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On separation axioms in fuzzy rough topological spaces

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Abstract. The purpose of this paper is to introduce the concept of fuzzy rough regular semi α-open sets and study its applications on fuzzy rough regular semi α-connected spaces, fuzzy rough regular super semi α-connected spaces, fuzzy rough regular semi α-ultranormal spaces, fuzzy rough regular semi α-completely ultranormal spaces and fuzzy rough regular semi α-quasi-normal spaces.

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1. Introduction

Zadeh [14] innovated the concept of a fuzzy set. The concept of fuzzy topological spaces was introduced and developed by Chang [4]. The definition of rough set was introduced by Pawlak [12]. Nanda and Majumdar [7] studied the concept of fuzzy rough set. The concept of fuzzy rough topological space was introduced by Padmapriya, Uma and Roja [9]. The concepts of fuzzy semi-closed sets and fuzzy α-open sets were introduced by Bin Shahna [2] and the concept of fuzzy regular closed set was introduced by Azad [1]. The concept of fuzzy normality was introduced by Bruce Hutton[3]. The concept of fuzzy connectedness was introduced by Fatteh and Bassan [5]. The concept of fuzzy semi-connectedness was introduced by Uma, Roja and Balasubramanian [13]. The concept of topology of intuitionistic fuzzy rough sets was introduced by Hazra, Samanta and Chattopadhyay [6]. The purpose of this paper is to introduce the concept of fuzzy rough regular semi α-open sets and study its applications on fuzzy rough regular semi α-connected spaces, fuzzy rough regular super semi α-connected spaces, fuzzy rough regular semi α-ultranormal spaces, fuzzy rough regular semi α-completely ultranormal spaces and fuzzy rough regular semi α-quasi-normal spaces.

2. Preliminaries

Unless otherwise stated consider $(V, \mathcal{B})$ to be a rough universe where V is a non-empty set and $\mathcal{B}$ is a Boolean subalgebra of the Boolean algebra of all subsets of V. Also consider a rough set $X = (X_L, X_U) \in \mathcal{B}^2$ with $X_L \subset X_U$.

Definition 2.1 ([6])**.** A fuzzy rough set (briefly FRS) in X is an object of the form $A = (A_L, A_U)$, where A_L and A_U are characterized by a pair of maps $A_L : X_L \to \mathcal{L}$ and $A_U : X_U \to \mathcal{L}$ with $A_L(x) \leq A_U(x) \forall x \in X_U$ where $(\mathcal{L}, \leq)$ is a fuzzy lattice (i.e complete and completely distributive lattice whose least and greatest elements are denoted by 0 and 1 respectively with an involutive order reversing operation $' : \mathcal{L} \to \mathcal{L}$).

Definition 2.2 ([8])**.** In particular $\mathcal{L}$ could be the closed interval $[0, 1]$.

Definition 2.3 ([6])**.** For any two fuzzy rough sets A $= (A_L, A_U)$ and B $= (B_L, B_U)$ in X,

(i) $A \subset B$ iff $A_L(x) \leq B_L(x)$, $\forall x \in X_L$ and $A_U(x) \leq B_U(x)$, $\forall x \in X_U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

If $\{A_i : i \in J\}$ be any family of fuzzy rough sets in X, where $A_i = (A_{iL}, A_{iU})$, then

(iii) $E = \bigcup_i A_i$ where $E_L(x) = \vee A_{iL}(x), \forall x \in X_L$ and $E_U(x) = \vee A_{iU}(x), \forall x \in X_U$.

(iv) $F = \bigcap_i A_i$ where $F_L(x) = \wedge A_{iL}(x), \forall x \in X_L$ and $F_U(x) = \wedge A_{iU}(x), \forall x \in X_U$.

Definition 2.4 ([6])**.** If A and B are fuzzy rough sets in X_L and X_U respectively where $X_L \subset X_U$. Then the restriction of B on X_L and the extension of A on X_U (denoted by $B_{>L}$ and $A_{<U}$ respectively) are denoted by $B_{>L}(x) = B(x), \forall x \in X_L$ and

$$A_{<U}(x) = \begin{cases} A(x), & \forall x \in X_L, \\ \vee_{\xi \in X_L}\{A(\xi)\}, & \forall x \in X_U - X_L. \end{cases}$$

Complement of a FRS $A = (A_L, A_U)$ in X are denoted by $\bar{A} = ((\bar{A})_L, (\bar{A})_U)$ and is defined by $(\bar{A})_L(x) = (A_{U>L})'(x), \forall x \in X_L$ and $(\bar{A})_U(x) = (A_{L<U})'(x), \forall x \in X_U$. For simplicity we write $\bar{A}_L, \bar{A}_U$ instead of $((\bar{A})_L, (\bar{A})_U)$.

Definition 2.5 ([10])**.** The null fuzzy rough set and whole fuzzy rough set in X is defined by $\tilde{0} = (0_L, 0_U)$ and $\tilde{1} = (1_L, 1_U)$.

Definition 2.6 ([10])**.** Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$. Let $A = (A_L, A_U)$ be a fuzzy rough set in X. Then $Y = f(X) \in \mathcal{B}_1^2$ and $Y_L = f(X_L)$, $Y_U = f(X_U)$. The image of A under f, denoted by $f(A) = (f(A_L), f(A_U))$ is defined by

$$f(A_L)(y) = \vee \left\{A_L(x) : x \in X_L \cap f^{-1}(y)\right\} \text{ for every } y \in Y_L$$

and

$$f(A_U)(y) = \vee \left\{A_U(x) : x \in X_U \cap f^{-1}(y)\right\} \text{ for every } y \in Y_U.$$

Definition 2.7 ([8]). Let $B = (B_L, B_U)$ be a fuzzy rough set in Y where $Y = (Y_L, Y_U) \in \mathcal{B}_1^1$ is a rough set. Then $X = f^{-1}(Y) \in \mathcal{B}_1^2$, where $X_L = f^{-1}(Y_L)$, $X_U = f^{-1}(Y_U)$. Then the inverse image of B under f, denoted by $f^{-1}(B) = (f^{-1}(B_L), f^{-1}(B_U))$ is defined by

$$f^{-1}(B_L)(x) = B_L(f(x)) \text{ for every } x \in X_L$$

and

$$f^{-1}(B_U)(x) = B_U(f(x)) \text{ for every } x \in X_U.$$

Proposition 2.8 ([8]). *If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping, then for all fuzzy rough sets $A, A_1, A_2 \in X$, we have*
(i) $f(A^{'}) \supset (f(B))^{'}$,
(ii) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

Proposition 2.9 ([11]). *Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that f^{-1}:$(V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all fuzzy rough sets, $B, B_i \in Y$, $i \in J$, we have*
(i) $f^{-1}(B^{'}) = f^{-1}(B)^{'}$,
(ii) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$,
(iii) $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$,
(iv) $f^{-1}(\cap_i B_i) = \cap_i f^{-1}(B_i)$,
(v) $f(\cup_i B_i) = \cup_i f(B_i)$,
(vi) $f(\cap_i B_i) \subset \cap_i f(B_i)$.

Proposition 2.10 ([11]). *Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that f^{-1}:$(V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all fuzzy rough sets A in X and B in Y, we have*
(i) $B = f(f^{-1}(B))$,
(ii) $A \subset f^{-1}(f(A))$.

Definition 2.11 ([8]). A fuzzy rough topology (in short, FRT) is a family τ of fuzzy rough sets in $X = (X_L, X_U)$ satisfying the following axioms:
(i) $\tilde{0}$ and $\tilde{1} \in \tau$,
(ii) $\lambda_1 \cap \lambda_2 \in \tau$ for any $\lambda_1, \lambda_2 \in \tau$,
(iii) $\cup \lambda_i \in \tau$ for any arbitrary family $\{\lambda_i, i \in J\} \in \tau$.

In this case the pair (X,τ) is called a fuzzy rough topological space (in short, $FRTS$) and any fuzzy rough set in τ is known as a fuzzy rough open set(in short, $FROS$) in X.

Definition 2.12 ([11]). A fuzzy rough set is a fuzzy rough closed set (in short, $FRCS$) if and only if its complement is a fuzzy rough open set (in short, $FROS$).

Definition 2.13 ([11]). Let (X,τ) be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough set in X =(X_L, X_U). Then the fuzzy rough interior of A is denoted by $FRint(A)$ and is defined by

$$FRint(A) = \cup \{B : B = (B_L, B_U) isafuzzyroughopensetinXandB \subseteq A\}.$$

Definition 2.14 ([11]). Let (X,τ) be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough set in X =(X_L, X_U). Then the fuzzy rough closure of A is denoted by $FRcl(A)$ and is defined by

$$FRcl(A) = \cap \{B : B = (B_L, B_U) isafuzzyroughclosedsetinXandB \supseteq A\}.$$

Definition 2.15 ([11]). Any fuzzy rough set A in a fuzzy topological space (X,τ) is said to be FR-clopen if it is both FR-open and FR-closed.

Remark 2.16 ([11]). $(i) FRint(A) \subseteq A \subseteq FRcl(A)$.
$(ii) A \subseteq FRint(FRcl(A)) \subseteq FRcl(A)$.
(iii) If A is a fuzzy rough open set, then $FRint(A) = A$.
(iv) If A is a fuzzy rough closed set, then $FRcl(A) = A$.

Remark 2.17. Let $A = (A_L, A_U)$ and $B = (B_L, B_U)$ be any two fuzzy rough sets in a fuzzy rough topological space (X, τ).
(i) If $A \subseteq B$, then $FRcl(A) \subseteq FRcl(B)$ and $FRint(A) \subseteq FRint(B)$.
(ii) $FRint(FRint(A)) = FRint(A)$.
(iii) $FRcl(FRcl(A)) = FRcl(A)$.

Proposition 2.18 ([11]). *A function f from a fuzzy rough topological space (X, τ) to a fuzzy rough topological space (Y, σ) is said to be a fuzzy rough continuous function if $f^{-1}(A)$ is a fuzzy rough open (resp. closed) set in X for each fuzzy rough open (resp. closed) set A in Y.*

Proposition 2.19 ([11]). *Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy rough continuous function. Then $f(FRcl(A)) \subseteq FRcl(f(A))$, for each fuzzy rough set A in (X, τ).*

Proposition 2.20 ([11]). *Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy rough continuous function. Then $FRcl(f^{-1}(A)) \subseteq f^{-1}(FRcl(A))$, for each fuzzy rough set A in (Y, σ).*

3. Separation axioms in fuzzy rough topological spaces

In this section, the concept of fuzzy rough regular semi α-open sets and its applications on fuzzy rough regular semi α-connected spaces, fuzzy rough regular super semi α-connected spaces, fuzzy rough regular semi α-ultra-normal spaces, fuzzy rough regular semi αcompletely ultra-normal spaces and fuzzy rough regular semi α-quasi-normal spaces are discussed.

Definition 3.1. Any fuzzy rough set A in a fuzzy rough topological space (X, τ) is said to be:
(i)Fuzzy rough α-closed (in short FR α-closed) if $A \supseteq FRcl(FRint(FRcl(A))$. Its complement is said to be a FR α-open set.
(ii)Fuzzy rough semi-closed (in short FR semi-closed) if $A \supseteq FRint(FRcl(A))$. Its complement is said to be a FR semi-open set.

Definition 3.2. Let (X, τ) be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough set in $X = (X_L, X_U)$. Then the fuzzy rough α-interior of A is denoted by $FR\alpha\text{-}int(A)$ and is defined by

$FR\alpha int(A) = \cup\{B : B = (B_L, B_U)$ is a fuzzy rough α-open set in X and $B \subseteq A\}$.

Definition 3.3. Let (X, τ) be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough set in $X = (X_L, X_U)$. Then the fuzzy rough α-closure of A is denoted by $FR\alpha\text{-}cl(A)$ and is defined by

$FR\alpha cl(A) = \cap\{B : B = (B_L, B_U)$ is a fuzzy rough α-closed set in X and $B \supseteq A\}$.

Definition 3.4. Let (X, τ) be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough set in $X = (X_L, X_U)$. Then the fuzzy rough semi-interior of A is denoted by $FRsint(A)$ and is defined by

$$FRsint(A) = \cup\{B : B = (B_L, B_U) \text{ is a fuzzy rough semi-open set in X and } B \subseteq A\}.$$

Definition 3.5. Let (X, τ) be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough set in $X = (X_L, X_U)$. Then the fuzzy rough semi-closure of A is denoted by $FRscl(A)$ and is defined by

$$FRscl(A) = \cap\{B : B = (B_L, B_U) \text{ is a fuzzy rough semi-closed set in X and } B \supseteq A\}.$$

Remark 3.6. $\mathcal{L}^X$ and $\mathcal{L}^Y$ denote collection of all fuzzy rough sets in X and Y.

Definition 3.7. Let (X, τ) be a fuzzy rough topological space. Any fuzzy rough set $A \in \mathcal{L}^X$ is said to be fuzzy rough regular semi α-open [in short $FRrs\alpha - open$] iff $A = FRsint(FR\alpha cl(A))$. The complement of a fuzzy rough regular semi α-open set is said to be a fuzzy rough regular semi α-closed set [in short $FRrs\alpha$-closed]. Any fuzzy rough set A in a fuzzy rough topological space (X, τ) is said to be a fuzzy rough regular semi α-clopen set [in short FRrsα-clopen] if it is both FRrsα-closed and FRrsα-open.

Example 3.8. Let $X = \{a, b, c\}$. Let $X_L = \{a, b\}$ and $X_U = \{a, b, c\}$. Let $A, B \in \mathcal{L}^X$ be defined as $A = ((a/0.2, b/0.3), (a/0.3, b/0.3, c/0.3))$ and $B = ((a/0.4, b/0.5), (a/0.4, b/0.5, c/0.5))$. Define the fuzzy rough topology τ on X as $\tau = \{\tilde{0}, \tilde{1}, A, B\}$. Let $C \in \mathcal{L}^X$ be defined as, $C = (C_L, C_U) = ((a/0.6, b/0.5), (a/0.6, b/0.5, c/0.5))$. Thus $FRsint(FR\alpha cl(C)) = C$. So C is a fuzzy rough regular semi α-open set.

Definition 3.9. Let (X, τ) be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough regular semi α-open set in $X = (X_L, X_U)$. Then the fuzzy rough regular semi α-interior of A is denoted by $FRrs\alpha int(A)$ and is defined by

$$FRrs\alpha int(A) = \bigcup\{B : B = (B_L, B_U) isaFRrs\ \alpha\text{-open set in } X \text{ and } B \subseteq A\}.$$

Definition 3.10. Let (X, τ) be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough regular semi α-open set in $X = (X_L, X_U)$. Then the fuzzy rough regular semi α-closure of A is denoted by $FRrs\alpha cl(A)$ and is defined by

$$FRrs\alpha cl(A) = \bigcap\{B : B = (B_L, B_U) isaFRrs\ \alpha\text{-closed set in } X \text{ and } B \supseteq A\}.$$

Definition 3.11. (i) If $A \neq \tilde{0}$ then either $A_L \neq 0_L$ or $A_U \neq 0_U$ or both.
(ii) If $A \neq \tilde{1}$ then either $A_L \neq 1_L$ or $A_U \neq 1_U$ or both.

Definition 3.12. Let (X, τ) be a fuzzy rough topological space. Let A be any $FRrs\alpha$-open set in (X, τ). Then A is said to be proper if $A \neq \tilde{0}$ and $A \neq \tilde{1}$.

Definition 3.13. A fuzzy rough topological space (X, τ) is said to be fuzzy rough regular semi α-connected [in short $FRrs\alpha$-connected] iff (X, τ) has no proper $FRrs\alpha$-open sets A and B such that $A_L(x) + (B_{U>L})(x) = 1$, $B_L(x) + (A_{U>L})(x) = 1 \forall x \in X_L$ and $A_U(x) + (B_{L<U})(x) = 1, B_U(x) + (A_{L<U})(x) = 1 \forall x \in X_U$.

Proposition 3.14. *For a fuzzy rough topological space (X, τ) the following statements are equivalent:*

(i) (X,τ) *is fuzzy rough regular semi* α*-connected.*

(ii) *There exist no* $FRrs\alpha$*-open sets* $A, B \in \mathcal{L}^X$ *where* $A \neq \tilde{0}$ *and* $B \neq \tilde{0}$ *such that* $A_L(x)+(B_{U>L})(x) = 1$, $B_L(x)+(A_{U>L})(x) = 1 \forall x \in X_L$ *and* $A_U(x)+(B_{L<U})(x) = 1, B_U(x) + (A_{L<U})(x) = 1 \forall x \in X_U$*.*

(iii) *There exist no* $FRrs\alpha$*-closed sets* $C, D \in \mathcal{L}^X$ *where* $C \neq \tilde{1}$ *and* $D \neq \tilde{1}$ *such that* $C_L(x) + (D_{U>L})(x) = 1$, $D_L(x) + (C_{U>L})(x) = 1 \forall x \in X_L$ *and* $C_U(x) + (D_{L<U})(x) = 1, D_U(x) + (C_{L<U})(x) = 1 \forall x \in X_U$*.*

(iv) (X,τ) *contains no fuzzy rough set* $A \neq \tilde{0}$ *and* $A \neq \tilde{1}$*, such that* A *is both* $FRrs\alpha$*-open and* $FRrs\alpha$*-closed.*

Proof. (i)⇒ (ii). Assume that (i) is true. Then (ii) follows from the Definition 3.13.

(ii) ⇒(iii). Assume that (ii) is true. Suppose that there exist fuzzy rough regular semi α-closed sets $A \neq \tilde{1}$ and $B \neq \tilde{1}$ such that

$$A_L(x) + (B_{U>L})(x) = 1,\ B_L(x) + (A_{U>L})(x) = 1 \forall x \in X_L$$

and

$$A_U(x) + (B_{L<U})(x) = 1,\ B_U(x) + (A_{L<U})(x) = 1 \forall x \in X_U.$$

Then, $C = \bar{A} \neq \tilde{0}$ is a $FRrs\alpha$-open set. Similarly, $D = \bar{B} \neq \tilde{0}$ is a $FRrs\alpha$-open set such that

$$C_L(x) + (D_{U>L})(x) = 1,\ D_L(x) + (C_{U>L})(x) = 1 \forall x \in X_L$$

and

$$C_U(x) + (D_{L<U})(x) = 1,\ D_U(x) + (C_{L<U})(x) = 1 \forall x \in X_U$$

which is a contradiction to our assumption. Thus (iii) holds.

(iii) ⇒(iv). Assume that (iii) is true. Suppose that (X,τ) contains a fuzzy rough set $A \neq \tilde{0}$ and $A \neq \tilde{1}$ which is $FRrs\alpha$-clopen. Then $\bar{A}$ is a proper $FRrs\alpha$-closed set. Also, by assumption, A is $FRrs\alpha$-closed. Thus,

$$A_L(x) + (\bar{A}_{U>L})(x) = 1, (\bar{A}_L)(x) + (A_{U>L})(x) = 1 \forall x \in X_L$$

and

$$A_U(x) + (\bar{A}_{L<U})(x) = 1, (\bar{A}_U)(x) + (A_{L<U})(x) = 1 \forall x \in X_U$$

which is a contradiction to our assumption. So (iv) holds.

(iv)⇒(i). Assume that (iv) is true. Suppose that (X,τ) is not $FRrs\alpha$-connected. Then, (X,τ) has proper $FRrs\alpha$-open sets A and B such that

$$A_L(x) + (B_{U>L})(x) = 1,\ B_L(x) + (A_{U>L})(x) = 1 \forall x \in X_L$$

and

$$A_U(x) + (B_{L<U})(x) = 1, B_U(x) + (A_{L<U})(x) = 1 \forall x \in X_U.$$

That is, there exist $FRrs\alpha$-closed set $\bar{A}$ and a $FRrs\alpha$-closed set $\bar{B}$ such that,

$$\bar{A}_L(x) + (\bar{B}_{U>L})(x) = 1, (\bar{B})_L(x) + (\bar{A}_{U>L})(x) = 1 \forall x \in X_L$$

and

$$(\bar{A})_U(x) + (\bar{B}_{L<U})(x) = 1, (\bar{B})_U(x) + (\bar{A}_{L<U})(x) = 1 \forall x \in X_U.$$

This implies that (X, τ) has a $FRrs\alpha$-open set $A = \bar{B} \neq \tilde{0}, \tilde{1}$ which is both $FRrs\alpha$-open and $FRrs\alpha$-closed which is a contradiction to our assumption. Thus (i) holds. □

Definition 3.15. A fuzzy rough topological space (X, τ) is said to be fuzzy rough regular super semi α-connected [in short $FRrss\alpha$-connected] if it has no proper $FRrs\alpha$-open set.

Proposition 3.16. *If (X, τ) is any fuzzy rough topological space, then ((i)$\Rightarrow$ ((ii)) and ((ii)$\Rightarrow$((iii)), where,*
(i) *(X, τ) is a $FRrss\alpha$-connected space.*
(ii) *If $\tilde{0} \neq A$ is a $FRrs\alpha$-open set then, $FRrs\alpha cl(A) = \tilde{1}$.*
(iii) *If $\tilde{1} \neq A$ is a $FRrs\alpha$-closed set then, $FRrs\alpha int(A) = \tilde{0}$.*

Proof. (i)$\Rightarrow$ (ii). Assume that (X, τ) is a fuzzy rough regular super semi α-connected space. Suppose that there exists a fuzzy rough regular semi α-open set $A \neq \tilde{0}$ such that $FRrs\alpha cl(A) \neq \tilde{1}$. Then $FRrs\alpha int(FRrs\alpha cl(A)) \neq \tilde{1}$. But since A is a fuzzy rough regular semi α-open set and $FRrs\alpha int(FRrs\alpha cl(A)) \neq \tilde{1}$, it is seen that $A \neq \tilde{1}$. Thus we find that (X, τ) has a proper $FRrs\alpha$-open set A, which is a contradiction.

(ii)$\Rightarrow$(iii). Assume that(ii) holds. Suppose that there exists a fuzzy rough regular semi α-closed set $A \neq \tilde{1}$ such that $FRrs\alpha int(A) \neq \tilde{0}$. Now $B = \bar{A} \neq \tilde{0}$ and B is a non-zero fuzzy rough regular semi α-open set. Then, $FRrs\alpha cl(B) = \tilde{1} - FRrs\alpha int(\bar{B}) = \tilde{1} - FRrs\alpha int(A) \neq \tilde{1}$, since $FRrs\alpha int(A) \neq \tilde{0}$. Thus we get a contradiction. □

Definition 3.17. Let (X, τ) and (Y, σ) be any two fuzzy rough topological spaces. Any function $f : (X, \tau) \to (Y, \sigma)$ is called fuzzy rough regular semi α-irresolute [in short, $FRrs\alpha$-irresolute] if $f^{-1}(A) \in \mathcal{L}^X$ is $FRrs\alpha$-open [resp. $FRrs\alpha$-closed] for every $FRrs\alpha$-open[resp. $FRrs\alpha$-closed] set $A \in \mathcal{L}^Y$.

Proposition 3.18. *Let (X, τ) and (Y, σ) be any two fuzzy rough topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ be a bijective $FRrs\alpha$-irresolute function. If (X, τ) is $FRrs\alpha$-connected then (Y, σ) is also $FRrs\alpha$-connected.*

Proof. Suppose that (Y, σ) is not $FRrs\alpha$-connected. Then there are $FRrs\alpha$-open sets $C = (f(A_L(x)), f(A_U(x)))$, $D = (f(B_L(x)), f(B_U(x))) \in \mathcal{L}^Y$ where $C, D \neq \tilde{0}$ such that

$$f(A_L(x)) + f((B_{U>L})(x)) = 1,\ f(B_L(x)) + f((A_{U>L})(x)) = 1 \forall x \in X_L$$

and

$$f(A_U(x)) + f((B_{L<U})(x)) = 1,\ f(B_U(x)) + f((A_{L<U})(x)) = 1 \forall x \in X_U.$$

Since f is bijective and $FRrs\alpha$-irresolute, $f^{-1}(C), f^{-1}(D) \in \mathcal{L}^X$ are $FRrs\alpha$-open sets where $f^{-1}(C), f^{-1}(D) \neq \tilde{0}$ such that

$$f^{-1}(f(A_L(x))) + f^{-1}(f(B_{U>L})(x)) = 1,$$

$$f^{-1}(f(B_L(x))) + f^{-1}(f(A_{U>L})(x)) = 1 \forall x \in X_L$$

and

$$f^{-1}(f(A_U(x))) + f^{-1}(f(B_{L<U})(x)) = 1,$$
$$f^{-1}(f(B_U(x))) + f^{-1}(f(A_{L<U})(x)) = 1 \forall x \in X_U.$$

This is a contradiction to the fact that (X,τ) is a $FRrs\alpha$-connected space. Hence, (Y,σ) is a $FRrs\alpha$-connected space. □

Definition 3.19. A fuzzy rough set A of a fuzzy rough topological space (X,τ) is called a fuzzy rough ζ- open set of (X,τ) if it is a finite union of $FRrs\alpha$- open sets of (X,τ). The complement of a fuzzy rough ζ- open set is called fuzzy rough ζ- closed set.

Definition 3.20. A fuzzy rough topological space (X,τ) is said to be

(i) fuzzy rough regular semi α-normal, [in short, $FRrs\alpha$-normal] if for any FR-closed set A and FR-open set B such that $A \subseteq B$, there exists a $FRrs\alpha$-open set C such that $A \subseteq FRrs\alpha int(C) \subseteq FRrs\alpha cl(C) \subseteq B$.

(ii) fuzzy rough regular semi α-quasi-normal, [in short, $FRrs\alpha$-quasi-normal] if for any $FR\zeta$-closed set A and $FR\zeta$-open set B such that $A \subseteq B$, there exists a $FRrs\alpha$-open set C such that $A \subseteq FRrs\alpha int(C) \subseteq FRrs\alpha cl(C) \subseteq B$.

(iii) fuzzy rough regular semi α-ultra-normal, [in short, $FRrs\alpha$-ultra-normal] if for any fuzzy rough closed set A and fuzzy rough open set B such that $A \subseteq B$, there exists a $FRrs\alpha$-clopen set C such that $A \subseteq C \subseteq B$.

(iv) fuzzy rough regular semi α-completely ultra-normal, [in short, $FRrs\alpha$-completely ultranormal] if for any fuzzy rough set A and fuzzy rough set B such that $FRcl(A) \subseteq B$ and $A \subseteq FRint(B)$ there exists a $FRrs\alpha$-clopen set C such that $A \subseteq C \subseteq B$.

Definition 3.21. Let (X,τ) and (Y,σ) be fuzzy rough topological spaces. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be a fuzzy rough regular semi α-open function, in short $FRrs\alpha$-open [resp. fuzzy rough regular semi α-closed function, in short $FRrs\alpha$-closed] function if $f(A)$ is $FRrs\alpha$-open [resp. $FRrs\alpha$-closed] set of (Y,σ) for each $FRrs\alpha$-open [resp. $FRrs\alpha$-closed] set of (X,τ).

Definition 3.22. Let (X,τ) and (Y,σ) be fuzzy rough topological spaces. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be a fuzzy rough regular semi α-clopen function [in short $FRrs\alpha$-clopen] if $f(A)$ is $FRrs\alpha$-clopen set of (Y,σ) for each $FRrs\alpha$-clopen set of (X,τ).

Proposition 3.23. *Let (X,τ) and (Y,σ) be any two fuzzy rough topological spaces. Let $f : (X,\tau) \to (Y,\sigma)$ be a fuzzy rough continuous and $FRrs\alpha$-clopen function from (X,τ) onto (Y,σ). If (X,τ) is $FRrs\alpha$-ultranormal then (Y,σ) is also $FRrs\alpha$-ultranormal.*

Proof. Let (X,τ) be a $FRrs\alpha$-ultranormal space and let A and B be, respectively FR-closed and FR-open sets of (Y,σ) such that $A \subseteq B$. Since f is fuzzy rough continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are respectively, fuzzy rough closed and fuzzy rough open sets of (X,τ) such that $f^{-1}(A) \subseteq f^{-1}(B)$. Since (X,τ) is $FRrs\alpha$-ultra-normal, there exists a $FRrs\alpha$-clopen set C in (X, τ) such that $f^{-1}(A) \subseteq C \subseteq f^{-1}(B)$. By surjectivity of f,

$$A = f(f^{-1}(A)) \subseteq f(C) \subseteq f(f^{-1}(B)) = B.$$

Since f is $FRrs\alpha$-clopen function, $f(C)$ is $FRrs\alpha$-clopen set of (Y, σ). Thus, (Y, σ) is $FRrs\alpha$-ultranormal. □

Proposition 3.24. *Let (X, τ) and (Y, σ) be any two fuzzy rough topological spaces. Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy rough continuous and $FRrs\alpha$-clopen function from (X, τ) onto (Y, σ). If (X, τ) is $FRrs\alpha$-completely ultranormal then (Y, σ) is also $FRrs\alpha$-completely ultranormal.*

Proof. Let (X, τ) be a $FRrs\alpha$-ultranormal space and let A and B be, respectively FR-closed and FR-open sets of (Y, σ) such that $FRcl(A) \subseteq B$ and $A \subseteq FRint(B)$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are fuzzy rough sets of (X, τ) such that $f^{-1}(FRcl(A)) \subseteq f^{-1}(B)$ and $f^{-1}(A) \subseteq f^{-1}(FRint(B))$. Since f is fuzzy rough continuous, $f^{-1}(FRcl(A))$ is fuzzy rough closed set in (X, τ) and $f^{-1}(FRint(B))$ is fuzzy rough open set in (X, τ). It is seen that $f^{-1}(FRcl(A))$ is a fuzzy rough closed set such that $f^{-1}(A) \subseteq f^{-1}(FRcl(A))$. Thus,

$$FRcl(f^{-1}(A)) \subseteq f^{-1}(FRcl(A)) \subseteq f^{-1}(B).$$

Similarly,

$$f^{-1}(A) \subseteq f^{-1}(FRint(B)) \subseteq FRint(f^{-1}(B)).$$

Since (X, τ) is $FRrs\alpha$-completely ultranormal, there exists a $FRrs\alpha$-clopen set C such that $f^{-1}(A) \subseteq C \subseteq f^{-1}(B)$. By surjectivity of f,

$$A = f(f^{-1}(A)) \subseteq f(C) \subseteq f(f^{-1}(B)) = B$$

such that

$$FRcl(A) = f(f^{-1}(FRcl(A))) \subseteq f(f^{-1}(B)) = B$$

and

$$A = f(f^{-1}(A)) \subseteq f(f^{-1}(FRint(B))) = FRint(B).$$

Also, $f(C)$ is a $FRrs\alpha$-clopen set, since f is a $FRrs\alpha$-clopen function. Thus, (Y, σ) is $FRrs\alpha$-completely ultranormal. □

Definition 3.25. Let (X, τ) and (Y, σ) be fuzzy rough topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be a fuzzy rough ζ- continuous, if $f^{-1}(A)$ is a $FR\zeta$-open [resp. $FR\zeta$-closed] set of (X, τ) for every fuzzy rough open [resp. fuzzy rough closed] set of (Y, σ).

Proposition 3.26. *Let (X, τ) and (Y, σ) be any two fuzzy rough topological spaces. Let (X, τ) be a $FRrs\alpha$-quasi-normal space and $f : (X, \tau) \to (Y, \sigma)$ be a surjective function such that f is fuzzy rough ζ-continuous, and f takes $FRrs\alpha$-open sets of (X, τ) to $FRrs\alpha$-clopen sets of (Y, σ) then (Y, σ) is $FRrs\alpha$-ultranormal.*

Proof. Let A and B be, respectively fuzzy rough closed and fuzzy rough open sets of (Y, σ) such that $A \subseteq B$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are respectively fuzzy rough ζ-closed and fuzzy rough ζ-open sets of (X, τ) such that $f^{-1}(A) \subseteq f^{-1}(B)$, since f is fuzzy rough ζ-continuous. By $FRrs\alpha$-quasi-normality of (X, τ), there exists a

$FRrs\alpha$-open set C such that $f^{-1}(A) \subseteq C \subseteq FRrs\alpha cl(C) \subseteq f^{-1}(B)$. By surjectivity of f,

$$A = f(f^{-1}(A)) \subseteq f(C) \subseteq f(FRrs\alpha cl(C)) \subseteq f(f^{-1}(B)) = B.$$

Thus, $A \subseteq f(C) \subseteq B$, where $f(C)$ is $FRrs\alpha$-clopen set of (Y, σ). So, (Y, σ) is $FRrs\alpha$-ultranormal. □

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