Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 12, No. 5, (November 2016), pp. 585–590
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

A note on interval-valued fuzzy metric spaces

Francisco Castro-Company, Pedro Tirado

Received 16 February 2016; Revised 2 April 2016; Accepted 11 April 2016

Abstract. In [11], Shen et al. introduced and studied a notion of interval-valued fuzzy metric space as a natural generalization of fuzzy metric spaces due to George and Veeramani[3]. In this note we show that each interval-valued fuzzy metric space $(X, \overline{M}, \overline{*})$ induces in a natural way two fuzzy metrics spaces $(X, M^-, *^-)$ and $(X, M^+, *^+)$ and that the topology generated by the interval-valued fuzzy metric $\overline{M}$ coincides with the topology generated by M^-, and hence the study of the space $(X, \overline{M}, \overline{*})$ reduces to the study of the fuzzy metric space $(X, M^-, *^-)$, so that Shen, Li and Wang's results follow directly from well-known results in fuzzy metric spaces.

2010 AMS Classification: 03E72, 54A10, 54A40

Keywords: Interval numbers, Interval-valued fuzzy sets, Interval-valued t-norm, Interval-valued fuzzy metric space, Topology, Metrizable.

Corresponding Author: Pedro Tirado (pedtipe@mat.upv.es)

1. Introduction and preliminaries

The concept of interval-valued fuzzy set was introduced by Zadeh in 1975 [12]. An interval-valued fuzzy set is characterized by an interval-valued membership function, and it is taken as a generalization of fuzzy sets. Some recent results on interval-valued fuzzy sets can be found in [9] and [13].

Throughout this paper the letters $\mathbb{N}$, I and $[I]$ will denote the set of all positive integers, the closed unit interval, i.e $I = [0, 1]$, and all interval numbers on I, i.e $[I] = \{\overline{a} = [a^-, a^+] : 0 \leq a^- \leq a^+ \leq 1\}$, respectively. If $a^- = a^+$, then the interval number $\overline{a}$ degenerates into an ordinary real number on I. Conversely, every $a \in I$ induces the interval number $[a, a]$ that we will denote as $\overline{a}$ if no confusion arises, so that we will write $(I] = [I] - \{\overline{0}\}$ and $(I) = [I] - \{\overline{0}, \overline{1}\}$.

Given $\overline{a}, \overline{b} \in [I]$ we will say that $\overline{a} \leq \overline{b}$ if $a^- \leq b^-$ and $a^+ \leq b^+$, $\overline{a} = \overline{b}$ if $a^- = b^-$ and $a^+ = b^+$ and $\overline{a} < \overline{b}$ if $\overline{a} \leq \overline{b}$ and $\overline{a} \neq \overline{b}$. It is obvious that $([I], \leq)$ is a partial ordered set.

For every $\overline{a}, \overline{b} \in [I]$ the following operations were introduced in [11]:
(i) $\overline{a} \wedge \overline{b} = [a^- \wedge b^-, a^+ \wedge b^+]$.
(ii) $\overline{a} \vee \overline{b} = [a^- \vee b^-, a^+ \vee b^+]$.
(iii) $\overline{a}^c = \overline{1} - \overline{a} = [1 - a^+, 1 - a^-]$.
In general (see [8]), given $\overline{a} = [a^-, a^+]$ and $\overline{b} = [b^-, b^+]$ we have $\overline{b} - \overline{a} = [b^- - a^+, b^+ - a^-]$ and $\overline{b} + \overline{a} = [b^- + a^-, b^+ + a^+]$.

Recall [10] that a t-norm is a binary operation $* : I \times I \to I$ that satisfies the following conditions:
(i) $*$ is associative and commutative,
(ii) $a * 1 = a$ for every $a \in I$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in I$.
If, in addition, $*$ is continuous, then $*$ is called a continuous t-norm.

Paradigmatic examples of continuous t-norms are the minimum, denoted by $\wedge$, the usual product, denoted by $\cdot$ and the Lukasiewicz t-norm, denoted by $*_L$, where $a *_L b = \max\{a + b - 1, 0\}$. They satisfy the following well-known inequalities: $a *_L b \leq a \cdot b \leq a \wedge b$. In fact, $a * b \leq a \wedge b$ for each t-norm $*$.

Our basic reference for continuous t-norms is [6].

Shen et al. extended in [11] the concept of t-norm to interval-valued fuzzy sets and defined the notion of interval-valued t-norm ($\mathcal{IV}$-t-norm for short) as follows:

Definition 1.1 ([11])**.** An $\mathcal{IV}$-t-norm is a binary operation $\overline{*} : [I] \times [I] \to [I]$ that satisfies the following conditions :
(i) $\overline{*}$ is associative and commutative,
(ii) $\overline{a} \,\overline{*}\, \overline{1} = \overline{a}$ and $\overline{a} \,\overline{*}\, I = [0, a^+]$ for every $\overline{a} = [a^-, a^+] \in [I]$,
(iii) $\overline{a} \,\overline{*}\, \overline{b} \leq \overline{c} \,\overline{*}\, \overline{d}$ whenever $\overline{a} \leq \overline{c}$ and $\overline{b} \leq \overline{d}$, for $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in [I]$.
If, in addition, $\overline{*}$ is continuous, then $\overline{*}$ is called a continuous $\mathcal{IV}$-t-norm.

Definition 1.2 ([11])**.** A sequence $\{\overline{a}_n\}_{n\in\mathbb{N}} = \{[a_n^-, a_n^+]\}_{n\in\mathbb{N}}$ of interval numbers converges to $\overline{a} = [a^-, a^+]$ if $\lim_{n\to\infty} a_n^- = a^-$ and $\lim_{n\to\infty} a_n^+ = a^+$. In this case, we write $\lim_{n\to\infty} \overline{a}_n = \overline{a}$ (or $\{\overline{a}_n\} \to \overline{a}$).

In [11, Definition 4] the authors define an $\mathcal{IV}$-t-norm $\overline{*}$ as continuous if it is continuous in its first component, i.e., if for each $\overline{b} \in [I]$ and $\lim_{n\to\infty} \overline{a}_n = \overline{a}$, then $\lim_{n\to\infty}(\overline{a}_n \,\overline{*}\, \overline{b}) = (\lim_{n\to\infty} \overline{a}_n \,\overline{*}\, \overline{b}) = \overline{a} \,\overline{*}\, \overline{b}$, where $\{\overline{a}_n\}_{n\in\mathbb{N}} \subseteq [I]$, $\overline{a} \subseteq [I]$. As in the case of continuous t-norms (see [6, Proposition 1.19]), the following proposition shows that the continuity of $\mathcal{IV}$-t-norms is equivalent to its continuity in the first component. As usually we say that $\overline{*} : [I] \times [I] \to [I]$ is continuous if for all convergent sequences $\{\overline{x}_n\}_{n\in\mathbb{N}}$, $\{\overline{y}_n\}_{n\in\mathbb{N}} \in [I]$ we have $\lim_{n\to\infty} \overline{x}_n \,\overline{*} \lim_{n\to\infty} \overline{y}_n = \lim_{n\to\infty} \overline{x}_n \,\overline{*}\, \overline{y}_n$.

Proposition 1.3. *An $\mathcal{IV}$-t-norm $\overline{*}$ is continuous if and only if it is continuous in its first component.*

Proof. If $\overline{*}$ is continuous, then it is obviously continuous in its firs component.

Conversely, if $\overline{*}$ is continuous in its first component, due to the commutativity of $\overline{*}$, then it is continuous in each component. Now fix $(\overline{x}_0, \overline{y}_0) \in [I] \times [I]$, $\overline{\varepsilon} = [\varepsilon, \varepsilon]$, $\varepsilon > 0$ and let $\{\overline{x}_n\}_{n\in\mathbb{N}}$ and $\{\overline{y}_n\}_{n\in\mathbb{N}}$ be sequences in $[I]$ converging to $\overline{x}_0$ and $\overline{y}_0$ respectively. From this, we construct the monotone sequences $\{\overline{a}_n\} \to \overline{x}_0$, $\{\overline{b}_n\} \to \overline{x}_0$, $\{\overline{c}_n\} \to \overline{y}_0$, $\{\overline{d}_n\} \to \overline{y}_0$ such that for all $n \in \mathbb{N}$ $\overline{a}_n \le \overline{x}_n \le \overline{b}_n$ and $\overline{c}_n \le \overline{y}_0 \le \overline{d}_n$. As $\overline{*}$ is continuous in its second component and by its monocyte ((iii), Definition 1.1), there exists an $N \in \mathbb{N}$ such that for all $n \ge N$,

$$\overline{x}_0 \,\overline{*}\, \overline{y}_0 - \overline{\varepsilon} < \overline{x}_0 \,\overline{*}\, \overline{c}_N \le \overline{x}_0 \,\overline{*}\, \overline{y}_n \le \overline{x}_0 \,\overline{*}\, \overline{d}_N < \overline{x}_0 \,\overline{*}\, \overline{y}_0 + \overline{\varepsilon}.$$

Since $\overline{*}$ is continuous in its first component, there exists $M \in \mathbb{N}$ such that for all $m \ge M$ and $n \ge N$, taking into account de monocyte of $\overline{*}$ we get:

$$\overline{x}_0 \,\overline{*}\, \overline{c}_N - \overline{\varepsilon} < \overline{a}_M \,\overline{*}\, \overline{c}_N \le \overline{x}_m \,\overline{*}\, \overline{y}_n \le \overline{b}_M \,\overline{*}\, \overline{d}_N < \overline{x}_0 \,\overline{*}\, \overline{d}_N + \overline{\varepsilon}.$$

Let $K = \max(M, N)$. Then for all $k \ge K$ we have:

$$\overline{x}_0 \,\overline{*}\, \overline{y}_0 - \overline{2\varepsilon} < \overline{x}_k \,\overline{*}\, \overline{y}_k < \overline{x}_0 \,\overline{*}\, \overline{y}_0 + \overline{2\varepsilon}.$$

Thus $\lim_{n\to\infty} \overline{x}_n \,\overline{*} \lim_{n\to\infty} \overline{y}_n = \lim_{n\to\infty} \overline{x}_n \,\overline{*}\, \overline{y}_n = \overline{x}_0 \,\overline{*}\, \overline{y}_0$ and $\overline{*}$ is continuous. □

Some examples of $\mathcal{IV}$-t-norms are:
(1) $\overline{a} \overline{\wedge} \overline{b} = [a^-, a^+] \overline{\wedge} [b^-, b^+] = [a^- \wedge b^-, a^+ \wedge b^+]$.
(2) $\overline{a} \,\overline{\cdot}\, \overline{b} = [a^-, a^+] \overline{\cdot} [b^-, b^+] = [a^- \cdot b^-, a^+ \cdot b^+]$.

Proposition 1.4. *Every $\mathcal{IV}-$t-norm $\overline{*}$ acts componentwise.*

Proof. Let $\overline{*}$ be an $\mathcal{IV}$-t-norm and $\overline{a}, \overline{b} \in [I]$. The second component of $\overline{a} \,\overline{*}\, \overline{b}$ coincides with the second component of $\overline{a} \,\overline{*}\, \overline{b} \overline{*} I \overline{*} I$, where

$$\begin{aligned} [a^-, a^+]\overline{*}[b^-, b^+]\overline{*}[0,1]\overline{*}[0,1] &= [a^-, a^+]\overline{*}[0,1]\overline{*}[b^-, b^+]\overline{*}[0,1] \\ &= [0, a^+]\overline{*}[0, b^+] \end{aligned}$$

which does not depend on a^- or b^-.

On the other hand, taking into account that every $\mathcal{IV}$-t-norm is distributive over $\vee$, we have:

$$\begin{aligned} [a^-, a^+]\overline{*}[b^-, b^+] &= [a^-, a^+]\overline{*}([b^-, b^-] \vee [0, b^+]) \\ &= [a^-, a^+]\overline{*}[b^-, b^-] \vee [a^-, a^+]\overline{*}[0, b^+] \\ &= [a^-, a^+]\overline{*}[b^-, b^-] \vee [0, c]. \end{aligned}$$

Thus the first component of $\overline{a} \,\overline{*}\, \overline{b}$ does not depend on b^+ (and similarly, by commutativity, does not depend on a^+). So $[a^-, a^+]\overline{*}[b^-, b^+] = [a^- *^- b^-, a^+ *^+ b^+]$. □

It is easy to see that $*^-$ and $*^+$ are two continuous t-norms such that $*^- \le *^+$. So, given an $\mathcal{IV}$-t-norm $\overline{*}$ we can write $\overline{*} = [*^-, *^+]$ were $*^-$ and $*^+$ are two continuous t-norms such that $*^- \le *^+$. In fact $\overline{\wedge} = [\wedge, \wedge]$ and $\overline{\cdot} = [\cdot, \cdot]$.

Following the ideas of interval-valued fuzzy set and continuous $\mathcal{IV}$-t-norm Y.Shen, H.Li and F.Wang introduced in [11] a notion of interval-valued fuzzy metric space (in the following $\mathcal{IV}$-fuzzy metric space) which is a generalization of fuzzy metric space in the sense of George and Veeramani [3] and they showed, as in the case of fuzzy metric spaces, that every $\mathcal{IV}$-fuzzy metric space generates a Hausdorff first countable topology. In the next section we show that every $\mathcal{IV}$-fuzzy metric space $(X, \overline{M}, \overline{*})$ induces two fuzzy metrics spaces $(X, M^-, *^-)$ and $(X, M^+, *^+)$ and that the topology $\tau_{\overline{M}}$ generated by the $\mathcal{IV}$-fuzzy metric space $(X, \overline{M}, \overline{*})$ coincides with the topology τ_{M^-} generated by the fuzzy metric space $(X, M^-, *^-)$, and thus, the results obtained in [11] are consequences of well-known results for fuzzy metric spaces.

2. Interval-valued fuzzy metric spaces

Recall [3] that a fuzzy metric space is a triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ such that for all $x, y, z \in X$; $t, s > 0$:
(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
(v) $M(x, y, \_) : (0, \infty) \to (0, 1]$ is continuous.
An alternative definition of fuzzy metric space can be found in [1].

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ (or simply M) is a fuzzy metric on X.

Our basic reference for general topology is [2].

George and Veeramani proved in [3] that every fuzzy metric $(M, *)$ on X generates a Hausdorff first countable topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, r, t) : x \in X,\ r \in (0, 1),\ t > 0\}$, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for all $x \in X$, $r \in (0, 1)$ and $t > 0$. Actually, the following result is proved in [4].

Theorem 2.1 ([4]). *Let $(X, M, *)$ be a fuzzy metric space. Then (X, τ_M) is a metrizable topological space.*

As a natural generalization of fuzzy metric space Y.Shen, H.Li and F.Wang gave in [11] the following definition of $\mathcal{IV}$-fuzzy metric space.

Definition 2.2 ([11]). An $\mathcal{IV}$-fuzzy metric space is a triple $(X, \overline{M}, \overline{*})$ such that X is a non-empty set, $\overline{*}$ is a continuous $\mathcal{IV}$ t-norm and $\overline{M}$ is a fuzzy set on $X \times X \times (0, \infty)$ such that for all $x, y, z \in X$; $t, s > 0$:
(i) $\overline{M}(x, y, t) > \overline{0}$,
(ii) $\overline{M}(x, y, t) = \overline{1}$ if and only if $x = y$,
(iii) $\overline{M}(x, y, t) = \overline{M}(y, x, t)$,
(iv) $\overline{M}(x, z, t + s) \geq \overline{M}(x, y, t) \overline{*} \overline{M}(y, z, s)$,
(v) $\overline{M}(x, y, \_) : (0, \infty) \to (I]$ is continuous,
(vi) $\lim_{t \to \infty} \overline{M}(x, y, t) = \overline{1}$.

In the previous definition $\overline{M} = [M^-, M^+]$ is called an interval-valued fuzzy metric on X ($\mathcal{IV}$-fuzzy metric in short). Following [11] the functions $M^-(x, y, t)$ and $M^+(x, y, t)$ can be interpreted as the lower nearness degree and the upper nearness degree between x and y with respect to t, respectively. This interpretation is consistent with the original one of $M(x, y, t)$ in the case of fuzzy metric spaces in the sense of [7] and [3] (see for instance [3, Remark 2.3]). Taking into account that an interesting class of fuzzy metric spaces were defined in [5] where M does not depend on t and that the topology generated by a ($\mathcal{IV}$-)fuzzy metric space can be defined having $t \in (0, \varepsilon)$, $\varepsilon > 0$, to our purposes here we are going to consider a more general definition of $(X, \overline{M}, \overline{*})$ without condition (f). In fact, the equivalent condition is not considered in the original definition of fuzzy metric space given by George and Veeramani.

Conditions in Definition 2.2 together with Proposition 1.4, where $\overline{*} = [*^-, *^+]$, imply that $(X, M^-, *^-)$ and $(X, M^+, *^+)$ are fuzzy metric spaces.

In [11] the authors proved that each $\mathcal{IV}$-fuzzy metric $\overline{M}$ on X generates a Hausdorff first countable topology $\tau_{\overline{M}}$ on X which has as a base the family of open sets of the form $\{B_{\overline{M}}(x, \overline{r}, t) : x \in X, \overline{0} < \overline{r} < \overline{1}, t > 0\}$, where $B_{\overline{M}}(x, \overline{r}, t) = \{y \in X : \overline{M}(x, y, t) > \overline{1} - \overline{r}\}$ for all $x \in X$, $\overline{0} < \overline{r} < \overline{1}$ and $t > 0$.

Proposition 2.3. *Let $(X, \overline{M}, \overline{*}) = (X, [M^-, M^+], [*^-, *^+])$ be an $\mathcal{IV}$-fuzzy metric space. Then, for each $x \in X$, $r \in (0, 1)$, $t > 0$ we have $B_{\overline{M}}(x, \overline{r}, t) = B_{M^-}(x, r, t)$.*

Proof. If $y \in B_{\overline{M}}(x, \overline{r}, t)$, then $\overline{M}(x, y, t) = [M^-(x, y, t), M^+(x, y, t)] > \overline{1} - \overline{r}$. Thus $M^-(x, y, t) > 1 - r$ and $y \in B_{M^-}(x, r, t)$. Now suppose that $y \in B_{M^-}(x, r, t)$. Then $M^-(x, y, t) > 1 - r$. Since $M^+(x, y, t) \geq M^-(x, y, t) > 1 - r$, we have $\overline{M}(x, y, t) = [M^-(x, y, t), M^+(x, y, t)] > \overline{1} - \overline{r}$. So $y \in B_{\overline{M}}(x, \overline{r}, t)$. □

From Proposition 2.3 we deduce the following.

Theorem 2.4. *Let $(X, \overline{M}, \overline{*}) = (X, [M^-, M^+], [*^-, *^+])$ be an $\mathcal{IV}$-fuzzy metric space. Then the topologies $\tau_{\overline{M}}$ and τ_{M^-} coincide on X.*

By Theorem 2.1 and Theorem 2.4 we obtain the following improvement of Theorem 5 of [11].

Corollary 2.5. *Let $(X, \overline{M}, \overline{*}) = (X, [M^-, M^+], [*^-, *^+])$ be an $\mathcal{IV}$-fuzzy metric space. Then $(X, \tau_{\overline{M}})$ is a metrizable topological space.*

Acknowledgements. The authors thank to referees for their comments which have helped in improving the quality of the paper. The second author thanks the support of Ministry of Economy and Competitiveness of Spain, grant MTM2015-64373-P

References

[1] M. Adabitabar Firozja and S. Firouzian, Definition of fuzzy metric spaces with t-conorms, Ann. Fuzzy Math. Inform. 10 (2015) 649–655.
[2] R. Engelking, General Topology, PWN-Polish Sci.Publ., Warsaw 1977.

[3] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.
[4] V. Gregori and S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets and Systems 115 (2000) 485–489.
[5] V. Gregori and S. Romaguera, Characterizing completable fuzzy metric spaces, Fuzzy Sets and Systems 114 (2004) 411–420.
[6] E. Klement, R. Mesiar and E. Pap, Triangular Norms, Kluwer Academic Publishers 2000.
[7] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975) 326–334.
[8] R. Moore, Interval Analysis. Pretince-Hall, Englewood Cliffs 1996.
[9] A. Mukherjee and M. Datta, Rough interval-valued intuitionistic fuzzy sets, Ann. Fuzzy Math. Inform. 9 (2015) 111–123.
[10] B. Schweizer, A. Sklar, Satatistical metric spaces, Pacific J. Math. 10 (1960) 314–334.
[11] Y. Shen, H. Li and F. Wang, On Interval-Valued Fuzzy Metric Spaces, Int. J. Fuzzy Syst. 14 (2012) 35–44.
[12] L. A. Zadeh, The concept of a linguistic variable and its application to approximation reasoning I, Inform. Sci. 8 (1975) 199–249.
[13] G. Zhang and G. Yu, IVF topologies based on IVF relations, Ann. Fuzzy Math. Inform. 10 (2015) 619–629.

Francisco Castro-Company (fracasco@mat.upv.es)
Gilmation S.L., Calle 232, 66. Paterna, 46182 Spain

Pedro Tirado (pedtipe@mat.upv.es)
Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022 Valencia, Spain