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Topological structure of covering rough sets

Shanshan Wang, Zhaohao Wang

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Abstract. In this study, we induce some topological structures in the covering rough set models, and construct their closure operators by using the covering upper approximation operators. Furthermore, we show that the minimum set of each of these topological structures is their base, and a partition on the universe of discourse. Finally, we discuss the relationships between some topologies generated by some approximation operators and unary covering.

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Corresponding Author: Shanshan Wang (wss26862@163.com)

1. Introduction

Rough set theory is an extension of set theory to deal with vagueness and uncertainty of imprecise data. It was firstly proposed by Pawlak [15]. This theory has been widely applied to feature selection, decision analysis, pattern recognition and knowledge discovery in databases [13, 14, 19]. In Pawlak's rough set theory, equivalence relation is the foundation of its object classification, the lower and upper approximation operators are two core notions. In real word databases, equivalence relations is not suitable in several situations because it can only deal with complete data [13, 15, 17, 27]. So from the angle of applications, it is important to extend equivalence relation to some other relations, such as arbitrary binary relation [9, 16, 21], fuzzy relation [1, 8, 12] and coverings [1, 2, 3, 11, 18, 26].

Topology theory has independent theoretical framework, which is an important mathematical tool for the study of information systems [5]. Such theory played an important role and presented various research perspectives in rough set theory study. Hence, the combination of the rough set theory and topology theory have obtained a lot of attention [4, 6, 11, 18, 20, 24, 28, 31, 32]. Zakowski used coverings instead of partitions to generate the classical rough set theory [26]. Such generalization leads to various covering approximation operators, then the combination of covering rough set and topology have both theoretical and practical important [5, 10, 16, 21, 22, 25, 30]. In [34],Zhao constructed a type of topology called the topology induced by the covering on a covering approximation space, and investigated the topological properties of the space such as separation, connectedness. In [30], Zhu explored the topological properties of rough sets and constructed axiomatic systems for the lower approximation operation and the upper approximation operation. In [31, 33], Zhu and Wang discussed the relationship among four types of covering rough sets by using properties of covering upper approximation operators. These four operators are denoted by FH, SH, TH and RH respectively. Meanwhile, characterizations of the closure operators by the covering upper approximation operators was studied. In [5], Ge et al. discussed the problems on characterizations of coverings for upper approximation operators being closure operators in depth. Specially, the general, topological and intuitive characterizations of coverings for these operators being closure operators were given. These researchers focused on exploring the characterizations of covering for covering-based upper approximation operators to be closure operators [5, 29, 31], but they did not investigated the corresponding topological structures. In this paper, we will induce some topological structures in the covering-based rough set models in order to combine the topology with the covering rough sets.

Pawlak [15] indicated that

$$T = \{X \subseteq U | \overline{R}(X) = X\}. \tag{1.1}$$

is a clopen topology on U, which is induced by equivalence relation R, and Pawlak rough upper approximation operator $\overline{R}$ is the closure operator of topology T. Generally, if we replace $\overline{R}$ with covering upper approximation operators FH, SH, TH and RH, respectively, in the formula (1.1), we can obtain four types of sets. Some natural questions thus arise: Is it possible for each of the four sets to be a topology? If the answer is"yes", then what are the conditions under which covering-based upper approximation operator to be closure operator? In this study, we will discuss these problems.

The rest of the paper is organized as follows. In Section 2, some basic concepts and results about covering rough sets and topology are reviewed. In Section 3, we investigate some topological structures in covering approximation spaces. Conclusion is given in Section 4.

2. Preliminaries

In this section, we review some fundamental concepts related to covering rough sets and topology.

In the following discussion, we always assume that the universe of discourse U is finite. $P(U)$ is the family of all subsets of U. For every $X \subseteq U$, $\sim X$ denotes the complement set of X in U, i.e., $\sim X = U \setminus X$.

Definition 2.1 (Covering [29]). Let U be a universe and $\mathcal{C}$ is a family of subsets of U. If none of subsets in $\mathcal{C}$ is empty, and $\cup\mathcal{C} = U$, then $\mathcal{C}$ is called a covering of U.

The ordered pair $(U, \mathcal{C})$ is called a covering approximation space.

Definition 2.2 ($Md(x)$, $Friends(x)$ and $CFriends(x)$ [2, 29, 31]). Let $\mathcal{C}$ be a covering of U, $x \in U$. Denote

$Md(x) = \{C \in \mathcal{C} | x \in C \wedge (\forall S \in \mathcal{C} \wedge x \in S \subseteq C \Rightarrow S)\}$,

$Friends(x) = \bigcup\{C | x \in C \in \mathcal{C}\}$,

$CFriends(x) = \bigcup Md(x)$.

We call $Md(x)$, $Friends(x)$ and $CFriends(x)$ are the minimal description of x, the indiscernible neighborhood of x and the closed friends of x, respectively.

Definition 2.3 (Unary [29]). Let $\mathcal{C}$ be a covering of U, $x \in U$. If $|Md(x)| = 1$, $\mathcal{C}$ is called unary.

Let $(U, \mathcal{C})$ be a covering-based approximation space. In this paper, we will discuss the following types of covering approximation operators.

Definition 2.4 (Covering lower approximation operators $CL_{\mathcal{C}}$ [26]). Let $\mathcal{C}$ be a covering of U. The lower approximation operators $CL_{\mathcal{C}}$ is defined as follows:

$$\forall X \subseteq P(U),\ CL_{\mathcal{C}}(X) = \bigcup\{C \in \mathcal{C} \mid C \subseteq X\}.$$

Definition 2.5 (Covering upper approximation operators $FH_{\mathcal{C}}$ [31], $SH_{\mathcal{C}}$ [31], $TH_{\mathcal{C}}$ [31] and $RH_{\mathcal{C}}$ [29]). Let $\mathcal{C}$ be a covering of U. Operators $FH_{\mathcal{C}}$, $SH_{\mathcal{C}}$, $TH_{\mathcal{C}}$, $RH_{\mathcal{C}}$: $P(U) \to P(U)$ are defined as follows: $\forall X \subseteq P(U)$,

$$\begin{aligned} FH_{\mathcal{C}}(X) =& CL_{\mathcal{C}}(X) \cup (\bigcup\{\bigcup Md(x) : x \in (X - CL_{\mathcal{C}}(X))\}) \\ =& CL_{\mathcal{C}}(X) \cup (\bigcup\{CFriends(x) : x \in (X - CL_{\mathcal{C}}(X))\}), \end{aligned}$$

$$SH_{\mathcal{C}}(X) = \bigcup\{C \in \mathcal{C} \mid C \cap X \neq \varnothing\} = \bigcup\{Friends(x) : x \in X\},$$

$$TH_{\mathcal{C}}(X) = \bigcup\{\bigcup Md(x) : x \in X\} = \bigcup\{CFriends(x) : x \in X\},$$

$$\begin{aligned} RH_{\mathcal{C}}(X) =& CL_{\mathcal{C}}(X) \cup (\bigcup\{C \in \mathcal{C} : C \cap (X - CL_{\mathcal{C}}(X)) \neq \varnothing\}) \\ =& CL_{\mathcal{C}}(X) \cup (\bigcup\{Friends(x) : x \in (X - CL_{\mathcal{C}}(X))\}). \end{aligned}$$

The basic concepts of topology have been widely used in many areas. The following topological concepts and facts are elementary and we list them below to facilitate our discussion. For more details, we refer to [7, 16, 23].

Definition 2.6 (Topology [25]). Let U be a non-empty set. Let T be a family of subsets of U, which satisfies the three conditions:

(T1) $\varnothing, U \in T$,

(T2) If $A, B \in T$, then $A \cap B \in T$,

(T3) If $\mathcal{A} \subseteq T$, then $\bigcup_{A \in \mathcal{A}} A \in T$.

Then we call T is a topology on U. The pair (U, T) is called a topological space.

The members of the topology T are called open set, and a subset of U is called closed if its complement is open set. A topology T is called a clopen topology if every open set is also closed.

A subset X in a topological space (U, T) is a neighborhood of a point $x \in U$ if X contains an open set to which x belongs.

Definition 2.7 (Closure operator [16, 23]). An operator Cl on $P(U)$ is a closure operator on U, if Cl satisfies the following conditions: $\forall A, B \subseteq U$,
(C1) $Cl(\varnothing) = \varnothing$,
(C2) $A \subseteq Cl(A)$,
(C3) $Cl(Cl(A)) = Cl(A)$,
(C4) $Cl(A \cup B) = Cl(A) \cup Cl(B)$.

In a topological space (U, T). A family $\mathcal{B} \subseteq T$ is called a base for T if every non-empty open subset of T can be represented as union of a subfamily of $\mathcal{B}$.

Theorem 2.8 ([21]). *Let (U, T) be a topological space. A family $\mathcal{B} \subseteq T$ is a base for T if and only if for each point x of the space, and each neighborhood X of x, there is a member V of $\mathcal{B}$ such that $x \in V \subseteq X$.*

Definition 2.9 (Topological covering [20]). Let T is a topology on U. $T \setminus \{\varnothing\}$ is a covering, then we call $T \setminus \{\varnothing\}$ is a topological covering, and $(U, T \setminus \{\varnothing\})$ is called a topological covering approximation space.

3. The topological structure induced by covering-based upper approximation operators

In this section, we investigate topological structures induced by covering-based upper approximation operators $FH_{\mathcal{C}}$, $SH_{\mathcal{C}}$, $TH_{\mathcal{C}}$ and $RH_{\mathcal{C}}$ respectively. In particular, topological structure induced by $SH_{\mathcal{C}}$ is discussed in depth.

3.1. The topological structure induced by $SH_{\mathcal{C}}$.

According to Definition 2.5 and the formula (1.1), we introduce a topological structure as follows.

$$T_{S\mathcal{C}} = \{X \subseteq U \mid SH_{\mathcal{C}}(X) = X\}. \tag{3.1}$$

Definition 3.1 (The minimal set [33]). Let $\mathcal{A} \subseteq \mathcal{P}(U)$. One can denote

$$Min(\mathcal{A}) = \{X \in \mathcal{A} \mid \forall Y \in \mathcal{A}, if Y \subseteq X, then X = Y\}.$$

Proposition 3.2. *Let $(U, \mathcal{C})$ be a covering-based approximation space. Then $T_{S\mathcal{C}}$ is a topology on U, where $T_{S\mathcal{C}}$ is defined by the formula (3.1).*

Proof. We only need to prove that $T_{S\mathcal{C}}$ satisfies the conditions in Definition 2.6.
(T1) It is easy to check that $\varnothing, U \in T_{S\mathcal{C}}$.
(T2) Let $X, Y \in T_{\mathcal{C}}$. By Proposition 6 of [30], $SH_{\mathcal{C}}(X \cap Y) \subseteq SH_{\mathcal{C}}(X) \cap SH_{\mathcal{C}}(Y) = X \cap Y$ and $X \cap Y \subseteq SH_{\mathcal{C}}(X \cap Y)$. Thus, we have $X \cap Y \in T_{S\mathcal{C}}$.
(T3) Let $X, Y \in T_{\mathcal{C}}$. By Proposition 6 of [30], $SH_{\mathcal{C}}(X \cup Y) = SH_{\mathcal{C}}(X) \cup SH_{\mathcal{C}}(Y) = X \cup Y$. Thus, we have $X \cup Y \in T_{S\mathcal{C}}$. So, $T_{S\mathcal{C}}$ is a topology on U. □

Proposition 3.2 guarantees that $T_{S\mathcal{C}}$ is a topology for any covering $\mathcal{C}$ of U. Moreover, the following results illustrates that $T_{S\mathcal{C}}$ is a clopen topology.

Proposition 3.3. *Let $(U, \mathcal{C})$ be a covering-based approximation space. If $X \in T_{S\mathcal{C}}$, then $\sim X \in T_{S\mathcal{C}}$.*

Proof. By the formula (3.1), we only need to prove that $SH_{\mathcal{C}}(\sim X) =\sim X$. Obviously, $\sim X \subseteq SH_{\mathcal{C}}(\sim X)$. If we assume that $\sim X \subsetneqq SH_{\mathcal{C}}(\sim X)$, then there is $t \in U$ such that $t \in SH_{\mathcal{C}}(\sim X)$ and $t \notin \sim X$. Since $t \in SH_{\mathcal{C}}(\sim X)$, there exists $C \in \mathcal{C}$ such that $t \in C$ and $C \cap (\sim X) \neq \varnothing$. By $t \notin \sim X$, $t \in X$. Thus, $t \in C \cap X$, i.e., $C \cap X \neq \varnothing$. So, $C \subseteq SH_{\mathcal{C}}(X) = X$, which means that $C \cap (\sim X) = \varnothing$. This contradicts $C \cap (\sim X) \neq \varnothing$. Hence, $SH_{\mathcal{C}}(\sim X) =\sim X$. Therefore, $\sim X \in T_{S\mathcal{C}}$. □

Corollary 3.4. *Let $(U, \mathcal{C})$ be a covering-based approximation space. If $X, Y \in T_{S\mathcal{C}}$ and $Y \subseteq X$, then $X - Y \in T_{S\mathcal{C}}$.*

Proof. Since $Y \in T_{S\mathcal{C}}$, by Proposition 3.3, $\sim Y \in T_{S\mathcal{C}}$. By Proposition 3.2, $T_{S\mathcal{C}}$ is a topology. Thus $X \cap (\sim Y) \in T_{S\mathcal{C}}$, i.e., $X - Y \in T_{S\mathcal{C}}$. □

Proposition 3.5. *Let $(U, \mathcal{C})$ be a covering-based approximation space. For any $x \in U$, there is an $X_i \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ such that $x \in X_i$.*

Proof. Let $SH_{\mathcal{C}}(\{x\}) = X_1$, $SH_{\mathcal{C}}(X_1) = X_2$, $SH_{\mathcal{C}}(X_2) = X_3$, ..., $SH_{\mathcal{C}}(X_{n-1}) = X_n$. For every $X \subseteq U$, $X \subseteq SH_{\mathcal{C}}(X)$. Then $\{x\} \subseteq X_1 \subseteq X_2 \subseteq ... \subseteq X_n$. Since U is finite, there exists a $t \in U$ such that $X_t = X_{t+1} = X_{t+2} = ... = X_n$. Thus $SH_{\mathcal{C}}(X_t) = X_{t+1} = X_t$. So, we prove that $x \in X_t$ and $X_t \in T_{S\mathcal{C}}$.

Take X_t as the minimum set of $T_{S\mathcal{C}}$, which contains x. Then we only need to prove that $X_t \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$. If $X_t \notin Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$, then there exists $Y \subsetneqq X_t$ such that $Y \in T_{S\mathcal{C}} \setminus \{\varnothing\}$. Since X_t is the minimum set of $T_{S\mathcal{C}}$, which contains x, then $x \notin Y$, and it follows that $x \in X_t - Y$. By Corollary 3.4, $X_t - Y \in T_{S\mathcal{C}}$. It is obvious that $x \in X_t - Y \subsetneqq X_t$, which contradicts the fact that X_t is the minimum set of $T_{S\mathcal{C}}$, which contains x. Hence, $X_t \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$. □

Proposition 3.6. *Let $(U, \mathcal{C})$ be a covering-based approximation space. Then $Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ is a partition on U.*

Proof. According to Proposition 3.5, it is evident that $U \subseteq \bigcup Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$. Obviously, $\bigcup Min(T_{S\mathcal{C}} \setminus \{\varnothing\}) \subseteq U$. Then, we have $\bigcup Min(T_{S\mathcal{C}} \setminus \{\varnothing\}) = U$. Now, we start to prove that if $X, Y \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ and $X \cap Y \neq \varnothing$, then $X = Y$. Assume that $X \neq Y$. Then $X \cap Y \subsetneqq X$ or $X \subsetneqq Y$. If $X, Y \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$, then $X, Y \in T_{S\mathcal{C}}$. Thus, by Proposition 3.2, $X \cap Y \in T_{S\mathcal{C}}$. Furthermore, by $X \cap Y \neq \varnothing$, $X \cap Y \in T_{S\mathcal{C}} \setminus \{\varnothing\}$. If $X \cap Y \subsetneqq X$, by Definition 3.1, $X \notin Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$, which contradicts $X \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$. If $X \subsetneqq Y$, by Definition 3.1, $Y \notin Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$, which contradicts $Y \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$. These contradictions show that if $X \cap Y \neq \varnothing$, we have $X = Y$. So, we prove that $Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ is a partition on U. □

The following result illustrates that $Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ is a base for topology $T_{S\mathcal{C}}$.

Proposition 3.7. *Let $(U, \mathcal{C})$ be a covering-based approximation space. Then $Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ is a base for topology space $(U, T_{S\mathcal{C}})$.*

Proof. Obviously, $Min(T_{S\mathcal{C}} \setminus \{\varnothing\}) \subseteq T_{S\mathcal{C}}$. $\forall X \in T_{S\mathcal{C}}$ and $x \in X$, since $Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ is a partition of U, there exists $Y \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ such that $x \in Y$. Thus $x \in X \cap Y$ and $Y \in T_{S\mathcal{C}}$. By Definition 3.1, we find that $x \in Y \subseteq X$. So, by Theorem 2.8, $Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ is a base for $(U, T_{S\mathcal{C}})$. □

Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two different coverings of U. If $SH_{\mathcal{C}_1}$ and $SH_{\mathcal{C}_2}$ generate the same topology on U, that is, $T_{S\mathcal{C}_1} = T_{S\mathcal{C}_2}$, then what is the relationship between $\mathcal{C}_1$ and $\mathcal{C}_2$?

In order to investigate this problem, we give the following example.

Example 3.1. Let $U = \{a, b, c, d\}, \mathcal{C}_1 = \{\{a, b\}, \{b, c\}, \{d\}\}, \mathcal{C}_2 = \{\{a, b\}, \{a, c\}, \{d\}\}$. By the formula (3.1), $T_{S\mathcal{C}_1} = T_{S\mathcal{C}_2} = \{\varnothing, \{a, b, c\}, \{d\}, U\}$. By Definition 3.1, we have $Min(T_{S\mathcal{C}_1} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}_2} \setminus \{\varnothing\}) = \{\{a, b, c\}, \{d\}\}$.

Proposition 3.8. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two coverings of U. $T_{S\mathcal{C}_1} = T_{S\mathcal{C}_2}$ if and only if $Min(T_{S\mathcal{C}_1} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}_2} \setminus \{\varnothing\})$.*

Proof. The necessity holds obviously. Now we consider the sufficient condition. $\forall X \in T_{S\mathcal{C}_1} \setminus \{\varnothing\}$, by Proposition 3.7, there exist $X_1, X_2, \ldots, X_k \in Min(T_{S\mathcal{C}_1} \setminus \{\varnothing\})$ such that $X = \bigcup_{i=1}^{k} X_i (1 \leq k \leq n)$. Since $Min(T_{S\mathcal{C}_1} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}_2} \setminus \{\varnothing\})$, $X_1, X_2, \ldots, X_k \in Min(T_{S\mathcal{C}_2} \setminus \{\varnothing\})$, which means that $X \in T_{S\mathcal{C}_2} \setminus \{\varnothing\}$. Thus, $T_{S\mathcal{C}_1} \setminus \{\varnothing\} \subseteq T_{S\mathcal{C}_2} \setminus \{\varnothing\}$. Similarly, we can prove that $T_{S\mathcal{C}_2} \setminus \{\varnothing\} \subseteq T_{S\mathcal{C}_1} \setminus \{\varnothing\}$. So, $T_{S\mathcal{C}_1} = T_{S\mathcal{C}_2}$. □

Proposition 3.8 shows a necessary and sufficient condition under which different coverings generate the same topology. Based on this conclusion, we define the family $[\mathcal{C}]$ as follows:

$$[\mathcal{C}] = \{\mathcal{C}_i \mid T_{S\mathcal{C}_i} = T_{S\mathcal{C}}, \forall C \in \mathcal{C}_i, C \neq \varnothing\}. \tag{3.2}$$

Proposition 3.9. *Let $(U, \mathcal{C})$ be a covering-based approximation space. For any $\mathcal{C}_i \in [\mathcal{C}]$, $\mathcal{C}_i$ is a covering of U.*

Proof. Since $\mathcal{C}$ is a covering of U, by Proposition 3.2, $T_{S\mathcal{C}}$ is a topology, and it follows that $U \in T_{S\mathcal{C}}$. Since $\mathcal{C}_i \in [\mathcal{C}]$, $T_{S\mathcal{C}_i} = T_{S\mathcal{C}}$, so $U \in T_{S\mathcal{C}_i}$, which implies $SH_{\mathcal{C}_i}(U) = \bigcup\{C \in \mathcal{C}_i | C \cap U \neq \varnothing\} = U$. Thus, $\bigcup_{C \in \mathcal{C}_i} C = U$. So, by the definitions of covering and $[\mathcal{C}]$, $\mathcal{C}_i$ is a covering of U. □

Combining with Propositions 3.8, 3.9 and the formula (3.2), we have the following conclusion:

Proposition 3.10. *Let $(U, \mathcal{C})$ be a covering-based approximation space. For any $\mathcal{C}_i \in [\mathcal{C}]$, then $Min(T_{S\mathcal{C}_i} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$.*

Proposition 3.10 guarantees that if two different covering generate the same topology, then the minimum sets of their topological coverings are the same partition on U.

Proposition 3.11. *Let $(U, \mathcal{C})$ be a covering-based approximation space. For any $\mathcal{C}_1$, $\mathcal{C}_2 \in [\mathcal{C}]$, then $\mathcal{C}_1 \cup \mathcal{C}_2 \in [\mathcal{C}]$.*

Proof. In order to prove this conclusion, by the formula (3.2) and Proposition 3.8, it is enough to prove that $Min(T_{S(\mathcal{C}_1 \cup \mathcal{C}_2)} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$. $\forall \mathcal{C}_1, \mathcal{C}_2 \in [\mathcal{C}]$, by Proposition 3.10, we have $Min(T_{S\mathcal{C}_1} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}_2} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$. Since $Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ is a partition,

$$\begin{aligned} Min(T_{S\mathcal{C}} \setminus \{\varnothing\}) &= Min(T_{S\mathcal{C}_1} \setminus \{\varnothing\}) \subseteq Min(T_{S(\mathcal{C}_1 \cup \mathcal{C}_2)} \setminus \{\varnothing\}) \\ &\subseteq Min(T_{S\mathcal{C}_1} \setminus \{\varnothing\}) \cup Min(T_{S\mathcal{C}_2} \setminus \{\varnothing\}) \\ &= Min(T_{S\mathcal{C}} \setminus \{\varnothing\}). \end{aligned}$$

Therefore, $Min(T_{S(\mathcal{C}_1 \cup \mathcal{C}_2)} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$, which follows that $T_{S(\mathcal{C}_1 \cup \mathcal{C}_2)} = T_{S\mathcal{C}}$. Thus, by the definition of $[\mathcal{C}]$, we have $\mathcal{C}_1 \cup \mathcal{C}_2 \in [\mathcal{C}]$. □

Proposition 3.11 illustrates that $[\mathcal{C}]$ is additive. However, $[\mathcal{C}]$ is not multiplicative. The following example demonstrates this point.

Example 3.2. Let $U = \{a, b, c, d, \}$, $\mathcal{C} = \{\{a, b\}, \{b, c\}, \{a, c\}, \{d\}\}$, $\mathcal{C}_1 = \{\{a, b\}, \{b, c\}, \{d\}\}$ and $\mathcal{C}_2 = \{\{a, b\}, \{a, c\}, \{d\}\}$. We have $\mathcal{C}_1 \cap \mathcal{C}_2 = \{\{a, b\}, \{d\}\}$. Obviously, $\mathcal{C}_1 \cap \mathcal{C}_2$ is not a covering of U. Therefore, by Proposition 3.9, $\mathcal{C}_1 \cap \mathcal{C}_2 \notin [\mathcal{C}]$.

By Proposition 3.2, for any covering $\mathcal{C}$ of U, $T_{S\mathcal{C}}$ is a topology. However, $SH_{\mathcal{C}}$ may not be the closure operator of $T_{S\mathcal{C}}$. The following example demonstrates this point.

Example 3.3. Let $U = \{a, b, c, d\}$ and $\mathcal{C} = \{\{a, b\}, \{b, c\}, \{d\}\}$. We have $SH_{\mathcal{C}}(\{a\}) = \{a, b\}$ and $SH_{\mathcal{C}}(\{a, b\}) = \{a, b, c\}$. Obviously, $SH_{\mathcal{C}}(SH_{\mathcal{C}}(\{a\})) \neq SH_{\mathcal{C}}(\{a\})$. By Definition 2.7, $SH_{\mathcal{C}}$ is not a closure operator.

It is an interesting question of whether we can give a covering $\mathcal{C}_0$ induced by covering $\mathcal{C}$, such that $SH_{\mathcal{C}_0}$ is the closure operator of $T_{S\mathcal{C}}$? Furthermore, what's the relationship between $\mathcal{C}_0$ and $\mathcal{C}$? In the following section, we will discuss these problems.

We denote:

$$\mathcal{C}_0 = \bigcup [\mathcal{C}]. \tag{3.3}$$

$$\mathcal{C}' = \bigcup \{\mathcal{P}(X) \setminus \{\varnothing\} | X \in Min(T_{\mathcal{C}} \setminus \{\varnothing\})\}. \tag{3.4}$$

Obviously, $\mathcal{C}_0$ and $\mathcal{C}'$ are coverings of U. It is easy to verify that $\mathcal{C}_0 \in [\mathcal{C}]$. Then, we will discuss the relationship between $\mathcal{C}_0$ and $\mathcal{C}'$.

In [5], the intuitive characterization for $SH_{\mathcal{C}}$ to be a closure operator was given.

Definition 3.12 (Triangle Chain with three points [5]). Let $\mathcal{C}$ be a covering of U. $\forall x, y, z \in U$, if either there exist $C_1, C_2, C_3 \in \mathcal{C}$, such that $x, y \in C_1, y, z \in C_2$ and $z, x \in C_3$, then we say that there is a triangle chain with three points x, y, z.

Theorem 3.13 (Triangle Chain condition [5]). *$SH_{\mathcal{C}}$ is a closure operator if and only if covering $\mathcal{C}$ divides U into disjoint parts $U_1, U_2, \ldots, U_n$ on U, such that for each $U_i (1 \leq i \leq n)$ and $\forall x, y, z \in U_i$, there is a Triangle Chain with three points x, y, z.*

The description in Theorem 3.13 is illustrated by the following example.

Example 3.3. Let $\mathcal{C} = \{\{a, b\}, \{b, c\}, \{a, c\}, \{d, e, f\}, \{e, f, g\}, \{d, g\}\}$ and $U = \{a, b, c, d, e, f, g\}$. Covering $\mathcal{C}$ divides U into two disjoint parts $U_1 = \{a, b, c\}$ and $U_2 = \{d, e, f, g\}$. It is obvious that points of U_1 or U_2 satisfy the Triangle Chain condition.

In order to explore the relationship between $\mathcal{C}_0$ and $\mathcal{C}'$, we prove the following Lemmas first.

Lemma 3.14. *Let $\mathcal{C}$ be a covering of U. For any $X \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$, then $SH_{\mathcal{C}'}(X) = X$, where $\mathcal{C}'$ is defined by the formula (3.4).*

Proof. $\forall x \in SH_{\mathcal{C}'}(X)$, there exists $C \in \mathcal{C}'$ such that $x \in C$ and $C \cap X \neq \varnothing$. Since $C \in \mathcal{C}'$, by the formula(3.4), there exists $Y \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ such that $C \in \mathcal{P}(Y) \setminus \{\varnothing\}$, which implies that $C \subseteq Y$. In order to prove $SH_{\mathcal{C}'}(X) \subseteq X$, it is enough to verify $C \subseteq X$. Assume that $C \nsubseteq X$. Then $X \neq Y$. Due to $C \cap X \neq \varnothing$ and $C \subseteq Y$, then $Y \cap X \neq \varnothing$, which contradicts the fact that $Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ is a partition on U. Thus, we have $C \subseteq X$, and it follows that $x \in X$. So, $SH_{\mathcal{C}'}(X) \subseteq X$. It is obvious that $X \subseteq SH_{\mathcal{C}'}(X)$. Hence, $SH_{\mathcal{C}'}(X) = X$. □

Lemma 3.15. *Let $\mathcal{C}$ be a covering of U, for any $\mathcal{C}_i \in [\mathcal{C}]$ and $C \in \mathcal{C}_i$. Then there exists an $X \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ such that $C \subseteq X$.*

Proof. Assume the contrary. If there exists $\mathcal{C}_i \in [\mathcal{C}]$ and $C \in \mathcal{C}_i$, $\forall X \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$, we have $C \nsubseteq X$. Since $\mathcal{C}_i \in [\mathcal{C}]$, by Proposition 3.10, $Min(T_{S\mathcal{C}_i} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$, and it follows that $\forall X \in Min(T_{S\mathcal{C}_i} \setminus \{\varnothing\})$, we have $C \nsubseteq X$. Since $Min(T_{S\mathcal{C}_i} \setminus \{\varnothing\})$ is a partition on U, there exist $Y, Z \in Min(T_{S\mathcal{C}_i} \setminus \{\varnothing\})$ and $Y \neq Z$ such that $C \cap Y \neq \varnothing$ and $C \cap Z \neq \varnothing$. By Definition 2.5, we find that $C \subseteq SH_{\mathcal{C}_i}(Y)$ and $C \subseteq SH_{\mathcal{C}_i}(Z)$. It follows that $C \subseteq SH_{\mathcal{C}_i}(Y) \cap SH_{\mathcal{C}_i}(Z) = Y \cap Z = \varnothing$, which contradicts to the definition of covering. □

Based on the above lemmas, we can prove the following result.

Proposition 3.16. *Let $(U, \mathcal{C})$ be a covering-based approximation space. Then*

$$\mathcal{C}_0 = \bigcup\{\mathcal{P}(X) \setminus \{\varnothing\} | X \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})\}.$$

Proof. To prove the conclusion, it is enough to prove $\mathcal{C}' = \mathcal{C}_0$. Firstly, we prove $\mathcal{C}' \subseteq \mathcal{C}_0$, by the formula (3.3), which is enough to prove $\mathcal{C}' \in [\mathcal{C}]$. Then, by Proposition 3.10, we only need to prove $Min(T_{S\mathcal{C}'} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$.

$\forall X \in Min(T_{S\mathcal{C}'} \setminus \{\varnothing\})$, we have $SH_{\mathcal{C}'}(X) = X$. Since $Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ is a partition on U, there exists $Y \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ such that $X \cap Y \neq \varnothing$. We can take an $x \in X \cap Y$, by Proposition 6 of [30], $SH_{\mathcal{C}'}(\{x\}) \subseteq SH_{\mathcal{C}'}(X) = X$. Since $Y \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$, by the formula (3.4), $Y \in \mathcal{C}'$. Meanwhile, we have $x \in Y$, by the definition of SH, $Y \subseteq SH_{\mathcal{C}'}(\{x\})$, and it follows that $Y \subseteq X$. By Lemma 3.14, $SH_{\mathcal{C}'}(Y) = Y$. So $Y \in T_{S\mathcal{C}'} \setminus \{\varnothing\}$. Since $X \in Min(T_{S\mathcal{C}'} \setminus \{\varnothing\})$, $Y \in T_{S\mathcal{C}'} \setminus \{\varnothing\}$ and $X \cap Y \neq \varnothing$, $X \subseteq Y$. Therefore, $X = Y \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$. It follows that $Min(T_{S\mathcal{C}'} \setminus \{\varnothing\}) \subseteq Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$.

$\forall X \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$, by the formula (3.4), $X \in \mathcal{C}'$. By Lemma 3.14, $SH_{\mathcal{C}'}(X) = X$. Then $X \in T_{S\mathcal{C}'} \setminus \{\varnothing\}$. If $Y \in T_{S\mathcal{C}'} \setminus \{\varnothing\}$ and $Y \subseteq X$, we have $SH_{\mathcal{C}'}(Y) = Y$. Since $X \in \mathcal{C}'$ and $Y \subseteq X$, $X \subseteq SH_{\mathcal{C}'}(Y) = Y$. Thus, $X = Y$. So, by the definition of the minimum set of $T_{S\mathcal{C}'} \setminus \{\varnothing\}$, $X \in Min(T_{S\mathcal{C}'} \setminus \{\varnothing\})$. Hence, $Min(T_{S\mathcal{C}} \setminus \{\varnothing\}) \subseteq Min(T_{S\mathcal{C}'} \setminus \{\varnothing\})$. Therefore, $Min(T_{S\mathcal{C}'} \setminus \{\varnothing\}) = Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$.

Next, we prove that $\mathcal{C}_0 \subseteq \mathcal{C}'$, which is enough to prove that $\forall \mathcal{C}_i \in [\mathcal{C}]$, $\mathcal{C}_i \subseteq \mathcal{C}'$.

$\forall \mathcal{C}_i \in [\mathcal{C}]$ and $C \in \mathcal{C}_i$, by Lemma 3.15, there exists an $X \in Min(T_{S\mathcal{C}} \setminus \{\varnothing\})$ such that $C \subseteq X$, so $C \in \mathcal{P}(X) \setminus \{\varnothing\}$. Thus, by the formula(5), $C \in \mathcal{C}'$, which implies $\mathcal{C}_i \subseteq \mathcal{C}'$. So, $\bigcup\{\mathcal{C}_i | \mathcal{C}_i \in [\mathcal{C}]\} \subseteq \mathcal{C}'$. □

According to Proposition 3.16, covering $\mathcal{C}_0$ induced by covering $\mathcal{C}$ satisfies the condition of Theorem 3.13, then we have the following result.

Proposition 3.17. *Let $(U, \mathcal{C})$ be a covering-based approximation space. $SH_{\mathcal{C}_0}$ is a closure operator.*

Next, we will prove that $SH_{\mathcal{C}_0}$ is the closure operator of topology $T_{S\mathcal{C}}$.

Proposition 3.18. *Let $(U, \mathcal{C})$ be a covering-based approximation space. $SH_{\mathcal{C}_0}$ is the closure operator of topology $T_{S\mathcal{C}_0}$.*

Proof. By Proposition 3.17, $SH_{\mathcal{C}_0}$ is a closure operator. Then $SH_{\mathcal{C}_0}$ is the closure operator of topology $T = \{\sim X \subseteq U \mid SH_{\mathcal{C}_0}(X) = X\}$. By Proposition 3.3, $T_{S\mathcal{C}_0}$ is a clopen topology. By the formula (3.1), we have $T_{S\mathcal{C}_0} = \{\sim X \subseteq U \mid SH_{\mathcal{C}_0}(X) = X\}$, and it follows that $T_{S\mathcal{C}_0} = T$. Therefore, $SH_{\mathcal{C}_0}$ is the closure operator of topology $T_{S\mathcal{C}_0}$. □

By the formula (3.4) and Proposition 3.11, we find that $\mathcal{C}_0 \in [\mathcal{C}]$. Thus, by the formula (3.2), $T_{S\mathcal{C}} = T_{S\mathcal{C}_0}$. Based on these conclusions and Proposition 3.18, we have the following result.

Corollary 3.19. *Let $(U, \mathcal{C})$ be a covering-based approximation space. Then $SH_{\mathcal{C}_0}$ is the closure operator of topology $T_{S\mathcal{C}}$.*

At the end of this section, we present the relationship among the approximation operators CL and SH generated by the coverings of $[\mathcal{C}]$.

Remark 3.20. Denote $\mathcal{C}_1 = Min(T_{S\mathcal{C}} \setminus \{\varnothing\}) \in [\mathcal{C}]$ and let $X \subseteq U$. Then for any $\mathcal{C}_i \in [\mathcal{C}]$, it is easy to verify that:
$CL_{\mathcal{C}_1}(X) \subseteq CL_{\mathcal{C}_i}(X) \subseteq CL_{\mathcal{C}_0}(X) \subseteq X \subseteq SH_{\mathcal{C}_i}(X) \subseteq SH_{\mathcal{C}_0}(X) = SH_{\mathcal{C}_1}(X)$.

3.2. **The topological structures induced by $FH_{\mathcal{C}}$, $TH_{\mathcal{C}}$ and $RH_{\mathcal{C}}$.**

In the above subsection, $T_{S\mathcal{C}}$ is generated by the upper approximation operator $SH_{\mathcal{C}}$. Similarly, we can construct the sets $T_{F\mathcal{C}}$, $T_{T\mathcal{C}}$ and $T_{R\mathcal{C}}$ using upper approximation operators $FH_{\mathcal{C}}$, $TH_{\mathcal{C}}$ and $RH_{\mathcal{C}}$ respectively. Whether the three sets are topologies? In the following subsection, we will investigate this problem.

$$T_{F\mathcal{C}} = \{X \subseteq U \mid FH_{\mathcal{C}}(X) = X\}. \tag{3.5}$$

$$T_{T\mathcal{C}} = \{X \subseteq U \mid TH_{\mathcal{C}}(X) = X\}. \tag{3.6}$$

$$T_{R\mathcal{C}} = \{X \subseteq U \mid RH_{\mathcal{C}}(X) = X\}. \tag{3.7}$$

Similar to the proof in Proposition 3.2, we can easily prove the following result.

Proposition 3.21. *Let $(U, \mathcal{C})$ be a covering-based approximation space. Then $T_{T\mathcal{C}} = \{X \subseteq U \mid TH_{\mathcal{C}}(X) = X\}$ is a topology on U.*

The above result illustrates that for any covering $\mathcal{C}$ of U, $T_{T\mathcal{C}}$ is a topology. However, $T_{F\mathcal{C}}$ or $T_{R\mathcal{C}}$ may not be a topology. The following example demonstrates this point.

Example 3.4. Let $U = \{a, b, c, d\}$ and $\mathcal{C} = \{\{a, b\}, \{b, c\}, \{d\}\}$. By formulas (3.5), (3.7) and Definition 2.5, $T_{F\mathcal{C}} = T_{R\mathcal{C}} = \{\varnothing, \{d\}, \{a, b\}, \{b, c\}, U\}$. According to the definition of topology, we have that $T_{F\mathcal{C}}$ and $T_{R\mathcal{C}}$ are not topologies.

In order to explore the conditions for $FH_{\mathcal{C}}$ or $RH_{\mathcal{C}}$ to be a topology, we introduce the following theorems.

Theorem 3.22 ([29]). *Let $\mathcal{C}$ be a covering of U. $\forall X, Y \in U$, $FH_{\mathcal{C}}(X \cup Y) = FH_{\mathcal{C}}(X) \cup FH_{\mathcal{C}}(Y)$ if and only if $\mathcal{C}$ is unary.*

Theorem 3.23 ([29]). *Let $\mathcal{C}$ be a covering of U. $\forall X, Y \in U$ and $X \subseteq Y$, $FH_{\mathcal{C}}(X) \subseteq FH_{\mathcal{C}}(Y)$ if and only if $\mathcal{C}$ is unary.*

Theorem 3.24 ([29]). *Let $\mathcal{C}$ be a covering of U. $\forall X, Y \in U$, $RH_{\mathcal{C}}(X \cup Y) = RH_{\mathcal{C}}(X) \cup RH_{\mathcal{C}}(Y)$ if and only if $\mathcal{C}$ is unary.*

Theorem 3.25 ([29]). *Let $\mathcal{C}$ be a covering of U. $\forall X, Y \in U$ and $X \subseteq Y$, $RH_{\mathcal{C}}(X) \subseteq RH_{\mathcal{C}}(Y)$ if and only if $\mathcal{C}$ is unary.*

Proposition 3.26. *Let $\mathcal{C}$ be a covering of U. $T_{F\mathcal{C}} = \{X \subseteq U \mid FH_{\mathcal{C}}(X) = X\}$ is a topology on U if and only if $\mathcal{C}$ is unary.*

Proof. Since $T_{F\mathcal{C}}$ is topology, $\forall X, Y \in T_{F\mathcal{C}}$, $FH_{\mathcal{C}}(X \cup Y) = FH_{\mathcal{C}}(X) \cup FH_{\mathcal{C}}(Y)$. By Theorem 3.22, we find that $\mathcal{C}$ is unary.

For the sufficient condition, we only need to prove that $T_{F\mathcal{C}}$ satisfies the conditions in Definition 2.6.

(T1) It is easy to check that $\varnothing, U \in T_{F\mathcal{C}}$.

(T2) $\forall X, Y \in T_{F\mathcal{C}}$, by Theorem 3.23, $FH_{\mathcal{C}}(X \cap Y) \subseteq FH_{\mathcal{C}}(X) \cap FH_{\mathcal{C}}(Y) = X \cap Y$. Obviously, $X \cap Y \subseteq FH_{\mathcal{C}}(X \cap Y)$. Thus $X \cap Y \in T_{F\mathcal{C}}$.

(T3) Since $\mathcal{C}$ is unary, $\forall X, Y \in T_{F\mathcal{C}}$, by Theorem 3.22, $FH_{\mathcal{C}}(X \cup Y) = FH_{\mathcal{C}}(X) \cup FH_{\mathcal{C}}(Y) = X \cup Y$. Thus, $X \cup Y \in T_{F\mathcal{C}}$. □

With a similar argument, by Theorems 3.24 and 3.25, we can prove the following result.

Proposition 3.27. *Let $\mathcal{C}$ be a covering of U. $T_{R\mathcal{C}} = \{X \subseteq U \mid RH_{\mathcal{C}}(X) = X\}$ is a topology on U if and only if $\mathcal{C}$ is unary.*

4. Conclusions

In this study, we investigated four classes of topological structures induced by covering upper approximation operators FH, SH, TH and RH. We obtained that the minimum set of topological structure generated by SH is not only a base for the topology, but also a partition on U, and we constructed the closure operator of the topology by means of SH. The relationships between topological structures and unary covering were discussed in the last part. This study only investigate a class of topological structure induced by covering rough sets. In [21], Wang and Wang presented four methods to induce topological structures by relation-based rough sets. Similarly, we can get the other topological structures using these methods in covering-based rough set model. In the future, we will study the relationship between these topological structures and covering rough sets.

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Shanshan Wang (wss26862@163.com)
College of Mathematics and Computer Science, Shanxi Normal University, 041000 Linfen, China

Zhaohao Wang (nysywzh@163.com)
College of Mathematics and Computer Science, Shanxi Normal University, 041000 Linfen, China