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Fuzzy hollow submodules

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Abstract. We define fuzzy hollow submodules of a module. We attempt to investigate various properties of fuzzy hollow submodules of a module. Fuzzy hollow submodules of a module are characterized in terms of fuzzy quotient modules. A relation between hollow submodule, and indecomposable fuzzy submodule is established. We investigated the nature of equivalent conditions of fuzzy small submodules, and fuzzy hollow submodules. We show that L-local modules are L-hollow modules, and finitely generated.

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1. Introduction

Fuzzy set which deals with degree of membership of an element in a set is tolerant of impresision, uncertainty, partial truth, and approximation. It was introduced by Zadeh [19] in the year 1965. After that a number of generalizations of this fundamental concept is done. It is worth pointing out that fuzzy sets and fuzzy logics have been applied in almost all branches of mathematical sciences. Many applications, and its significances can be seen on fuzzy algebra [8, 9, 14], fuzzy topology [6], fuzzy measure, fuzzy analysis [3, 12], fuzzy optimization, decision making [5], and fuzzy reasoning [7] etc. In the year 1971, Rosenfeld introduced the concept of fuzzy subgroups of a group. It is important point to note that in algebraic structure, commutative algebra (specifically rings and modules) play a vital role. As its generalization, in the year 1975 [10], fuzzy submodules of a module was introduced. Some of the important properties of modules theory are: finitely generated modules, quotient modules, maximal submodules, radical, simple modules, essential submodules, small submodules, hollow modules, and goldie dimension etc. As a consequence of [10], fuzzy finitely generated submodules, fuzzy quotient modules [11], radical of fuzzy submodules, primary fuzzy submodules [16, 4] were investigated. In the year 2009, Saikia and Kalita [15] define fuzzy essential submodules, and their various properties were investigated. These modules play a prominent role in fuzzy Goldie dimension of modules. Dual to the notion of fuzzy essential submodule, fuzzy small submodule was introduced by Rahman and Saikia [13] in the year 2011, and some of its important results were discussed there in. These motivate us to introduce fuzzy hollow submodules, and investigate their properties. In 2014 and 2015, Fuzzy cosmall submodules [1], and on fuzzy supplement submodules [17] were investigated.

In this paper, we have defined fuzzy hollow submodule of a module. We have established some necessary and sufficient conditions for fuzzy hollow submodules. Fuzzy hollow submodule of a module is characterized in terms of its cut set. It is shown that hollow submodules of a module are indecomposable fuzzy submodule. We investigate properties of fuzzy hollow submodules of a module in terms of fuzzy quotient modules. Finally, we investigate the nature of equivalent conditions of fuzzy small submodules and fuzzy hollow submodules, and we show that local modules are hollow submodules and finitely generated.

2. Preliminaries

We will denote by R a commutative ring with unity 1, and M denotes an $R-$module. We will write the zero elements of R and M are 0 and θ, respectively.

A submodule S of a module M over a ring R is called a small submodule of M if for every submodule N of M with $N \neq M$ implies $S + N \neq M$. The notation $S << M$ indicates that S is a small submodule of M.

Recall that a nonzero R-module M is hollow if every proper submodule is superfluous (small) in M. For the details we refer the reader to [18].

A mapping μ from a universal set X to $[0,1]$ is called a fuzzy subset of X. We will denote by $[0,1]^X$ the set of all fuzzy subsets of X.

A complete Heyting algebra L is a complete lattice such that for all $A \subset L$, and for all $b \in L$, $\vee\{a \wedge b | a \in A\} = (\vee\{a | a \in A\}) \wedge b$ and $\wedge\{a \vee b | a \in A\} = (\wedge\{a | a \in A\}) \vee b$.

If μ is a mapping from X to L, then μ is called an L-subset of X. We will denote by L^X the set of all L-subsets of X.

For each fuzzy set μ of X, and every $\alpha \in [0,1]$, we define two sets $U(\mu,\alpha) = \{x \in X \mid \mu(x) \geq \alpha\}$, $L(\mu,\alpha) = \{x \in X \mid \mu(x) \leq \alpha\}$, which are called an upper level cut and a lower level cut of μ, respectively. The support of μ, denoted by μ^*, is the crisp subset $\{x \in X | \mu(x) > 0\}$. Let us denote by μ_* the set defined as $\{x \in X | \mu(x) = 1\}$. The complement of μ, denoted by μ^c, is the fuzzy set on X defined by $\mu^c(x) = 1 - \mu(x)$. The range set or image of μ, denoted by $Im(\mu)$, is the set $\{t \in [0,1] \mid \mu(x) = t \text{ for some } x \in X\}$.

Definition 2.1 ([8])**.** If $Y \subseteq X$, and $\alpha \in [0,1]$, then α_Y is defined as

$$\alpha_Y(x) = \begin{cases} \alpha, & \text{if } x \in Y \\ 0, & \text{otherwise.} \end{cases}$$

If $Y = \{x\}$, then $\alpha_{\{x\}}$ is often called a fuzzy point, and we will denote by x_α the fuzzy point $\alpha_{\{x\}}$. Moreover, if $\alpha = 1$, then 1_Y is known as the characteristic function of Y. Here and subsequently, χ_Y stands for characteristic function of Y.

If $\mu, \sigma \in [0,1]^X$, then $\mu \subseteq \sigma$ if and only if $\mu(x) \leq \sigma(x)$ for all $x \in X$.

Definition 2.2 ([10])**.** Let M be a module over a ring R. A fuzzy subset μ of M is called a fuzzy submodule of M if for every $x, y \in M$, and $r \in R$ the following conditions hold:
(i) $\mu(\theta) = 1$,
(ii) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$,
(iii) $\mu(rx) \geq \mu(x)$.

The set of all fuzzy submodules of M will be denoted by $F(M)$.

If we replace $[0,1]$ by a complete Heyting algebra L in the above definition, then μ is called a L-submodule of M.

Definition 2.3 ([8])**.** Let μ, $\nu \in F(M)$ be such that $\mu \subseteq \nu$. Then the quotient of ν with respect to μ is a fuzzy submodule of M/μ^*, denoted by ν/μ, and is defined as follows:

$$(\nu/\mu)([x]) = \vee\{\, \nu(z) \,|\, z \in [x] \,\} \;\; \forall x \in \nu^*, \tag{2.1}$$

where $[x]$ denotes the coset $x + \mu^*$.

Definition 2.4 ([13])**.** A fuzzy submodule σ in M is called a fuzzy direct sum of two fuzzy submodules μ, and ν if $\sigma = \mu + \nu$ and $\mu \cap \nu = \chi_{\{\theta\}}$.

Definition 2.5 ([8])**.** Let $\mu \in [0,1]^M$. Then $\cap\{\nu | \mu \subseteq \nu,\, \nu \in F(M)\,\}$ is a fuzzy submodule of M and it is called the fuzzy submodule generated by the fuzzy subset μ. We will denote it by $< \mu >$, i.e.,

$$< \mu >= \cap\{\nu | \mu \subseteq \nu,\, \nu \in F(M)\,\}.$$

Let $\xi \in F(M)$. If $\xi =< \mu >$ for some $\mu \in [0,1]^M$, then μ is called a generating fuzzy subset of ξ.

A fuzzy submodule ξ of a module M is called finitely generated if $\xi =< \mu >$ for some $\mu \in [0,1]^M$, and μ_* is finite.

Remark 2.6. (a) If A is a nonempty subset of M, then $< \chi_A >= \chi_{<A>}$, where $< A >$ is the submodule of M generated by A.

(b) If $x \in M$, then $\chi_R \odot \chi_{\{x\}}$ is a fuzzy submodule of M generated by $\chi_{\{x\}}$, and in this case,

$$\chi_R \odot \chi_{\{x\}} =< \chi_{\{x\}} >= \chi_{<\{x\}>} = \chi_{Rx}.$$

Lemma 2.7 ([8])**.** *Let $\mu_i \in F(M)$ for each $i \in I$, where $|I| > 1$. Then $\sum_{i \in I} \mu_i \in F(M)$, and $< \cup_{i \in I} \mu_i >= \sum_{i \in I} \mu_i$.*

Definition 2.8 ([13])**.** Let M be a module over a ring R, and let $\mu \in F(M)$. Then μ is said to be a fuzzy small submodule of M if for any $\nu \in F(M)$ satisfying $\nu \neq \chi_M$ implies $\mu + \nu \neq \chi_M$. Let $\mu <<_f M$ denote that μ is a fuzzy small submodule of M. We will indicate that μ is a small L-submodule of M by the notation $\mu <<_L M$.

Definition 2.9 ([13])**.** Let μ and σ be any two fuzzy submodules of M such that $\mu \subseteq \sigma$, then μ is called a fuzzy submodule of σ, and μ is called a fuzzy small submodule in σ, denoted by $\mu <<_f \sigma$, if $\mu <<_f \sigma^*$ in the sense that for every submodule γ in M satisfying $\gamma_{|\sigma^*} \neq \chi_{\sigma^*}$ implies $\mu_{|\sigma^*} + \gamma_{|\sigma^*} \neq \chi_{\sigma^*}$ (by $\mu_{|\sigma^*}$, $\gamma_{|\sigma^*}$ we mean the restriction mapping of μ, γ on σ^*, respectively).

Definition 2.10 ([2]). A $\mu(\neq \chi_{\{\theta\}}) \in F(M)$ is said to be fuzzy indecomposable if there do not exist $\sigma, \gamma \in F(M)$ with $\sigma \neq \chi_{\{\theta\}}$, $\gamma \neq \chi_{\{\theta\}}$ and $\sigma \neq \mu$, $\gamma \neq \mu$ such that $\mu = \sigma \oplus \gamma$.

Theorem 2.11 ([13]). *Any finite sum of fuzzy small submodules of M is a fuzzy small submodule in M.*

Theorem 2.12 ([13]). *Let M be a module, and let $N \leq M$. Then $N << M$ if and only if $\chi_N <<_f M$.*

Theorem 2.13 ([13]). *Let $\mu \in F(M)$. Then $\mu <<_f M$ if and only if $\mu_* << M$.*

Theorem 2.14 ([13]). *Let $\mu, \nu \in F(M)$. Then $\mu <<_f \nu$ if and only if $\mu_* << \nu_*$.*

3. Fuzzy hollow submodules

Definition 3.1. A fuzzy submodule ν with $\nu_* \neq \{\theta\}$ of M is said to be a fuzzy hollow submodule if for every fuzzy submodule μ of ν with $\mu_* \neq \nu_*$, μ is a fuzzy small submodule of ν. We say that an R-module $M \neq \{\theta\}$ is fuzzy hollow module if for every $\sigma \in F(M)$ with $\sigma_* \neq M$ implies $\sigma <<_f M$.

Example 3.2. Consider $M = Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ under addition modulo 8. Then M is a module over the ring Z. Let $S = \{0, 2, 4, 6\}$. Define $\nu \in [0,1]^M$ as follows:

$$\nu(x) = \begin{cases} 1, & \text{if } x \in S \\ \alpha, & \text{otherwise,} \end{cases}$$

where $0 \leq \alpha < 1$. Let $K = \{0, 4\}$, and define μ as follows:

$$\mu(x) = \begin{cases} 1, & \text{if } x \in K \\ \beta, & \text{otherwise,} \end{cases}$$

where $\beta \leq \alpha < 1$. Then clearly ν, μ are the only fuzzy submodules of M with $\mu_*, \nu_* \neq M$, and μ is the only fuzzy submodule of ν with $\mu_* \neq \nu_*$. It is clear that ν, μ are fuzzy small submodules of M, and μ is a fuzzy small submodule of ν. It follows that ν is a fuzzy hollow submodule, and M is a fuzzy hollow module.

Theorem 3.3. *A nonzero R-module M is a fuzzy hollow module if and only if for every μ, $\sigma \in F(M)$ with μ, $\sigma \neq \chi_M$ implies $\mu + \sigma \neq \chi_M$.*

Proof. Proof is straight forward, and it directly follows from the definition. □

Theorem 3.4. *A nonzero R-module M is a hollow module if and only if M is a fuzzy hollow module.*

Proof. Let M be a hollow module, and let μ be a fuzzy submodule of M with $\mu \neq \chi_M$. Then μ_* is a submodule of M with $\mu_* \neq M$. Since M is a hollow submodule, $\mu_* << M$. It follows from Theorem 2.13 that $\mu <<_f M$. Hence M is a fuzzy hollow module.

Conversely, we assume that M is a fuzzy hollow module. Let N be a submodule of M such that $N \neq M$. Then $\chi_N(\neq \chi_M)$ is a fuzzy submodule of M. Since M is a fuzzy hollow module, $\chi_N <<_f M$ (it follows from Theorem 2.12). This shows that $N << M$. □

Theorem 3.5. *Let $\sigma \in F(M)$ be such that $\sigma \neq \chi_{\{\theta\}}$. Then σ is a fuzzy hollow submodule of M if and only if σ_* is a hollow submodule of M.*

Proof. Let σ be a fuzzy hollow submodule of M. To show σ_* is a hollow submodule of M. Let N be a proper submodule of σ_*. Then $\chi_N \subset \sigma$ with $(\chi_N)_* \neq \sigma_*$. Since σ is a fuzzy hollow submodule of M, $\chi_N <<_f \sigma$ which is equivalent to $N << \sigma_*$ (Theorem 4.15 of [13]). Hence σ_* is a hollow submodule of M.

Conversely, we assume σ_* is a hollow submodule of M. Let $\mu \in F(M)$ be such that $\mu \subset \sigma$, and $\mu_* \neq \sigma_*$. Then μ_* is a proper submodule of σ_*, and so $\mu_* << \sigma_*$. Therefore $\mu <<_f \sigma$ (Theorem 4.15 of [13]). Hence σ is a fuzzy hollow submodule of M. □

Theorem 3.6. *Every fuzzy hollow submodule is indecomposable.*

Proof. Let μ be a fuzzy hollow submodule of a module M. If μ is not indecomposable, then there exist ν, $\sigma \in F(M)$ with ν, $\sigma(\neq \chi_{\{\theta\}}, \mu)$ such that $\mu = \nu \oplus \sigma$. This implies that $\mu_* = \nu_* \oplus \sigma_*$, and σ_*, $\nu_* \neq \mu_*$. This implies that μ_* in not indecomposable. But, hollow submodules are indecomposable, and by Theorem 3.5 μ_* is a hollow submodule of M. It follows that μ_* is indecomposable, a contradiction. This contradiction leads us to the conclusion that μ is a fuzzy indecomposable submodule of M. □

Corollary 3.7. *If M is a fuzzy hollow module, then χ_M is a fuzzy indecomposable module.*

Theorem 3.8. *Let ν be a fuzzy hollow submodule of M, and let $\mu \in F(M)$ such that $\mu \subset \nu$ with $\mu_* \neq \nu_*$. Then ν/μ is a fuzzy hollow submodule of M/μ.*

Proof. Let $\sigma \in F(M)$ such that $\mu \subset \sigma \subset \nu$ satisfying $(\sigma/\mu)_* \neq (\nu/\mu)_*$. We claim $(\sigma/\mu) <<_f (\nu/\mu)$. Since $(\sigma/\mu)_* \neq (\nu/\mu)_*$, $\sigma_* \neq \nu_*$. It follows that $\sigma \subset \nu$ with $\sigma_* \neq \nu_*$. Since ν is a fuzzy hollow submodule, $\sigma <<_f \nu$. Thus $\mu \subset \sigma \subset \nu$ with $\sigma <<_f \nu$. It follows from Theorem 4.22 of [13] that $(\sigma/\mu) <<_f (\nu/\mu)$. Therefore ν/μ is a fuzzy hollow submodule of M/μ. □

Theorem 3.9. *Let $\nu \in F(M)$. Then ν is a fuzzy hollow submodule of M if and only if for every $\mu \in F(M)$ with $\mu \subset \nu$, and $\mu_* \neq \nu_*$, ν/μ is a fuzzy hollow submodule of M/μ and $\mu <<_f \nu$.*

Proof. Let ν be a fuzzy hollow submodule of M. It is given that $\mu \in F(M)$ with $\mu \subset \nu$, and $\mu_* \neq \nu_*$. It follows that $\mu <<_f \nu$, and hence from Theorem 3.8, we get ν/μ is a fuzzy hollow submodule of M/μ.

Conversely, we assume $\mu <<_f \nu$, and ν/μ is a fuzzy hollow submodule of M/μ. Let $\sigma \in F(M)$ with $\sigma \subset \nu$, and $\sigma_* \neq \nu_*$. Since $\mu <<_f \nu$, $\mu_* << \nu_*$ (Theorem 2.14). Thus $\mu_* + \sigma_* \neq \nu_*$. Since $\mu_* + \sigma_* = (\mu + \sigma)_*$, we have $(\mu + \sigma)_* \neq \nu_*$. It follows that $(\mu + \sigma) \subset \nu$ with $(\mu + \sigma)_* \neq \nu_*$, which gives $((\mu + \sigma)/\mu)_* \neq (\nu/\mu)_*$. As ν/μ is a hollow submodule, and $(\mu + \sigma)/\mu \subset \nu/\mu$ we have

$$(\mu + \sigma)/\mu <<_f \nu/\mu. \tag{3.1}$$

Let $\gamma \in F(M)$ be such that $\sigma_{|_{\nu^*}} + \gamma_{|_{\nu^*}} = \chi_{\nu^*}$. Then

$$(\mu + \sigma_{|_{\nu^*}} + \gamma_{|_{\nu^*}})/\mu = \chi_{\nu^*}/\mu = \chi_{(\nu/\mu)^*}. \tag{3.2}$$

Since $\sigma \subset \nu$,

$$(\mu+\sigma)/\mu + (\mu+\gamma_{|_{\nu^*}})/\mu = \chi_{(\nu/\mu)^*}. \tag{3.3}$$

Also from $\mu \subset \nu$ we have

$$(\mu+\sigma)/\mu + ((\mu+\gamma)/\mu)_{|_{(\nu/\mu)^*}} = \chi_{(\nu/\mu)^*}. \tag{3.4}$$

Now, equation 3.1 and 3.4 together imply that

$$\begin{aligned} &((\mu+\gamma)/\mu)_{|_{(\nu/\mu)^*}} = \chi_{(\nu/\mu)^*} \\ &\Rightarrow (\mu+\gamma_{|_{\nu^*}})/\mu = \chi_{(\nu/\mu)^*} = \chi_{\nu^*}/\mu \\ &\Rightarrow \mu+\gamma_{|_{\nu^*}} = \chi_{\nu^*} \\ &\Rightarrow \gamma_{|_{\nu^*}} = \chi_{\nu^*} \text{ (since } \mu <<_f \nu \text{).} \end{aligned}$$

Thus for $\gamma \in F(M)$ with $\sigma + \gamma|_{\nu^*} = \chi_{\nu^*}$, it follows that $\gamma_{|_{\nu^*}} = \chi_{\nu^*}$. So $\sigma <<_f \nu$. Hence ν is a fuzzy hollow submodule of M. □

Theorem 3.10. *Let $\nu \in F(M)$. Then ν is a fuzzy hollow submodule of M if and only if for every $\mu \in F(M)$ with $\mu \subset \nu$, and $\mu_* \neq \nu_*$, ν/μ is a fuzzy indecomposable submodule of M/μ.*

Proof. Let ν be a fuzzy hollow submodule of M. Let $\mu \in F(M)$ with $\mu \subset \nu$, and $\mu_* \neq \nu_*$. Then, by Theorem 3.8, ν/μ is a hollow submodule of $M/\mu(= \chi_M/\mu^*)$. It follows from Theorem 3.6 that ν/μ is a fuzzy indecomposable submodule of M/μ.

Conversely, let us assume that for every $\mu \in F(M)$ with $\mu \subset \nu$, and $\mu_* \neq \nu_*$, ν/μ is a fuzzy indecomposable submodule of M/μ. Let $\sigma \in F(M)$ be such that $\sigma \subset \nu$ with $\sigma_* \neq \nu_*$. As ν/μ is a fuzzy indecomposable submodule of M/μ, and $\sigma/\mu \subset \nu/\mu$ with $(\sigma/\mu)_* \neq (\nu/\mu)_*$ we have for every $\gamma \in F(M)$, $\gamma \subset \nu$ with $\gamma_* \neq \nu_*$ implies that

$$\begin{aligned} &\sigma/\mu \oplus \gamma/\mu \neq \nu/\mu \\ &\Rightarrow (\sigma+\gamma)/\mu \neq \nu/\mu \\ &\Rightarrow \sigma+\gamma \neq \nu \subset \chi_{\nu^*}. \end{aligned}$$

Thus $\sigma + \gamma \neq \chi_{\nu^*}$, and so $\sigma <<_f \nu$. From this we conclude that ν is a fuzzy hollow submodule of M. □

As a consequence of Theorem 3.9 and 3.10, we have the following result.

Theorem 3.11. *Let $\nu \in F(M)$. Then the following statements are equivalent:*

(i) *ν is a fuzzy hollow submodule of M.*
(ii) *for every $\mu \in F(M)$ with $\mu \subset \nu$, and $\mu_* \neq \nu_*$, ν/μ is a fuzzy hollow submodule of M/μ and $\mu <<_f \nu$.*
(iii) *for every $\mu \in F(M)$ with $\mu \subset \nu$, and $\mu_* \neq \nu_*$, ν/μ is a fuzzy indecomposable submodule of M/μ.*

An L-submodule μ of a module M is called maximal if $\mu \neq \chi_M$, and if σ is any other proper L-submodule of M containing μ, then $\mu = \sigma$.

An element $a \in L - \{1\}$ is called maximal element if $c \in L - \{1\}$ such that $a \leq c$, then $a = c$.

We note that maximal L-submodule exist only when an maximal element exist in L, and hence M does not possess an maximal fuzzy submodule as $[0, 1)$ does not provide maximal element.

The Jacobson L-radical (simply radical) of M, denoted by $JLR(M)$, is defined as the intersection of all maximal L-sumodules of M. We refer the reader to [13] for details.

An R-module $M \neq \{\theta\}$ is called L-hollow module if for every L-submodule σ of M with $\sigma_* \neq M$ implies that σ is an L-small submodule of M.

A module M is called L-local if M has exactly one maximal L-submodule. Recall the following theorem.

Theorem 3.12 ([13])**.** *Let $\mu \in L^M$. Then μ is a maximal L-submodule of M if and only if μ can be expressed as $\mu = \chi_{\mu_*} \cup \alpha_M$, where μ_* is a maximal submodule of M and α is a maximal element of $L - \{1\}$.*

Theorem 3.13. *Let M be an R-module. If M is L-local, then M is an L-hollow module, and finitely generated.*

Proof. Suppose that M is L-local. Then M has exactly one maximal L-submodule. In that case, $JLR(M)$ is that maximal submodule. Let $\mu = JLR(M)$. Then, by Theorem 3.12, μ can be expressed as $\mu = \chi_{\mu_*} \cup \alpha_M$, where μ_* is a maximal submodule of M, and α is a maximal element of $L - \{1\}$. Now we claim that M is local. If possible, let M is not a local module. Then M has atleast two distinct maximal submodules. Let M_1 and M_2 be two distinct maximal submodules of M. Define

$$\mu_1(x) = \begin{cases} 1, & \text{if } x \in M_1 \\ \alpha, & \text{otherwise} \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & \text{if } x \in M_2 \\ \alpha, & \text{otherwise.} \end{cases}$$

It is easy see that $\mu_1 = \chi_{\mu_{1_*}} \cup \alpha_M$ and $\mu_2 = \chi_{\mu_{2_*}} \cup \alpha_M$. Thus, by Theorem 3.12, we have μ_1 and μ_2 are two distinct maximal L-submodules of M, a contradiction that M is L-local. Thus M is local. Now $\mathrm{Rad}(M)$ being proper submodule of M is a small submodule of M (as local modules are hollow modules). But $\mu_* = JLR(M)_* = \mathrm{Rad}(M)$. It follows that $JLR(M)_*$ is a small submodule of M. So, by Theorem 2.13, $JLR(M)$ is a small L-submodule of M. Since every proper L-submodule is a submodule of $JLR(M)$ as $JLR(M)$ is the only maximal L-submodule of M, and since submodule of a small submodule is small in M, every proper submodule is a small submodule of M, and hence M is L-hollow module. To show M is finitely generated, let $m \in M$ such that $\chi_{\{m\}} \nsubseteq JLR(M)$. Since $JLR(M)$ is an maximal submodule, $\chi_M =< \chi_{\{m\}} > +JLR(M)$. It follows that $\chi_M =< \chi_{\{m\}} >= \chi_{Rm}$ as $JLR(M)$ is small L-submodule of M. Thus M is cyclic, and hence finitely generated. □

4. Conclusions

In this article, we have defined fuzzy hollow submodules of a module, and some of their properties were investigated. This may help toward the study of the fuzzy finite spanning, and fuzzy hollow dimension of a module which dualize the notion of Goldie, and uniform dimension of a module, respectively.

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