermore, $\alpha \subset \beta$ if and only if $\alpha \subseteq \beta$, $\alpha \neq \beta$, and

$$A \uplus B = \bigcup_{\substack{\alpha\in A\\ \beta\in B}} \alpha \uplus \beta,$$

for $A, B \subseteq \Sigma^{\uplus}$. We also use the notation $\langle y\rangle$ for a singleton multiset:

$$\langle y\rangle(x) = \begin{cases} 0, & \text{if } y = x \\ 1, & \text{if } y \neq x \end{cases}$$

for all $x \in \Sigma$. For any set A, let $\overline{A} = \{\langle a\rangle \mid a \in A\}$. Assume that $\Sigma = \{a, b, c\}$. Then, the multiset $\alpha = \{a, a, b, c, c\}$ is equivalent to $\langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle \uplus \langle c\rangle$. The definitions that follow have been introduced in [5].

Definition 2.2. A multiset finite automaton (MFA) is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where

- Q and Σ are nonempty finite sets called the state-set and input-set, respectively,
- $\delta : Q \times \Sigma^{\uplus} \to 2^Q$ is a map called transition map,
- $q_0 \in Q$ is called the initial state, and
- $\tau \subseteq Q$ is called the set of final states.

A configuration of a MFA M is a pair (p, α), where p and α denote current state and current multiset, respectively. The transition in a multiset finite automaton are described with the help of configurations. The transition from configuration (p, α) leads to configuration (q, β) if there exists a multiset $\gamma \in \Sigma^{\uplus}$ with $\gamma \subseteq \alpha$, $q \in \delta(p, \gamma)$ and $\beta = \alpha \ominus \gamma$, and is denoted by $(p, \alpha) \to (q, \beta)$. We shall denote by $\to *$, the reflexive and transitive closure of this operation.

Definition 2.3. For a given set Σ, a multiset language L is a subset of $\Sigma^{\uplus}$. In addition, a multiset language $L \subseteq \Sigma^{\uplus}$ is accepted by a MFA $M = (Q, \Sigma, \delta, q_0, F)$ if

$$L(M) = \Big\{\alpha \in \Sigma^{\uplus} : \delta(q_0, \alpha) \cap F \neq \phi\Big\},$$

for all $\alpha \in \Sigma^{\uplus}$.

Definition 2.4. A multiset grammar is a structure $G = (V_N, V_T, S, P)$, where V_N and V_T are finite sets of non-terminal and terminal symbols, with $V_N \cap V_T = \varnothing$, $S \in V_N$ is the starting symbol, and $P \subseteq (V^{\uplus} \uplus V_N) \times V^{\uplus}$ is a finite set of production rules, where $V = V_N \cup V_T$.

We have recollected the basic definitions and ideas of the theory of multisets as well as the notions a multiset automaton and of a multiset language. The next step is to generalize these ideas in the fuzzy theoretic framework.

3. Fuzzy multiset finite automata

In this section, we study the concepts of fuzzy multiset finite automata and fuzzy multiset languages. Let us start with the following definitions and results, which are borrowed from [21, 22].

Definition 3.1. A fuzzy multiset finite automaton ($FMFA$) is a quintuple $M = (Q, \Sigma, \delta, \sigma, \tau)$, where

- Q and Σ are nonempty finite sets called the state-set and input-set, respectively,
- $\delta : Q \times \Sigma^{\uplus} \times Q \to [0, 1]$ is a map called transition map,
- $\sigma : Q \to [0, 1]$ is a map called the fuzzy set of initial states, and
- $\tau : Q \to [0, 1]$ is a map called the fuzzy set of final states.

A configuration of a fuzzy multiset finite automaton M is a pair (p, α), where p and α denote the current state and the current multiset, respectively. Transitions in a fuzzy multiset finite automaton are described with the help of configurations. The transition from configuration (p, α) leads to configuration (q, β) with membership degree $k \in [0, 1]$ if there exists a multiset $\gamma \in \Sigma^{\uplus}$ with $\gamma \subseteq \alpha$, $\delta(p, \gamma, q) = k$ and $\beta = \alpha \ominus \gamma$. This transition is written as $(p, \alpha) \xrightarrow{k} (q, \beta)$.

The expression $\xrightarrow{k'} *$ denotes the reflexive and transitive closure of $\xrightarrow{k'}$. This means that for (p, α), $(q, \beta) \in Q \times \Sigma^{\uplus}$, $(p, \alpha) \xrightarrow{k'} *(q, \beta)$, if for some $n \geq 0$, there exist $(n + 1)$ states $q_0, \ldots, q_n$ and $(n + 1)$ multisets $\alpha_0, \ldots, \alpha_n$ such that $q_0 = p$, $q_n = q$, $\alpha_0 = \alpha$, $\alpha_n = \beta$, and $(q_i, \alpha_i) \xrightarrow{k_i} (q_{i+1},\ \alpha_{i+1})$, for all $i = 0, n - 1$, where $k' = k_0 \wedge k_1 \wedge \cdots \wedge k_{n-1}$. Next, we define

$$\begin{aligned} \mu_M\big((p, \alpha) \to *(q, \beta)\big) \quad = \quad & \bigvee \Big\{ \mu_M((p, \alpha) \to *(r, \alpha \ominus \gamma)) \wedge \mu_M((r, \alpha \ominus \gamma) \\ & \to *(q, \beta)) : r \in Q, \gamma \in \Sigma^{\uplus} \text{ and } \gamma \subseteq \alpha \Big\}, \end{aligned}$$

and

$$\mu_M\big((p, \alpha) \to *(q, \alpha)\big) = \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{if } p \neq q. \end{cases}$$

Example 3.2. Let $M = (Q, \Sigma, \delta, \sigma, \tau)$ be a $FMFA$, where $Q = \{q_1, q_2, q_3, q_4\}$, $\Sigma = \{a, b, c\}$. Also, assume that the fuzzy transition function is defined as follows:

$$\begin{aligned}
\delta(q_1, \langle b\rangle, q_3) &= 0.5\\
\delta(q_3, \langle b\rangle \uplus \langle c\rangle, q_4) &= 0.3\\
\delta(q_4, \langle a\rangle, q_2) &= 0.7\\
\delta(q_1, \langle b\rangle \uplus \langle c\rangle, q_3) &= 0.3\\
\delta(q_3, \langle b\rangle, q_4) &= 0.6\\
\delta(q_2, \langle a\rangle, q_4) &= 0.2.
\end{aligned}$$

If $\alpha = \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle b\rangle \uplus \langle c\rangle$ and $\beta = 0_\Sigma$, then, the transition steps, $\big((q_1, \alpha) \to *(q_2, \beta)\big)$ are "expanded" as follows.

(1) $(q_1, \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle b\rangle \uplus \langle c\rangle) \xrightarrow{0.5} (q_3, \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle) \xrightarrow{0.3} (q_4, \langle a\rangle \uplus \langle a\rangle) \xrightarrow{0.7} (q_2, \langle a\rangle) \xrightarrow{0.2} (q_4, 0_\Sigma)$,

(2) $(q_1, \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle b\rangle \uplus \langle c\rangle) \xrightarrow{0.3} (q_3, \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle) \xrightarrow{0.7} (q_2, \langle a\rangle \uplus \langle b\rangle) \xrightarrow{0.2} (q_4, \langle b\rangle) \xrightarrow{0.6} (q_3, 0_\Sigma)$.

Thus

$$\begin{aligned}
\mu_M((q_1, \alpha) \to *(q_2, \beta)) &= \bigvee\Big\{0.5 \wedge 0.3 \wedge 0.7 \wedge 0.2, 0.3 \wedge 0.7 \wedge 0.2 \wedge 0.6\Big\}\\
&= \bigvee\{0.2, 0.2\} = 0.2.
\end{aligned}$$

Definition 3.3. For a given set Σ, a fuzzy multiset language is a map $L : \Sigma^{\uplus} \to [0, 1]$. Let $M = (Q, \Sigma, \delta, \sigma, \tau)$ be a fuzzy multiset finite automaton. Then, the set

$$L(M) = \Big\{\alpha \in \Sigma^{\uplus} \;\Big|\; \alpha \text{ is accepted by } M\Big\}$$

is called the fuzzy multiset language of M. A fuzzy multiset language $L : \Sigma^{\uplus} \to [0, 1]$ is accepted by a FMFA $M = (Q, \Sigma, \delta, \sigma, \tau)$ if

$$L(\alpha) = \bigvee\Big\{\sigma(q) \wedge \mu_M((q, \alpha) \to *(p, 0_\Sigma)) \wedge \tau(p) : q, p \in Q\Big\},$$

for all $\alpha \in \Sigma^{\uplus}$. Furthermore, a fuzzy multiset language $L : \Sigma^{\uplus} \to [0, 1]$ is called regular if there exists a FMFA M such that $L = L(M)$.

We shall denote by $L(M)$ a fuzzy multiset language L, if L is accepted by a FMFA M.

Example 3.4. Assume that $M = (Q, \Sigma, \delta, \sigma, \tau)$ is a FMFA. Also assume that $Q = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8\}$, $\Sigma = \{a, b, c\}$, $\sigma(q_1) = 0.2$, $\tau(q_6) = 0.3$, $\tau(q_8) = 0.6$, and that δ is defined as follows:

$$\begin{aligned} \delta(q_1,\langle c\rangle,q_2)&=0.2, & \delta(q_4,\langle 0_\Sigma\rangle,q_7)&=0.2,\\ \delta(q_1,\langle a\rangle,q_3)&=0.3, & \delta(q_5,\langle a\rangle,q_6)&=0.3,\\ \delta(q_2,\langle a\rangle,q_3)&=0.2, & \delta(q_6,\langle a\rangle,q_6)&=0.4,\\ \delta(q_2,\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle,q_4)&=0.9, & \delta(q_6,\langle 0_\Sigma\rangle,q_8)&=0.4,\\ \delta(q_3,\langle b\rangle\uplus\langle c\rangle,q_4)&=0.6, & \delta(q_7,\langle c\rangle,q_6)&=0.8,\\ \delta(q_3,\langle a\rangle\uplus\langle b\rangle,q_7)&=0.5, & \delta(q_7,\langle a\rangle,q_8)&=0.2,\\ \delta(q_4,\langle c\rangle,q_5)&=0.3, & \delta(q_8,\langle c\rangle,q_8)&=0.7. \end{aligned}$$

Then, if $\alpha=(2,1,2)$ can be written as $\langle a\rangle\uplus\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle\uplus\langle c\rangle$, the transition steps of α in M are as follows:

$$\begin{aligned} (1)\quad (q_1,\langle a\rangle\uplus\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle\uplus\langle c\rangle) &\xrightarrow{0.2} (q_2,\langle a\rangle\uplus\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle)\\ &\xrightarrow{0.2} (q_3,\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle)\xrightarrow{0.6}(q_4,\langle a\rangle)\\ &\xrightarrow{0.2} (q_7,\langle a\rangle)\xrightarrow{0.2}(q_8,\langle 0_\Sigma\rangle). \end{aligned}$$

$$\begin{aligned} (2)\quad (q_1,\langle a\rangle\uplus\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle\uplus\langle c\rangle) &\xrightarrow{0.2} (q_2,\langle a\rangle\uplus\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle)\\ &\xrightarrow{0.2} (q_3,\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle)\\ &\xrightarrow{0.5} (q_7,\langle c\rangle)\\ &\xrightarrow{0.8} (q_6,\langle 0_\Sigma\rangle). \end{aligned}$$

$$\begin{aligned} (3)\quad (q_1,\langle a\rangle\uplus\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle\uplus\langle c\rangle) &\xrightarrow{0.2} (q_2,\langle a\rangle\uplus\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle)\\ &\xrightarrow{0.9} (q_4,\langle a\rangle)\\ &\xrightarrow{0.2} (q_7,\langle a\rangle)\\ &\xrightarrow{0.2} (q_8,\langle 0_\Sigma\rangle). \end{aligned}$$

$$\begin{aligned} (4)\quad (q_1,\langle a\rangle\uplus\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle\uplus\langle c\rangle) &\xrightarrow{0.3} (q_3,\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle\uplus\langle c\rangle)\\ &\xrightarrow{0.6} (q_4,\langle a\rangle\uplus\langle c\rangle)\\ &\xrightarrow{0.3} (q_5,\langle a\rangle)\\ &\xrightarrow{0.3} (q_6,\langle 0_\Sigma\rangle). \end{aligned}$$

$$\begin{aligned} (5)\quad (q_1,\langle a\rangle\uplus\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle\uplus\langle c\rangle) &\xrightarrow{0.3} (q_3,\langle a\rangle\uplus\langle b\rangle\uplus\langle c\rangle\uplus\langle c\rangle)\\ &\xrightarrow{0.6} (q_4,\langle a\rangle\uplus\langle c\rangle)\\ &\xrightarrow{0.2} (q_7,\langle a\rangle\uplus\langle c\rangle)\\ &\xrightarrow{0.2} (q_8,\langle c\rangle)\\ &\xrightarrow{0.7} (q_8,\langle 0_\Sigma\rangle). \end{aligned}$$

$$\begin{aligned}
(6)\quad (q_1, \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle \uplus \langle c\rangle) &\xrightarrow{0.3} (q_3, \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle \uplus \langle c\rangle)\\
&\xrightarrow{0.6} (q_4, \langle a\rangle \uplus \langle c\rangle)\\
&\xrightarrow{0.2} (q_7, \langle a\rangle \uplus \langle c\rangle)\\
&\xrightarrow{0.8} (q_6, \langle a\rangle)\\
&\xrightarrow{0.4} (q_6, \langle 0_\Sigma\rangle).
\end{aligned}$$

$$\begin{aligned}
(7)\quad (q_1, \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle \uplus \langle c\rangle) &\xrightarrow{0.3} (q_3, \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle \uplus \langle c\rangle)\\
&\xrightarrow{0.5} (q_7, \langle c\rangle \uplus \langle c\rangle)\\
&\xrightarrow{0.8} (q_6, \langle c\rangle)\\
&\xrightarrow{0.4} (q_8, \langle c\rangle)\\
&\xrightarrow{0.7} (q_8, \langle 0_\Sigma\rangle).
\end{aligned}$$

Thus

$$\begin{aligned}
\mu_M(\alpha) &= \bigvee\Big\{\sigma(q_1)\wedge 0.2\wedge 0.2\wedge 0.6\wedge 0.2\wedge 0.2\wedge \tau(q_8),\\
&\qquad \sigma(q_1)\wedge 0.2\wedge 0.2\wedge 0.5\wedge 0.8\wedge \tau(q_6),\\
&\qquad \sigma(q_1)\wedge 0.2\wedge 0.9\wedge 0.2\wedge 0.2\wedge \tau(q_8),\\
&\qquad \sigma(q_1)\wedge 0.3\wedge 0.6\wedge 0.3\wedge 0.3\wedge \tau(q_6),\\
&\qquad \sigma(q_1)\wedge 0.3\wedge 0.6\wedge 0.2\wedge 0.2\wedge 0.7\wedge \tau(q_8),\\
&\qquad \sigma(q_1)\wedge 0.3\wedge 0.6\wedge 0.2\wedge 0.8\wedge 0.4\wedge \tau(q_6),\\
&\qquad \sigma(q_1)\wedge 0.3\wedge 0.5\wedge 0.8\wedge 0.4\wedge 0.7\wedge \tau(q_8)\Big\}\\
&= \bigvee\Big\{0.2\wedge 0.2\wedge 0.2\wedge 0.6\wedge 0.2\wedge 0.2\wedge 0.6,\\
&\qquad 0.2\wedge 0.2\wedge 0.2\wedge 0.5\wedge 0.8\wedge 0.3,\\
&\qquad 0.2\wedge 0.2\wedge 0.9\wedge 0.2\wedge 0.2\wedge 0.6,\\
&\qquad 0.2\wedge 0.3\wedge 0.6\wedge 0.3\wedge 0.3\wedge 0.3,\\
&\qquad 0.2\wedge 0.3\wedge 0.6\wedge 0.2\wedge 0.2\wedge 0.7\wedge 0.6,\\
&\qquad 0.2\wedge 0.3\wedge 0.6\wedge 0.2\wedge 0.8\wedge 0.4\wedge 0.3,\\
&\qquad 0.2\wedge 0.3\wedge 0.5\wedge 0.8\wedge 0.4\wedge 0.7\wedge 0.6\Big\}\\
&= \bigvee\Big\{0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2\Big\}\\
&= 0.2.
\end{aligned}$$

Definition 3.5. A language $L \subseteq \Sigma^{\uplus}$ is called a fuzzy multiset automaton language if there exists a fuzzy multiset automaton M such that $L(M) = L$.

Theorem 3.6. *If L_1 and L_2 are fuzzy multiset automata languages over Σ, then $L_1 \cap L_2$, $L_1 \uplus L_2$ and $L_1^{\uplus}$ are fuzzy multiset automata languages over Σ.*

Proof. The proof is identical to the one given by Wang, Yin and Gu [22] but we include it for reasons of completness. Assume that L_1 and L_2 are two fuzzy multiset finite automata languages accepted by the FMFAs $M_1 = (Q_1, \Sigma, \delta_1, \sigma_1, \tau_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, \sigma_2, \tau_2)$, respectively. Let $Q_1 \cap Q_2 = \varnothing$.

(1) First of all, we construct a FMFA $M = (Q, \Sigma, \delta, \sigma, \tau)$ such that $Q = Q_1 \cup Q_2$ as follows:

$$\sigma(q) = \begin{cases} \sigma_1(q), & \text{if } q \in Q_1, \\ \sigma_2(q), & \text{if } q \in Q_2 \end{cases} \quad \text{and} \quad \tau(q) = \begin{cases} \tau_1(q), & \text{if } q \in Q_1, \\ \tau_2(q), & \text{if } q \in Q_2 \end{cases}$$

for all $x \in \Sigma^{\uplus}$ and

$$\delta^*(q_1, x, p_1) = \begin{cases} \delta_1(q_1, x, p_1), & \text{if } q_1, p_1 \in Q_1 \\ \delta_2(q_1, x, p_1), & \text{if } q_1, p_1 \in Q_2 \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, for all $\omega \in \Sigma^{\uplus}$, we have

$$\begin{aligned} \mu_M(\omega) &= \max_{q,p \in Q} \{\sigma(q) \wedge \mu_M((q,\omega) \xrightarrow{*} (p, 0_\Sigma)) \wedge \tau(p)\} \\ &= \max_{q,p \in Q} \{\sigma_1(q) \wedge \mu_{M_1}((q,\omega) \xrightarrow{*} (p, 0_\Sigma)) \wedge \tau_1(p)\} \vee \\ &\quad \max_{q,p \in Q} \{\sigma_2(q) \wedge \mu_{M_1}((q,\omega) \xrightarrow{*} (p, 0_\Sigma)) \wedge \tau_2(p)\} \\ &= \mu_{M_1}(\omega) \vee \mu_{M_2}(\omega). \end{aligned}$$

Hence $L(M) = L(M_1) \cup L(M_2) = L_1 \cup L_2$ is a FMFA language.

(2) Now we construct a FMFA $M = (Q, \Sigma, \delta, \sigma, \tau)$ such that $Q = Q_1 \cap Q_2$

$$\sigma(q) = \begin{cases} \sigma_1(q), & \text{if } q \in Q_1, \\ 0, & \text{if } q \in Q_2 \end{cases} \quad \text{and} \quad \tau(q) = \begin{cases} 0, & \text{if } q \in Q_1, \\ \tau_2(q), & \text{if } q \in Q_2 \end{cases}$$

Also, for all $x \in \Sigma^{\uplus} \setminus 0_\Sigma$:

$$\delta^*(q_1, x, p_1) = \begin{cases} \delta_1(q_1, x, p_1), & \text{if } q_1, p_1 \in Q_1 \\ \delta_2(q_1, x, p_1), & \text{if } q_1, p_1 \in Q_2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\delta^*(q_1, 0_\Sigma, p_1) = \begin{cases} \delta_1(q_1, 0_\Sigma, p_1), & \text{if } q_1, p_1 \in Q_1 \\ \delta_2(q_1, 0_\Sigma, p_1), & \text{if } q_1, p_1 \in Q_2 \\ \tau_1(q_1) \wedge \tau_2(p_1), & \text{if } q_1 \in Q_1, p_1 \in Q_2 \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, for all $\omega \in \Sigma^{\uplus}$, we have

$$\begin{aligned}
\mu_M(\omega) &= \max_{q,p\in Q}\{\sigma(q) \wedge \mu_M((q,\omega) \xrightarrow{*} (p,0_\Sigma)) \wedge \tau(p)\} \\
&= \max_{\substack{\omega_1 \uplus \omega_2 = \omega \\ q,q_1,p_1,p\in Q}} \{\{\sigma(p) \wedge \mu_M((q,\omega) \xrightarrow{*} (q_1,\omega_1)) \\
&\qquad \wedge \mu_M((q_1,\omega_1) \to (p_1,\omega_2)) \wedge \mu_M((p_1,\omega_2) \xrightarrow{*} (p,0_\Sigma)) \wedge \tau(p)\} \\
&= \max_{\substack{\omega_1 \uplus \omega_2 = \omega \\ q,q_1\in Q_1,p_1 \\ p\in Q_2}} \{\{\sigma_1(p) \wedge \mu_{M_1}((q,\omega) \xrightarrow{*} (q_1,\omega_1)) \wedge \tau_1(q_1) \\
&\qquad \wedge \sigma_2(p_1) \wedge \mu_{M_2}((p_1,\omega_2) \xrightarrow{*} (p,0_\Sigma)) \wedge \tau(p)\} \\
&= \max_{\omega_1 \uplus \omega_2 = \omega}\{\max_{q,q_1\in Q_1}\{\sigma_1(q) \wedge \mu_{M_1}((q,\omega_1) \xrightarrow{*} (q_1,0_\Sigma)) \wedge \tau_1(q_1)\} \\
&\qquad \wedge \max_{p_1,p\in Q_2}\{\sigma_2(p_1) \wedge \mu_{M_2}((p_1,\omega_2) \xrightarrow{*} (p,0_\Sigma)) \wedge \tau_2(p)\}\} \\
&= \max_{\omega_1 \uplus \omega_2}\{\mu_{M_1}(\omega_1) \wedge \mu_{M_2}(\omega_2)\}.
\end{aligned}$$

Hence, $L(M) = L(M_1) \uplus L(M_2) = L_1 \uplus L_2$ is a FMFA language.

(3) We construct a FMFA $M = (Q,\Sigma,\delta,\sigma,\tau)$ such that $Q = Q_1 \cup \{q_0\}$ for all $x \in \Sigma^{\uplus} \setminus 0_\Sigma$:

$$\sigma(q) = \begin{cases} \sigma_1(q), & \text{if } q \in Q_1, \\ 1, & \text{if } q \in q_0 \end{cases} \quad \text{and} \quad \tau(q) = \begin{cases} \tau_1(q), & \text{if } q \in Q_1, \\ 1, & \text{if } q \in q_0 \end{cases}$$

Also, for all $\omega \in \Sigma^{\uplus}$:

$$\delta^*(q_1,x,p_1) = \begin{cases} \delta_1(q_1,x,p_1), & \text{if } q_1,p_1 \in Q_1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\delta^*(q_1,0_\Sigma,p_1) = \begin{cases} \delta_1(q_1,0_\Sigma,p_1), & \text{if } q_1,p_1 \in Q_1, \\ \tau_1(q_1) \wedge \tau_2(p_1), & \text{if } q_1,p_1 \in Q_1, \\ 0, & \text{otherwise.} \end{cases}$$

Now,

(a) if $\omega = 0_\Sigma$, then

$$\begin{aligned}
\mu_M(0_\Sigma) &= \max_{q,p\in Q}\{\sigma(q) \wedge \mu_M((q,0_\Sigma) \xrightarrow{*} (p,0_\Sigma)) \wedge \tau(p)\} \\
&= \sigma(q_0) \wedge \mu_M((q_0,0_\Sigma) \xrightarrow{*} (q_0,0_\Sigma)) \wedge \tau(q_0)\} \\
&= 1 \\
&= \mu_{L_1^{\uplus}}(0_\Sigma);
\end{aligned}$$

(b) if $\omega \neq 0_\Sigma$, then

$$\begin{aligned}
\mu_M(\omega) &= \max_{q,p\in Q}\{\sigma(q) \wedge \mu_M((q,\omega) \xrightarrow{*} (p,0_\Sigma)) \wedge \tau(p)\} \\
&= \max_{m\geq 1} \max_{\omega_1\uplus\omega_2\uplus\cdots\uplus\omega_m=\omega} \max_{p,q_1,q_2,\ldots,q_{2(m-1)},p_1\in Q}\{\sigma(q) \wedge \\
&\quad \mu_M((q,\omega_1) \xrightarrow{*} (q_1,\omega_2 \uplus\cdots\uplus \omega_m)) \wedge \cdots \wedge \\
&\quad \mu_M((q_{2(m-1)},\omega_m) \xrightarrow{*} (p,0_\Sigma)) \wedge \tau(p)\} \\
&= \max_{m\geq 1} \max_{\omega_1\uplus\omega_2\uplus\cdots\uplus\omega_m=\omega} \max_{p,q_1,q_2,\ldots,q_{2(m-1)},p_1\in Q}\{\sigma_1(q) \wedge \\
&\quad \mu_{M_1}((q,\omega_1) \xrightarrow{*} (q_1,\omega_2 \uplus\cdots\uplus \omega_m)) \wedge \tau_1(q_1) \wedge \sigma_1(q_2) \\
&\quad \wedge\cdots\wedge \mu_{M_1}((q_{2(m-1)},\omega_m) \xrightarrow{*} (p,0_\Sigma)) \wedge \tau_1(p)\} \\
&= \max_{m\geq 1} \max_{\omega_1\uplus\omega_2\uplus\cdots\uplus\omega_m=\omega}\{ \max_{p,q_1\in Q_1}\{\sigma_1(q) \wedge \mu_{M_1}((q,\omega_1) \\
&\quad \xrightarrow{*} (q_1,0_\Sigma)) \wedge \tau_1(q_1)\} \wedge \max_{q_2,q_3\in Q_1}\{\sigma_1(q_2) \wedge \mu_{M_1}((q_2,\omega_2) \\
&\quad \xrightarrow{*} (q_3,0_\Sigma)) \wedge \tau_1(q_3)\} \wedge\cdots\wedge \max_{q_{2(m-1)},q_m\in Q_1}\{\sigma_1(q_{2(m-1)}) \\
&\quad \wedge\mu_{M_1}((q_{2(m-1)},\omega_m) \xrightarrow{*} (p,0_\Sigma)) \wedge \tau_1(p)\}\} \\
&= \max_{m\geq 1} \max_{\omega_1\uplus\omega_2\uplus\cdots\uplus\omega_m=\omega}\{\mu_{M_1}(\omega_1) \wedge \mu_{M_1}(\omega_2) \wedge\cdots\wedge \\
&\quad \mu_{M_1}(\omega_m)\} \\
&= \mu_{L_1^{\uplus}}(\omega).
\end{aligned}$$

Hence, $L(M) = L_1^{\uplus}$ is a FMFA language.

□

Example 3.7. Assume that $M_1 = (Q_1, \Sigma, \delta_1, \sigma_1, \tau_1)$ is a FMFA, where $Q_1 = \{q_1, q_2, q_3\}$, $\Sigma = \{a, b, c\}$, $\sigma_1 = \{(q_1, 0.5)\}$, $\tau_1 = \big\{(q_2, 0.4), (q_3, 0.6)\big\}$, and δ_1 is defined as follows:

$$\begin{aligned}
\delta_1(q_1, \langle a\rangle \uplus \langle b\rangle, q_2) &= 0.4, \\
\delta_1(q_1, \langle c\rangle, q_3) &= 0.5,
\end{aligned}$$

and

$$\delta_1(q_2, \langle a\rangle \uplus \langle c\rangle, q_3) = 0.3.$$

Then, we define the language

$$L(M_1) = \big\{(\langle a\rangle \uplus \langle b\rangle, 0.4), (\langle a\rangle \uplus \langle b\rangle \uplus \langle a\rangle \uplus \langle c\rangle, 0.3), (\langle c\rangle, 0.5)\big\}.$$

Assume that $M_2 = (Q_2, \Sigma, \delta_2, \sigma_2, \tau_2)$ is a FMFA, where $Q_2 = \{p_1, p_2, p_3\}$, $\Sigma = \{a, b, c\}$, $\sigma_2 = \big\{(p_1, 0.6), (p_2, 0.4)\big\}$, $\tau_2 = \big\{(p_2, 0.4), (p_3, 0.5)\big\}$, and δ_2 is defined as follows:

$$\begin{aligned}
\delta_2(p_1, \langle a\rangle \uplus \langle c\rangle, p_2) &= 0.6, \\
\delta_2(p_1, \langle c\rangle, p_3) &= 1.0,
\end{aligned}$$

and

$$\delta_2(p_2, \langle a\rangle \uplus \langle b\rangle, p_3) = 0.4.$$

Then, we find the language

$$L(M_2) = \big\{(0_\Sigma, 0.4), (\langle a\rangle \uplus \langle c\rangle, 0.4), (\langle c\rangle, 0.5), (\langle a\rangle \uplus \langle c\rangle \uplus \langle a\rangle \uplus \langle b\rangle, 0.4), (\langle a\rangle \uplus \langle b\rangle, 0.4)\big\}.$$

From the proof of the above theorem, we have

(1) $M = (Q_1 \cup Q_2, \Sigma, \delta, \sigma, \tau)$, where

$$\sigma = \big\{(q_1, 0.5), (p_1, 0.6), (p_2, 04)\big\},$$
$$\tau = \big\{(q_2, 0.4), (q_3, 0.6), (p_2, 0.4), (p_3, 0.5)\big\},$$

and the δ's are defined as follows:

$$\begin{aligned} \delta_1(q_1, \langle a\rangle \uplus \langle b\rangle, q_2) &= 0.4,\\ \delta_1(q_1, \langle c\rangle, q_3) &= 0.5,\\ \delta_1(q_2, \langle a\rangle \uplus \langle c\rangle, q_3) &= 0.3, \end{aligned}$$

and

$$\begin{aligned} \delta_2(p_1, \langle a\rangle \uplus \langle c\rangle, p_2) &= 0.6,\\ \delta_2(p_1, \langle c\rangle, p_3) &= 1.0,\\ \delta_2(p_2, \langle a\rangle \uplus \langle b\rangle, p_3) &= 0.4. \end{aligned}$$

Now,

$$\begin{aligned} L(M) &= L(M_1) \cup L(M_2)\\ &= \big\{(\langle a\rangle \uplus \langle b\rangle, 0.4), (\langle a\rangle \uplus \langle b\rangle \uplus \langle a\rangle \uplus \langle c\rangle, 0.3), (\langle c\rangle, 0.5),\\ &\quad (0_\Sigma, 0.4), (\langle a\rangle \uplus \langle c\rangle, 0.4), (\langle c\rangle, 0.5),\\ &\quad (\langle a\rangle \uplus \langle c\rangle \uplus \langle a\rangle \uplus \langle b\rangle, 0.4), (\langle a\rangle \uplus \langle b\rangle, 0.4)\big\}. \end{aligned}$$

(2) $M' = (Q_1 \cup Q_2, \Sigma, \delta', \sigma', \tau')$, where $\sigma' = \big\{(q_1, 0.5)\big\}$, $\tau' = \big\{(p_2, 0.4), (p_3, 0.5)\big\}$, and δ' is defined as follows:

$$\begin{aligned} \delta'(q_2, 0_\Sigma, p_2) &= 0.6,\\ \delta'(q_2, 0_\Sigma, p_1) &= 0.5,\\ \delta'(q_3, 0_\Sigma, p_2) &= 0.7,\\ \delta'(q_3, 0_\Sigma, p_1) &= 0.8,\\ \delta'(q_1, \langle a\rangle \uplus \langle b\rangle, q_2) &= 0.4,\\ \delta'(q_1, \langle c\rangle, q_3) &= 0.5,\\ \delta'(q_2, \langle a\rangle \uplus \langle c\rangle, q_3) &= 0.3,\\ \delta'(p_1, \langle a\rangle \uplus \langle c\rangle, p_2) &= 0.6,\\ \delta'(p_1, \langle c\rangle, p_3) &= 1.0, \end{aligned}$$

and

$$\delta'(p_2, \langle a\rangle \uplus \langle b\rangle, p_3) = 0.4.$$

Then,

$$\begin{aligned} L(M') &= L(M_1) \uplus L(M_2) \\ &= \big\{(\langle a\rangle \uplus \langle b\rangle, 0.4), (\langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle b\rangle, 0.4), \\ &\quad (\langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle, 0.4), (\langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle, 0.4), \\ &\quad (\langle a\rangle \uplus \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle b\rangle \uplus \langle c\rangle, 0.4), (\langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle, 0.3), \\ &\quad (\langle a\rangle \uplus \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle b\rangle \uplus \langle c\rangle, 0.3), (\langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle \uplus \langle c\rangle, 0.3), \\ &\quad (\langle a\rangle \uplus \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle \uplus \langle c\rangle, 0.3), \\ &\quad (\langle a\rangle \uplus \langle a\rangle \uplus \langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle b\rangle \uplus \langle c\rangle \uplus \langle c\rangle, 0.3), \\ &\quad (\langle c\rangle, 0.4), (\langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle, 0.4), \\ &\quad (\langle c\rangle \uplus \langle c\rangle, 0.5), (\langle a\rangle \uplus \langle c\rangle \uplus \langle c\rangle, 0.4), \\ &\quad (\langle a\rangle \uplus \langle a\rangle \uplus \langle b\rangle \uplus \langle c\rangle \uplus \langle c\rangle, 0.4)\big\}. \end{aligned}$$

(3) $M'' = (Q_1 \cup \{q_0\}, \Sigma, \delta'', \sigma'', \tau'')$ is a FMFA, where $\sigma'' = \{(q_1, 0.5), (q_0, 1)\}$, $\tau'' = \{(q_2, 0.4), (q_3, 0.6), (q_0, 1)\}$ and δ'' is defined as follows:

$$\begin{aligned} \delta''(q_2, 0_\Sigma, q_1) &= 0.8, \\ \delta''(q_3, 0_\Sigma, q_1) &= 0.9, \\ \delta''(q_1, \langle a\rangle \uplus \langle b\rangle, q_2) &= 0.4, \\ \delta''(q_1, \langle c\rangle, q_3) &= 0.5, \end{aligned}$$

and

$$\delta''(q_2, \langle a\rangle \uplus \langle c\rangle, q_3) = 0.3.$$

Then, we get the following language

$$\begin{aligned} L(M'') &= L(M_1)^{\uplus} \\ &= \big\{(0_\Sigma, 1), (\langle a\rangle \uplus \langle b\rangle, 0.4), \\ &\quad (\langle a\rangle \uplus \langle b\rangle \uplus \langle b\rangle \uplus \langle c\rangle, 0.3), (\langle c\rangle, 0.5), \dots\big\}. \end{aligned}$$

Definition 3.8. An onto function $f : \Sigma \to \Sigma_1^{\uplus}$ is called a homomorphism if for all $x, y \in \Sigma$, $f(x \uplus y) = f(x) \uplus f(y)$. This homomorphism can be naturally extended to $f : \Sigma^{\uplus} \to \Sigma_1^{\uplus}$, where $f(0_\Sigma) = 0_\Sigma$, $f(a \uplus \omega) = f(a) \uplus f(\omega)$ for $a \in \Sigma$ and $\omega \in \Sigma^{\uplus}$.

The following theorem can be proved by induction on the length of b ($|b|$).

Theorem 3.9. *If $f : \Sigma \to \Sigma_1^{\uplus}$ is a homomorphism, then for all $a, b, \in \Sigma^{\uplus}$, $f(a \uplus b) = f(a) \uplus f(b)$.*

Theorem 3.10. *The class of all fuzzy multiset automaton languages is closed under homomorphism and inverse homomorphism.*

Proof. Let $f : \Sigma \to \Sigma_1^{\uplus}$ be a homomorphism and $L \subseteq \Sigma^{\uplus}$ be a fuzzy multiset finite automaton language. Also, let $M = (Q, \Sigma, \delta, \sigma, \tau)$ be a fuzzy multiset finite automaton such that $L(M) = L$. Then, we construct a fuzzy multiset finite automaton $M' = (Q, \Sigma_1, \delta', \sigma, \tau)$, where $\delta' : Q \times \Sigma_1 \times Q \to [0, 1]$ is defined by $\delta'(p, \alpha, q) = r$ if and only if there is an $\omega \in \Sigma^{\uplus}$ such that $f(\omega) = \alpha$ and $\delta'(p, \alpha, q) = r$, where $r \in [0, 1]$. Assume that $\alpha \in L(M')$. Then, $\{\sigma(p) \wedge \mu'_M((p, \alpha) \to (q, 0_{\Sigma_1})) \wedge \tau(q)\} \neq 0$, for some $p, q \in Q$. Also, $\mu'_M((p, \alpha) \to (q, 0_{\Sigma_1})) \neq 0$. Thus there is an $\omega \in \Sigma^{\uplus}$ such that $f(\omega) = \alpha$ and then $\{\sigma(p) \wedge \mu'_M((p, \omega) \to (q, 0_{\Sigma})) \wedge \tau(q)\} \neq 0$. Hence, $\omega \in L(M) = L$. Therefore, $\alpha \in f(L)$.

The converse is similar. Assume that $L \in \Sigma_1^{\uplus}$ is a fuzzy multiset finite automaton language and $M = (Q, \Sigma_1, \delta, \sigma, \tau)$ is a fuzzy multiset finite automaton such that $L(M) = L$. Then we construct a fuzzy multiset automaton $M' = (Q, \Sigma, \delta', \sigma, \tau)$, where $\delta' = Q \times \Sigma^{\uplus} \times Q \to [0, 1]$ is defined by $\mu_{M'}((p, \omega) \to^* (q, 0_{\Sigma})) = \mu_M((p, f(\omega)) \to^* (q, 0_{\Sigma_1}))$, for all $p, q \in Q$, $\omega \in \Sigma^{\uplus}$. Then,

$$\begin{aligned}
\omega \in L(M') &\Leftrightarrow \{\sigma(p) \wedge \mu_{M'}((p, \omega) \to (q, 0_{\Sigma})) \wedge \tau(q)\} \neq 0 \\
&\Leftrightarrow \{\sigma(p) \wedge \mu_M((p, f(\omega)) \to (q, 0_{\Sigma_1})) \wedge \tau(q)\} \neq 0 \\
&\Leftrightarrow f(\omega) \in L(M) = L \\
&\Leftrightarrow \omega \in f^{-1}(L).
\end{aligned}$$

□

Definition 3.11. Suppose that $L_1, L_2 \subseteq \Sigma^{\uplus}$. Then, the quotient of language L_1 with language L_2 is the following language:

$$L_1/L_2 = \{\omega \in \Sigma^{\uplus} \mid \exists v \in L_2 \text{ such that } \omega \uplus v \in L_1\}.$$

Theorem 3.12. *The class of all fuzzy multiset automata languages is closed under quotient with arbitrary multisets.*

Proof. Assume that $L_1 \subseteq \Sigma^{\uplus}$ is a fuzzy multiset automaton language and that $L_2 \subseteq \Sigma^{\uplus}$ is a multiset. Then, there exists a fuzzy multiset automaton $M = (Q, \Sigma, \delta, \sigma, \tau)$ such that $L(M) = L_1$. We construct a fuzzy multiset finite automaton $M' = (Q, \Sigma, \delta, \sigma, \tau')$, where $\tau' : Q \to [0, 1]$ is defined by

$$\tau'(q) = \max_{v \in L_2}\Big\{\mu_M\big((p, v) \to (q', 0_{\Sigma})\big) \wedge \tau(q')\Big\} \neq 0.$$

Also, $\omega \in L(M')$ if and only if there exist $p, q \in Q$ such that

$$\Big\{\sigma(p) \wedge \mu_M\big((p, \omega) \to (q, 0_{\Sigma})\big) \wedge \tau'(q)\Big\} \neq 0.$$

A different way to say the same thing are the following "equations":

$$\sigma(p) \neq 0, \quad \mu_M\big((p, \omega) \to (q, 0_{\Sigma})\big) \neq 0, \quad \text{and} \quad \tau'(q) \neq 0.$$

Again, these are equivalent to the following for $p, q, q' \in Q$

$$\Big\{\sigma(p) \wedge \mu_M\big((p, \omega) \to (q, 0_{\Sigma})\big) \wedge \mu_M\big((p, v) \to (q, 0_{\Sigma})\big) \wedge \tau(q')\Big\} \neq 0.$$

Now, for some $v \in L_2$ and since $f'(q) \neq 0$

$$\Big\{\sigma(p) \wedge \mu_M\big((p, \omega \uplus v) \to *(q', 0_{\Sigma})\big) \wedge \tau(q')\} \neq 0,$$

that is, for some $p, q' \in Q$ and $v \in L_2$ if and only if $\omega \uplus v \in L(M) = L_1$ for some $v \in L_2$ if and only if $\omega \in L_1/L_2$.
The converse part can be proved in a similar way. □

Theorem 3.13. *The class of all fuzzy multiset finite automaton languages is closed under right quotient by any multiset and reversal of a fuzzy multiset finite language.*

Proof. The proof is similar to the proof of the previous theorem. □

4. Fuzzy multiset regular grammar

The concept of regular grammar for both finite state automaton and fuzzy finite state automaton have been introduced in (see also [3, 6, 12, 26]). In this section, we introduce the notion of regular grammar and regular grammar in normal form for fuzzy multiset finite automata and then discuss some properties. The following definition is borrowed from [22].

Definition 4.1. A fuzzy multiset grammar is a quintuple $G = (V_N, V_T, S, P)$, where

(1) V_N is a finite alphabet of nonterminal symbols,
(2) V_T is a finite alphabet of terminal symbols, such that $V_N \cap V_T = \varnothing$,
(3) V is the total alphabet $(V = V_N \cup V_T)$,
(4) $S \in V_N$ is a starting nonterminal symbol,
(5) P is a finite set of fuzzy multiset production rules over V, that is,

$$P = \Big\{\alpha \to \beta, \mu_P(\alpha \to \beta) \mid \alpha \in V^{\uplus} \uplus V_N, \beta\ V^{\uplus}, \mu_P(\alpha \to \beta) \in [0,1]\Big\},$$

where $\mu_P : (V^{\uplus} \uplus V_N) \times V^{\uplus} \to [0,1]$ is called the fuzzy transition function.

For any $(\alpha \to \beta, r) \in P$, for convenience, we sometimes abbreviate $(\alpha \to \beta, r)$ to $\alpha \xrightarrow{r} \beta$.

Definition 4.2. Suppose that $G = (V_N, V_T, S, P)$ is a fuzzy multiset grammar, $B \in V^{\uplus}$, and $A \in V^{\uplus} \setminus \{0_V\}$. Then,

(1) B immediately derives from A written as $(A \xrightarrow{r} B)$ if there exist multisets α, β, x such that $A = \alpha \uplus x, \beta = \beta\uplus$ and $\alpha \xrightarrow{r} \beta \in P$,
(2) B derives from A (written $(A \xrightarrow{r} *B)$) if for some $n \geq 0$ there exist $n+1$ multisets $v_0, v_1, \ldots, v_n$ such that $v_0 = A$, $v_n = B$ and $v_i \xrightarrow{r_i} v_{i+1}$ for $i = 0, 1, \ldots n-1$, where $r = r_0 \wedge r_1 \wedge \cdots \wedge r_{n-1}$ is called the grade of membership of the derivation of B from A with the derivation chain $A \xrightarrow{r_0} v_1 \xrightarrow{r_1} \cdots \xrightarrow{r_{n-2}} v_{n-2} \xrightarrow{rn-1} v_{n-1} \xrightarrow{r_n-1} B$.

Definition 4.3. A fuzzy multiset grammar $G = (V_N, V_T, S, P)$ is said to be regular, if each of its fuzzy multiset productions is either of the form $\langle\alpha\rangle \xrightarrow{r} \langle a\rangle \uplus \langle\beta\rangle$ or $\langle\alpha\rangle \xrightarrow{r} \langle a\rangle$, where $\alpha, \beta \in V_N$, $a \in V_T$, and $r \in [0,1]$.

Definition 4.4. A string $w \in V_T^*$ is said to be generated by fuzzy multiset regular grammar G if

$$\max_{q_0 \in V_N} \Big\{S(q_0) \wedge \mu_G(q_0 \to *\omega)\Big\} \neq 0.$$

Definition 4.5. The set of all string $\omega \in V_T^*$ that are generated by a fuzzy multiset regular grammar G is called a fuzzy multiset regular language. We shall denote it by $L(G)$.

The following three theorems have been proved in [22].

Theorem 4.6. *For each fuzzy multiset finite automaton $M = (Q, \Sigma, \delta, \sigma, \tau)$, there exists a fuzzy multiset regular grammar G such that $L(G) = L(M) \setminus \{0_\Sigma\}$.*

Theorem 4.7. *For a fuzzy multiset regular grammar $G = (V_N, V_T, S, P)$ there exists a fuzzy multiset finite automaton M such that $L(M) = L(G)$.*

Definition 4.8. A fuzzy multiset $G = (V_N, V_T, S, P)$ is said to be in normal form if it has either of the following types of fuzzy multiset production rules:

$$\langle A\rangle \xrightarrow{r} \langle a\rangle \uplus \langle B\rangle \quad \text{or} \quad \langle A\rangle \xrightarrow{r} \langle 0_{V_T}\rangle,$$

where $A, B \in V_N$, $a \in V_T$, and $r \in [0, 1]$. Every fuzzy multiset regular grammar can be reduced to a fuzzy multiset grammar in normal form by changing its fuzzy multiset production $A \xrightarrow{r} \langle a\rangle$ by two fuzzy multiset productions $\langle A\rangle \xrightarrow{r} \langle a\rangle \uplus \langle C\rangle$, $C \notin V_N$ and $\langle C\rangle \xrightarrow{r} \langle 0_{V_T}\rangle$.

Theorem 4.9. *For every fuzzy multiset grammar $G = (V_N, V_T, S, P)$ in normal form, there exists a fuzzy multiset automata M, such that $L(M) = L(G)$.*

Proof. Let $Q = V_N$, $\sigma = S$. Also, for $\tau : Q \to [0, 1]$ it holds that $\tau(q) = r$ if and only if $\langle q\rangle \xrightarrow{r} 0_{V_T}$ is a fuzzy multiset production in P. In addition, for $\delta : Q \times \Sigma \times Q \to [0, 1]$, it holds that $\mu_M\big((q, a) \xrightarrow{r} (p, a)\big)$ if and only if $\langle q\rangle \xrightarrow{r} \langle a\rangle \uplus \langle p\rangle \in P$. Then, $M = (Q, \Sigma, \delta, \sigma, \tau)$ is a fuzzy multiset finite automaton. Also, let $\omega \in L(G)$ and $\omega = a_1a_2a_3 \dots a_{n-1}a_n$, $a_i \in \Sigma$. Then,

$$\max_{q \in V_N} \Big\{ S(q) \wedge \mu\big((q, \omega) \to *(q_f, 0_\Sigma)\big) \Big\} \neq 0,$$

that is, there exists $q_0 \in V_N$ such that $S(q_0) \neq 0$ and $\mu_M((q_0, \omega) \to *(q_f, 0_{V_T}))$, for all $r \in [0, 1]$. Now, $\langle q_0\rangle \xrightarrow{r} \langle \omega\rangle$, $r \in [0, 1]$ implies that there exist $q_1, q_2, \dots q_n \in V_N$ and $r_1, r_2, \dots r_{n-1} \in (0, 1]$ such that

$$\begin{aligned}
\langle q_0\rangle &\xrightarrow{r_1} \langle a_1\rangle \uplus \langle q_1\rangle \\
&\xrightarrow{r_2} \langle a_1\rangle \uplus \langle a_2\rangle \uplus \langle q_2\rangle \cdots \\
&\xrightarrow{r_n} \langle a_1\rangle \uplus \langle a_2\rangle \uplus \cdots \uplus \langle a_n\rangle \uplus \langle q_n\rangle \\
&\xrightarrow{r_{n+1}} \langle a_1\rangle \uplus \langle a_2\rangle \uplus \cdots \uplus \langle a_n\rangle \uplus \langle 0_{V_T}\rangle \\
&= \omega.
\end{aligned}$$

Thus for this derivation chain, P must have the following fuzzy multiset productions:

$$\begin{aligned}
\langle q_0\rangle &\xrightarrow{r_1} \langle a_1\rangle \uplus \langle q_1\rangle,\\
\langle q_1\rangle &\xrightarrow{r_2} \langle a_2\rangle \uplus \langle q_2\rangle,\\
&\vdots \qquad\qquad \vdots\\
\langle q_{n-1}\rangle &\xrightarrow{r_n} \langle a_n\rangle \uplus \langle q_n\rangle,\\
\langle q_n\rangle &\xrightarrow{r_{n+1}} 0_{V_T}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta(q_0, \langle a_1\rangle, q_1) &= r_1,\\
\delta(q_1, \langle a_2\rangle, q_2) &= r_2,\\
\vdots \quad &\quad \vdots\\
\delta(q_{n-1}, \langle a_n\rangle, q_n) &= r_n,\\
\tau(q_n) &= r_{n+1}.
\end{aligned}$$

But then,

$$\Big\{\mu_M\big((q_0, a_1) \to (q_1, a_2)\big) \wedge \cdots \wedge \mu_M\big((q_{n-1}, a_n) \to (q_n, 0_\Sigma)\big) \wedge \tau(q_n)\} \neq 0,$$

that is, there exist $q_0, q_n \in Q$, such that

$$\Big\{\sigma(q_0) \wedge \mu_M\big((q_0, \omega) \to^* (q_n, 0_\Sigma)\big) \wedge \tau(q_n)\Big\} \neq 0,$$

or that

$$\max_{q\in Q}\Big\{\sigma(q) \wedge \mu_M\big((q, \omega) \to *(p, 0_\Sigma)\big)\Big\} \neq 0.$$

Hence $\omega \in L(M)$.

The converse can be proved similarly. □

Corollary 4.10. *For every fuzzy multiset grammar G in normal form, there exists a fuzzy multiset regular grammar G_1 such that*

$$L(G_1) = L(G) - \{0_{V_T}\}.$$

Definition 4.11. A fuzzy multiset grammar $G = (V_N, V_T, S, P)$ is called a linear grammar, if it has either of the following fuzzy multiset production rules:

- $\langle A\rangle \xrightarrow{r} \langle \omega\rangle$ or
- $\langle A\rangle \xrightarrow{r} \langle \omega_1\rangle \uplus \langle B\rangle \uplus \langle \omega_2\rangle$,

where $A, B \in V_N$, $\omega_1, \omega_2, \omega \in V_{V_T}^{\uplus}$, $r \in [0, 1]$. When G contains production rule of the second case and $\omega_1 = \{0_{V_N}\}$, then G is called a left linear grammar, while if $\omega_2 \in \{0_{V_T}\}$, then it is called right linear grammar.

Definition 4.12. A language $L \subseteq V_T^{\uplus}$ is fuzzy multiset linear (left linear, right linear), if there is fuzzy multiset linear (left linear, right linear, respectively) grammar G such that

$$L(G) = L.$$

It is clear that the class of fuzzy multiset regular languages is a subclasses of the class of fuzzy linear languages.

Theorem 4.13. *A fuzzy left linear grammar and a fuzzy right linear grammar both generate the same languages.*

Proof. Let $G = (V_N, V_T, S, P)$ be a fuzzy left right linear grammar $G' = (V_N, V_T, S, P')$ whose fuzzy multiset production rule set P' is as follows:

(1) $\langle q_0\rangle \xrightarrow{r} \langle\omega\rangle$ in P' if and only if $\langle q_0\rangle \xrightarrow{r} \langle\omega\rangle$ in P, $S(q_0) \neq 0$.
(2) $\langle q_0\rangle \xrightarrow{r} \langle\omega\rangle \uplus \langle A\rangle$ in P' for $S(q_0) \neq 0$ if and only if $\langle A\rangle \xrightarrow{r} \langle\omega\rangle$ in P; $S(A) = 0$.
(3) $\langle A\rangle \xrightarrow{r} \langle\omega\rangle$ and $\langle A\rangle \xrightarrow{r} \langle\omega\rangle \uplus \langle q_0\rangle$ in P' if and only if $\langle q_0\rangle \xrightarrow{r} \langle A\rangle \uplus \langle\omega\rangle$ in P, $S(q_0) \neq 0$.
(4) $\langle A\rangle \xrightarrow{r} \langle\omega\rangle \uplus \langle B\rangle$ in P' if and only if $\langle B\rangle \xrightarrow{r} \langle A\rangle \uplus \langle\omega\rangle$ in P; $S(B) = 0$.

We prove that $L(G') = L(G)$. Assume that $\omega \in L(G)$, where $\omega = \omega_1 \dots \omega_n$, and $\omega_i \in V_T^{\uplus}$. Then,

$$\max_{q\in V_N}\Big\{S(q) \wedge (\langle q\rangle \xrightarrow{r} \langle w\rangle)\Big\} \neq 0,$$

where $r = r_0 \wedge r_1 \wedge \cdots \wedge r_n \in [0,1]$. Thus there exists $q_0 \in V_N$ such that $S(q_0) \neq 0$ and $\langle q\rangle \xrightarrow{r} \langle\omega\rangle$; $r \neq 0$ in G. If $\langle q_0\rangle \xrightarrow{r} \langle\omega\rangle$ is a fuzzy multiset production rule in P for some $r \in [0,1]$ then $\langle q_0\rangle \xrightarrow{r} \langle\omega\rangle$ in P' and $\omega \in L(G')$, otherwise these exist $A_n, A_{n-1}, \dots, A_2 \in V_N$ and $r_1, r_2, \dots, r_n \in (0,1]$ such that $\langle q_0\rangle \xrightarrow{r_n} \langle A_n\rangle \uplus \langle\omega_n\rangle \xrightarrow{r_{n-1}} \langle A_n\rangle \uplus \langle\omega_{n-1}\rangle \uplus \langle\omega_n\rangle \xrightarrow{r_{n-2}} \cdots \xrightarrow{r_2} \langle a_2\rangle \uplus \langle\omega_2\rangle \uplus \langle\omega_3\rangle \uplus \cdots \uplus \dots \langle\omega_{n_1}\rangle \uplus \langle\omega_n\rangle \xrightarrow{r_1} \langle\omega_1\rangle \uplus \langle\omega_2\rangle \uplus \dots \uplus \langle\omega_{n-1}\rangle \uplus \langle\omega_n\rangle = \langle\omega\rangle$. But, corresponding to above derivation chain, P must have following fuzzy multiset production rules.

$$\begin{aligned}
\langle q_0\rangle &\xrightarrow{r_n} \langle A_n\rangle \uplus \langle\omega_n\rangle,\\
\langle A_n\rangle &\xrightarrow{r_{n-1}} \langle A_{n-1}\rangle \uplus \langle\omega_{n-1}\rangle,\\
&\vdots \qquad \vdots \qquad \vdots\\
\langle A_3\rangle &\xrightarrow{r_2} \langle A_2\rangle \uplus \langle\omega_2\rangle
\end{aligned}$$

and

$$\langle A_2\rangle \xrightarrow{r_1} \langle\omega_1\rangle.$$

Therefore, P' should have the following fuzzy multiset production rules:

$$\begin{aligned}
\langle A_n\rangle &\xrightarrow{r_n} \langle\omega_n\rangle,\\
\langle A_{n-1}\rangle &\xrightarrow{r_{n-1}} \langle\omega_{n-1}\rangle \uplus \langle A_n\rangle,\\
&\vdots \qquad \vdots \qquad \vdots\\
\langle A_2\rangle &\xrightarrow{r_2} \langle\omega_2\rangle \uplus \langle A_3\rangle,
\end{aligned}$$

and

$$\langle q_0\rangle \xrightarrow{r_1} \langle\omega_1\rangle \uplus \langle A_2\rangle.$$

Thus, there is a derivation chain for ω in G' as:

$$\begin{aligned}\langle q_0\rangle &\xrightarrow{r_1} \langle \omega_1\rangle \uplus \langle A_2\rangle\\ &\xrightarrow{r_2} \langle \omega_1\rangle \uplus \langle \omega_2\rangle \uplus \langle A_3\rangle\\ &\quad\vdots\\ &\xrightarrow{r_{n-2}} \langle \omega_1\rangle \uplus \langle \omega_2\rangle \uplus \cdots \langle \omega_{n-2}\rangle \uplus \langle A_{n-1}\rangle\\ &\xrightarrow{r_{n-1}} \langle \omega_1\rangle \uplus \langle \omega_2\rangle \uplus \cdots \langle \omega_{n-1}\rangle \uplus \langle A_n\rangle\\ &\xrightarrow{r_n} \langle \omega_1\rangle \uplus \langle \omega_2\rangle \uplus \cdots \langle \omega_{n-1}\rangle \uplus \langle \omega_n\rangle\\ &= \omega.\end{aligned}$$

Hence

$$\max_{q \in V_N} \Big\{ S(q) \wedge (\langle q\rangle \xrightarrow{r} \langle \omega\rangle) \Big\} \neq 0,$$

that is, $\omega \in L(G')$.

The other part of the proof is similar. □

Theorem 4.14. *Every fuzzy multiset right linear language can be generated from a fuzzy multiset grammar in normal form.*

Proof. We first construct a fuzzy multiset regular grammar $G_1 = (V'_N, V_T, S, P_1)$ such that $G \sim G_1$. Next, we construct a fuzzy multiset grammar $G' = (V'_N, V_T, S, P')$ in normal form from G_1 such that $G' \sim G_1$. There are the two cases which are examined in what follows.

Case 1. Assume that $G = (V_N, V_T, S, P)$ is a fuzzy multiset right linear grammar. Then, any fuzzy multiset production $\langle \alpha\rangle \xrightarrow{r} \langle \omega\rangle \uplus \langle \beta\rangle$ or $\langle \alpha\rangle \xrightarrow{r} \langle \omega\rangle$ of P with $|\omega| \leq 1$, is a fuzzy multiset production rule in P_1. The fuzzy multiset production rule $\langle \alpha\rangle \xrightarrow{r} \langle \omega\rangle \uplus \langle \beta\rangle$ is in P with $|\omega| > 1$ and $\omega = a_1 \uplus a_2 \uplus \cdots \uplus a_{n-1} \uplus a_n$. We add in P_1 a set of fuzzy multiset production rules

$$\begin{aligned}\langle \alpha\rangle &\xrightarrow{r} \langle a_1\rangle \uplus \langle Z_1\rangle,\\ \langle Z_1\rangle &\xrightarrow{r} \langle a_2\rangle \uplus \langle Z_2\rangle,\\ \vdots\quad & \quad\vdots\\ \langle Z_{n-1}\rangle &\xrightarrow{r} \langle a_n\rangle \uplus \langle \beta\rangle\end{aligned}$$

where $Z_1, Z_2, \ldots, Z_{n-1}$ are new variables not in V_N. Similarly, for the fuzzy multiset production rules

$$\langle \alpha\rangle \xrightarrow{r} \langle a_1\rangle \uplus \langle a_2\rangle \uplus \cdots \uplus \langle a_m\rangle,$$

$m \geq 2$ and $r \in [0,1]$, we add a set of fuzzy multiset production rules

$$\begin{aligned}
\langle \alpha \rangle &\xrightarrow{r} \langle a_1 \rangle \uplus \langle Y_1 \rangle, \\
\langle Z_1 \rangle &\xrightarrow{r} \langle a_2 \rangle \uplus \langle Y_2 \rangle, \\
&\vdots \quad \vdots \\
\langle Y_{m-1} \rangle &\xrightarrow{r} \langle a_m \rangle \uplus \langle Y_m \rangle, \\
\langle Y_m \rangle &\xrightarrow{r} \langle 0_{VT} \rangle
\end{aligned}$$

are in P_1, where $Y_1, Y_2, \ldots, Y_{m-1}, Y_m$ are new variables not in V_N.

Let V'_N be the set of all such variables in V_N that includes all the new variables introduces in the previous process. Then, P_1 of the fuzzy multiset grammar $G_1 = (V'_N, V_T, S, P_1)$ contains the following types of fuzzy multiset production rules:

(1) $\langle \alpha \rangle \xrightarrow{r} \langle \omega \rangle \uplus \langle \beta \rangle$,
(2) $\langle \alpha \rangle \xrightarrow{r} \langle \beta \rangle$,
(3) $\langle \alpha \rangle \xrightarrow{r} \langle 0_{V_T} \rangle, \alpha, \beta \in V'_N, 0_{V_T} \in V_T^{\uplus}$ and $\omega \in V_T, r \in [0,1]$.

Next, we prove that $G \sim G_1$. Let $\omega \in L(G)$. Then, $\max\{S(q) \wedge (\langle q \rangle \xrightarrow{r} \langle \omega \rangle) \neq 0\}$, that is, there exists $q_0 \in V_N$ such that $S(q_0) \neq 0$ and $(\langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle) \neq 0$. If $\langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle$ is a fuzzy multiset production rule in P and $|\omega| = 1$, then clearly $\langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle$ in P_1. Now, if $|\omega| > 1$ and $\omega = a_1 \uplus a_2 \uplus \cdots \uplus a_{n-1} \uplus \langle a_n \rangle$, $a_i \in V_T$, then there exist $q_1, q_2, \ldots, q_{n-1} \in V'_N$ such that $\langle q_0 \rangle \xrightarrow{r} \langle \omega_1 \rangle \uplus \langle q_1 \rangle \xrightarrow{r} \langle a_1 \rangle \uplus \langle a_2 \rangle \uplus \langle q_2 \rangle, \ldots,$ $\langle q_{n-1} \xrightarrow{r} a_1 \uplus a_2 \uplus \cdots \uplus a_{n-1} \uplus \langle a_n \rangle = \omega$, that is, $\langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle$ in G_1. Hence $\omega \in L(G_1)$. Again, consider $\omega \in L(G_1)$. Then,

$$\max_{q \in V'_N} \Big\{ S(q) \wedge (\langle q \rangle \xrightarrow{r} \langle \omega \rangle) \neq 0 \Big\},$$

that is, there exists $q_0 \in V'_N$ such that $S(q_0) \neq 0$ and $(\langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle) \neq 0$ in G_1.

Now, $(\langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle) \neq 0$. Therefore, there exists a derivation chain of ω in G_1 as:

$$\begin{aligned}
\langle q_0 \rangle &\xrightarrow{r_1} \langle a_1 \rangle \uplus \langle \alpha'_1 \rangle \\
&\xrightarrow{r_2} \langle a_1 \rangle \uplus \langle a_2 \rangle \uplus \langle \alpha'_2 \rangle \\
&\xrightarrow{r_3} \cdots \xrightarrow{r_n} \langle a_1 \rangle \uplus \langle a_2 \rangle \uplus \cdots \uplus \langle a_n \rangle \uplus \langle \alpha'_n \rangle \\
&\xrightarrow{r_{n+1}} \langle \omega_1 \rangle \uplus \langle q_1 \rangle \\
&\xrightarrow{s_1} \langle \omega_1 \rangle \uplus \langle b_1 \rangle \uplus \langle \beta'_1 \rangle \\
&\xrightarrow{s_2} \langle \omega_1 \rangle \uplus \langle b_1 \rangle \uplus \langle b_2 \rangle \uplus \langle \beta'_2 \rangle \\
&\xrightarrow{s_3} \cdots \xrightarrow{s_m} \langle \omega_1 \rangle \uplus \langle b_1 \rangle \uplus \langle b_2 \rangle \uplus \cdots \uplus \langle b_m \rangle \uplus \langle \beta'_m \rangle \\
&\xrightarrow{s_{m+1}} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \langle q_2 \rangle
\end{aligned}$$

$$\begin{aligned}
&\xrightarrow{t_1} \cdots \xrightarrow{t_{r+1}} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \cdots \uplus \langle \omega_{n-1} \rangle \uplus \langle q_{n-1} \rangle \\
&\xrightarrow{u_1} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \cdots \uplus \langle \omega_{n-1} \rangle \uplus \langle c_1 \rangle \uplus \langle C_1' \rangle \\
&\xrightarrow{u_2} \cdots \xrightarrow{u_n} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \cdots \uplus \langle \omega_n \rangle \uplus \langle q_n \rangle \\
&\xrightarrow{u_n+1} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \cdots \uplus \langle \omega_n \rangle \uplus \langle 0_{V_T} \rangle \\
&= \omega.
\end{aligned}$$

Thus the production rules are

$$\begin{aligned}
\langle q_0 \rangle &\xrightarrow{r_1} \langle a_1 \rangle \uplus \langle \alpha_1' \rangle \\
\langle \alpha_1' \rangle &\xrightarrow{r_2} \langle a_2 \rangle \uplus \langle \alpha_2' \rangle \\
&\cdots\cdots \\
\langle \alpha_{i-1}' \rangle &\xrightarrow{r_n} \langle a_i \rangle \uplus \langle \alpha_i' \rangle \\
\langle \alpha_{i+1}' \rangle &\xrightarrow{r_{n+1}} \langle a_{i+1} \rangle \uplus \langle \alpha_{i+2}' \rangle \\
&\cdots\cdots \\
\langle \alpha_n' \rangle &\xrightarrow{r_{n+1}} \langle q_1 \rangle \\
\langle q_1 \rangle &\xrightarrow{s_1} \langle b_1 \rangle \uplus \langle \beta_1' \rangle \\
\langle \beta_1' \rangle &\xrightarrow{s_2} \langle b_2 \rangle \uplus \langle \beta_2' \rangle \\
&\cdots\cdots \\
\langle \beta_m' \rangle &\xrightarrow{s_{m+1}} \langle q_2 \rangle \\
&\cdots\cdots \\
\langle C_{n-1}' \rangle &\xrightarrow{u_n} \langle \omega_n \rangle \uplus \langle q_n \rangle \\
\langle q_n \rangle &\xrightarrow{u_{n+1}} \langle \{0_{V_T}\} \rangle
\end{aligned}$$

are in P_1, or that there is a derivation chain of ω in G as:

$$\begin{aligned}
\langle q_0 \rangle &\xrightarrow{r} \langle \omega_1 \rangle \uplus \langle q_1 \rangle \\
&\xrightarrow{s} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \langle q_2 \rangle \\
&\xrightarrow{t} \cdots \xrightarrow{u} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \cdots \uplus \langle \omega_n \rangle \uplus \langle q_n \rangle \\
&\xrightarrow{t_{n+1}} \langle \omega_1 \rangle \uplus \langle \omega_2 \rangle \uplus \cdots \uplus \langle \omega_n \rangle \uplus \langle 0_{V_T} \rangle \\
&= \omega,
\end{aligned}$$

where $r = r_1 \wedge r_2 \wedge \cdots \wedge r_{n+1}$, $s = s_1 \wedge s_2 \wedge \cdots \wedge s_{m+1}$, $t = t_1 \wedge t_2 \wedge \cdots \wedge t_{r+1}$, $u = u_1 \wedge u_2 \wedge \cdots \wedge u_{n+1}$. Thus there exits $q_0 \in V_N$ such that $S(q_0) \neq 0$ and $\langle q_0 \rangle \xrightarrow{r} \langle \omega \rangle) \neq 0$ in G. Thus $\omega \in L(G)$, and hence $G \sim G_1$.

Case 2. We note that V_N' contains the variables in V_N as well as the new variables introduced in the process of finding $G1_1$. Fuzzy multiset productions like this one

$$\langle \alpha \rangle \xrightarrow{r} \langle \beta \rangle$$

are called fuzzy multiset chain rules. Now, we describe an algorithm to eliminate all such fuzzy multiset chain rules.

Initially, we construct the set $\cup_i(\alpha) = \{\alpha\}$, for $\alpha \in V'_N$, and $\cup_{i+1} = \cup_i(\alpha) \cup \big\{\beta \mid \langle\beta\rangle \xrightarrow{r} \langle Z\rangle \in P_i$ for some $Z \in \cup_i(\alpha)$, $r \in [0,1]\big\}$. Since V'_N is finite, there exists an integer K such that $\cup_{k+j}(\alpha) = \cup_k(\alpha), j = 1, 2, \ldots$. Next, write $\cup(\alpha)$ instead of $\cup_k(\alpha)$ for all $\alpha \in V'_N$. Then, we construct the required fuzzy multiset grammar $G' = (V'_N, V_T, S, P')$, where

(1) $\langle\alpha\rangle \xrightarrow{s\wedge r} \langle a\rangle \uplus \langle\beta\rangle$, in P' if and only if there is a $Z \in V'_N$ such that $\alpha \in \cup(Z)$ and $\langle Z\rangle \xrightarrow{r} \langle a\rangle \uplus \langle\beta\rangle$ in P_1,

(2) $\langle\alpha\rangle \xrightarrow{s\wedge r} \langle 0_{V_T}\rangle$ in P' if and only if there is $Z \in V'_N$ such that $\alpha \in \cup(Z)$ and $\langle Z\rangle \xrightarrow{r} \langle\{0_{V_T}\}\rangle$ in P_1.

Clearly, the fuzzy multiset grammar $G' = (V'_N, V_T, s, P')$ is in the normal form. It is now obvious that $G_1 \sim G'$. □

Corollary 4.15. *Fuzzy multiset right linear grammar is equivalent to fuzzy multiset regular grammar except for* $\{0_V\}$.

Theorem 4.16. *The following are equivalent except for* $\{0_V\}$.

(1) *Fuzzy multiset regular grammar.*
(2) *Fuzzy multiset left linear grammar.*
(3) *Fuzzy multiset right linear grammar.*
(4) *Fuzzy multiset grammar in normal form.*

5. Conclusions

In this paper, we have introduced and studied fuzzy multiset grammars. In addition, we proved that fuzzy multiset finite automata, and all other related concepts are equivalent, in the sense that all generate the same languages. We plan to further investigate the properties of these automata but we also hope that members of the scientific community will find useful applications of both fuzzy multiset finite-state automata and fuzzy multiset grammars.

Acknowledgements. The authors are grateful to the Editor-in-Chief and the anonymous referees for their valuable remarks and suggestions that helped us to substantially improve the quality of the final paper.

References

[1] F. Blanchet-Sadri and C. Morgan, Multiset and Set Decipherable Codes, Computers & Mathematics with Applications 41 (10–11) (2001) 1257–1262.
[2] Matteo Cavalierea, Rudolf Freundb, Marion Oswald and Dragoş Sburlan, Multiset random context grammars, checkers, and transducers, Theoretical Computer Science 372 (2–3) (2007) 136–151.
[3] Shrikant R. Chaudhari and Digambar D. Komejwar, On Fuzzy Regular Grammars, Advances in Fuzzy Mathematics 6 (1) (2011) 89–103.
[4] Gabriel Ciobanu and Mijai Gontineac, Mealy Multiset Automata, International Journal of Foundations of Computer Science, 17 (1) (2006) 111–126.

[5] Erzsébet Csuhaj-Varjú, Carlos Martín-Vide and Victor Mitrana Multiset Automata. In Cristian S. Calude, Gheorghe Păun, Grzegorz Rozenberg, and Arto Salomaa, editors, Multiset Processing, number 2235 in Lecture Notes in Computer Science 69–83, Springer-Verlag, Berlin 2001.

[6] John E. Hopcroft, Rajeev Motwani and Jeffrey D. Ullman, Introduction to Automata Theory, Languages, and Computation, Pearson Education Ltd., 3rd edition 2014.

[7] Manfred Kudlek and Gheorghe Păun Carlos Martín-Vide, Toward a Formal Macroset Theory, In Cristian S. Calude, Gheorghe Păun, Grzegorz Rozenberg, and Arto Salomaa, editors, Multiset Processing, number 2235 in Lecture Notes in Computer Science 123–133, Springer-Verlag, Berlin 2001.

[8] Yongming Li, Approximation and robustness of fuzzy finite automata, International Journal of Approximate Reasoning, 47 (2) (2008) 247–257.

[9] Yongming Li and Witold Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-ordered monoids, Fuzzy Sets and Systems 156 (2005) 68–92.

[10] Zhihui Lia, Ping Lia and Yongming Li, The relationships among several types of fuzzy automata, Information Sciences 176 (2006) 2208–2226.

[11] Simon McGregor, Further results in multiset processing with neural networks, Neural Networks 21 (6) (2008) 830–837.

[12] John N. Mordeson and Davender S. Malik, Fuzzy Automata and Languages: Theory and Applications, Chapman & Hall/CRC, Boca Raton, FL, USA 2002.

[13] V.K. Murthy and E.V. Krishnamurthy, Probabilistic parallel programming based on multiset transformation, Future Generation Computer Systems 11 (3) (1995) 283–2935.

[14] Darrell Raymond, Derick Wood and Sheng Yu, editors, Automata Implementation, number 1260 in Lecture Notes in Computer Science Springer-Verlag, Berlin 1997.

[15] Arto Salomaa, Theory of Automata, Pergamon Press, Oxford 1969.

[16] Eugene S. Santos, Maximin automata, Information and Control, 13 (4) (1968) 363–377.

[17] Apostolos Syropoulos. Mathematics of Multisets, In Cristian S. Calude, Gheorghe Păun, Grzegorz Rozenberg, and Arto Salomaa, editors, Multiset Processing, number 2235 in Lecture Notes in Computer Science 347–358, Springer-Verlag, Berlin 2001.

[18] Apostolos Syropoulos, Theory of Fuzzy Computation, Springer, New York 2014.

[19] Apostolos Syropoulos, Fuzzy Multisets and Fuzzy Computing, In Sunil Jacob John, editor, Handbook of Research on Generalized and Hybrid Set Structures and Applications for Soft Computing 23–43, IGI Global, Hershey, PA 17033, USA 2016.

[20] R. Tauler, M. Maeder and A. de Juan, Multiset Data Analysis: Extended Multivariate Curve Resolution, In Romà Tauler, Beata Walczak, and Steven D. Brown, editors, Comprehensive Chemometrics: Chemical and Biochemical Data Analysis, volume 2 473–505, Elsevier, Amsterdam 2009.

[21] S. P. Tiwari, Vinay Gautam and M. K. Dubey, On fuzzy multiset automata, Journal of Applied Mathematics and Computing 51 (1) (2016) 643–657.

[22] Jinyan Wang, Minghao Yin and Wenxiang Gu, Fuzzy multiset finite automata and their languages, Soft Computing 17 (3) (2013) 381–390.

[23] William G. Wee, On generalizations of adaptive algorithm and application of the fuzzy sets concepts to pattern classification, PhD thesis, Purdue University 1967.

[24] William G. Wee and K. S. Fu, A Formulation of Fuzzy Automata and Its Application as a Model of Learning Systems, IEEE Transactions on Systems Science and Cybernetics 5 (3) (1969) 215–223, 1969

[25] Ronald R. Yager, On the Theory of Bags, International Journal of General Systems 13 (1986) 23–37.

[26] Sheng Yu, Regular Languages, In Grzegorz Rozenberg and Arto Salomaa, editors, Handbook of Formal Languages, volume 1 41–110. Sringer, Berlin 1997.

[27] Lotfi Askar Zadeh, Fuzzy Sets, Information and Control 8 (1965) 338–353.

B. K. Sharma (`bksharma1@gmail.com`)
Department of Computer Science and Engineering , Birla Institute of Technology,

Mesra, Ranchi-834001, India

Apostolos Syropoulos (asyropoulos@yahoo.com)
Greek Molecular Computing Group, Xanthi, Greece

S.P. Tiwari (sptiwarimaths@gmail.com) – Department of Applied Mathematics, Indian School of Mines, Dhanbad-826004, India