Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 12, No. 4, (October 2016), pp. 585–596
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Strength of line graph of certain fuzzy graphs

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Received 22 January 2016; Revised 1 March 2016; Accepted 23 April 2016

Abstract. In this paper we find the strength of the line graphs of strong fuzzy butterfly graph, strong fuzzy star graph, strong fuzzy bull graph and strong fuzzy diamond graph, strong fuzzy path, strong fuzzy cycle in terms of the respective graphs.

2010 AMS Classification: 05C72

Keywords: Strength of fuzzy graphs, Strong fuzzy butterfly graph, Strong fuzzy star graph, Strong fuzzy bull graph, Strong fuzzy diamond graph, Weakest path, M weakest paths, Consecutive paths in a fuzzy graph.

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1. Introduction

We introduce, in this paper, the concept of extra strong path to find the strength of line graph of various fuzzy graphs. The notion of a fuzzy subset was introduced for the first time in 1965 by Lofti A. Zadeh [13]. Azriel Rosenfeld [9], in 1975, defined the fuzzy graph based on definitions of fuzzy sets and relations. He was the one who developed the theory of fuzzy graphs. The crux of this paper, fuzzy line graph was introduced by J. N. Mordeson in the year 1993 [5]. He together with Premchand S. Nair [7] studied different operations on fuzzy graphs and their properties. The concept of strength of connectivity between two vertices of a fuzzy graph was introduced by M. S. Sunitha [10] and extended by Sheeba M. B. [11], [12] to arbitrary fuzzy graphs. Sheeba called it, strength of the fuzzy graph and determined it, in two different ways, of which one is by introducing weight matrix of a fuzzy graph and other by introducing the concept of extra strong path between its vertices.

2. Preliminaries

A fuzzy graph $G = (V, \mu, \sigma)$ [1, 9] is a nonempty set V together with a pair of functions $\mu : V \longrightarrow [0, 1]$ and $\sigma : V \times V \longrightarrow [0, 1]$ such that for all $u, v \in V$, $\sigma(u, v) = \sigma(uv) \leq \mu(u) \wedge \mu(v)$. We call μ the fuzzy vertex set of G and σ the fuzzy edge set of G. Here after we denote the fuzzy graph $G(\mu, \sigma)$ simply by G and the underlying crisp graph of G by $G^*(V, E)$ with V as vertex set and $E = \{(u, v) \in V \times V : \sigma(u, v) > 0\}$ as the edge set or simply by G^*. The fuzzy graph $H = (\nu, \tau)$ is called a partial fuzzy subgraph of $G = (\mu, \sigma)$ if $\nu \subseteq \mu$ and $\tau \subseteq \sigma$. The fuzzy graph [8] $H = (P, \nu, \tau)$ is called a fuzzy subgraph of $G = (V, \mu, \sigma)$ if $P \subseteq V$, $\nu(x) = \mu(x)$ for all $x \in P$ and $\tau(x, y) = \sigma(x, y)$ for all $x, y \in P$. If for $(u, v) \in E$, we say that u and v are adjacent in G^*. In that case we also say that u and v are adjacent in G. An edge uv is strong [2] if and only if $\sigma(uv) = \mu(u) \wedge \mu(v)$. A fuzzy graph G is complete [7] if $\sigma(uv) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$. A fuzzy graph G is a strong fuzzy graph [6] if $\sigma(uv) = \mu(u) \wedge \mu(v)$, $\forall uv \in E$. The fuzzy line graph [5] $L(G)$, of G, is the graph with vertex set Z and edge set W, where $Z = \{\{x\} \cup \{u_x, v_x\} | x \in X, u_x, v_x \in V, x = \{u_x, v_x\}\}$ and $W = \{(S_x, S_y) | S_x \cap S_y \neq \phi, x \in X, x \neq y\}$ and where $S_x = \{x\} \cup \{u_x, v_x\}, x \in X$. Let (μ, σ) be a partial fuzzy subgraph of G. Define the fuzzy subsets λ, ω of Z, W, respectively, as follows: $\forall S_x \in Z, \lambda(S_x) = \sigma(x)$; $\forall (S_x, S_y) \in W, \omega(S_x, S_y) = \sigma(x) \wedge \sigma(y)$. A fuzzy subgraph (λ, ω) of $L(G)$ is called the line graph corresponding to (μ, σ). The line graph of a fuzzy graph is always a strong fuzzy graph [3].

A path P of length $n-1$ in a fuzzy graph G [9] is a sequence of distinct vertices $v_1, v_2, v_3, \ldots, v_n$, such that $\sigma(v_i, v_{i+1}) > 0$, $i = 1, 2, 3, \ldots, n-1$. If $v_1 = v_n$ and $n \geq 3$ we call P a cycle and cycle P is called a fuzzy cycle if it contains more than one weakest edge. The strength of a path in a fuzzy graph is defined as the weight of the weakest edge of the path [7] which is $\wedge_{i=1}^{n} \sigma(v_{i-1} v_i)$. A path P is said to connect the vertices u and v of G strongly if its strength is maximum among all paths between u and v. Such paths are called strong paths [7]. Any strong path between two distinct vertices u and v in G with minimum length is called an extra strong path between them [11]. There may exists more than one extra strong paths between two vertices in a fuzzy graph G. But, by the definition of an extra strong path each such path between two vertices has the same length. The maximum length of extra strong paths between every pair of distinct vertices in G is called the strength of the graph G [12]. For a fuzzy graph G, if G^* is the path $P = v_1 v_2 \ldots v_n$ on n vertices then the strength of the graph G is its length $(n-1)$ [11].

Here after we denote the strength of a fuzzy graph G by $\mathscr{S}(G)$.

Theorem 2.1. [12] *If G is a complete fuzzy graph, then $\mathscr{S}(G)$ is one.*

The following theorems determine the strength of a fuzzy cycle in terms of the order of its crisp graph and the number of weakest edges.

Theorem 2.2 ([11]). *In a fuzzy cycle G of length n, suppose there are l weakest edges, where $l \leq [\frac{n+1}{2}]$. If these weakest edges altogether form a subpath then $\mathscr{S}(G)$ is $n - l$.*

Theorem 2.3 ([11]). *Let G be a fuzzy cycle with crisp graph G^* a cycle of length n, having l weakest edges which altogether form a subpath. If $l > [\frac{n+1}{2}]$, then $\mathscr{S}(G)$ is $[\frac{n}{2}]$.*

Theorem 2.4 ([11]). *Let G be a fuzzy cycle with crisp graph G^* a cycle of length n, having l weakest edges which do not altogether form a subpath. If $l > [\frac{n}{2}] - 1$, then the strength of the graph is $[\frac{n}{2}]$.*

Theorem 2.5 ([11]). *In a fuzzy cycle of length n suppose there are $l < [\frac{n}{2}] - 1$ weakest edges which do not altogether form a subpath. Let s denote the maximum length of a subpath which does not contain any weakest edge. If $s \leq [\frac{n}{2}]$, then the strength of the graph is $[\frac{n}{2}]$ and if $s > [\frac{n}{2}]$, then the strength of the graph is s.*

A fuzzy wheel graph W_n [4] is the join [7] of the fuzzy cycle C_{n-1} and the fuzzy vertex K_1. The following theorems determine the strength of a strong fuzzy wheel graph.

Theorem 2.6 ([4]). *For $n \geq 4$, let $W_n = C_{n-1} \vee K_1$ be a strong fuzzy wheel graph with the fuzzy hub h and $u_1 u_2 \ldots u_{n-1} u_1$ the fuzzy cycle C_{n-1}. If $\mu(h) < \mu(u_i), \forall i = 1, 2, \ldots, n-1$, then the strength of W_n is the strength of C_{n-1}.*

Theorem 2.7 ([4]). *Let W_n be as in Theorem 2.6. If $\mu(h) \geq \mu(u_i), \forall i = 1, 2, \ldots, n-1$, then the strength of W_n is one when $n = 4$ and two when $n > 4$.*

Theorem 2.8 ([4]). *Let W_n be as in Theorem 2.2. Suppose that $\mu(h) \leq \mu(u_i)$ for some but not all the vertices $u_i, i = 1, 2, \ldots, n-1$. Let P be one of the maximal paths of C_{n-1} with the property that each edge of which has strength greater than $\mu(h)$. Let l be the length of P. Then*

$$\mathscr{S}(W_n) = \begin{cases} l & \text{if } l \geq 2 \\ 2 & \text{if } l = 1. \end{cases}$$

Definition 2.9 ([4]). A finite sequence of fuzzy graphs $G_1, G_2, \ldots, G_m$ with the property that $V(G_i) \cap V(G_j)$ is nonempty only if $|j - i| \leq 1, 1 \leq i, j \leq m$ is called a properly linked sequence or simply properly linked. It is $n-$ linked if the crisp graph induced by $V(G_i) \cap V(G_j)$ is K_n, a complete graph on n vertices, if $|j - i| = 1, 1 \leq i, j \leq m$.

Theorem 2.10 ([4]). *Let G be a properly linked fuzzy graph with the complete fuzzy graphs $G_1, G_2, \ldots, G_m$ as its parts. Suppose for $i = 1, 2, \ldots, m-1$, $< V(G_i) \cap V(G_{i+1}) >= K_{n_i}$, a complete graph on n_i vertices. Then the strength of G is m, the diameter of G.*

3. Main results

First of all we consider the strength of line graph of a strong fuzzy butterfly graph.

Theorem 3.1. *The strength of the line graph of a strong fuzzy butterfly graph [4] is three.*

Proof. The line graph $L(G)$ of a strong fuzzy butterfly graph G is a $2-$linked fuzzy graph with parts G_1, G_2, and G_3, where G_1 and G_3 are fuzzy triangles and G_2 is fuzzy complete graph on 4 vertices (A butterfly graph and its line graph are shown in figure 1). Then by Theorem 2.10, the strength of $L(G)$ is 3.

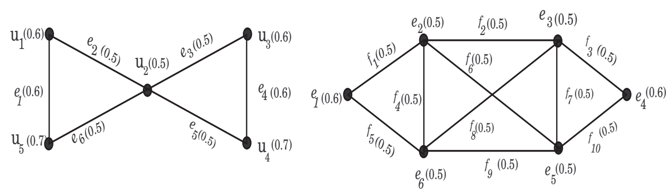

FIGURE 1. A strong fuzzy Butterfly graph G (left) and its line graph L(G)(right)

□

Theorem 3.2. *The strength of the line graph of a strong fuzzy star graph* [4] *is one.*

Proof. In a strong fuzzy star graph S_n, all the edges are adjacent. Then the line graph of the strong fuzzy star graph is a complete fuzzy graph. Thus by Theorem 2.1, the strength of the line graph of a strong fuzzy star graph is one. □

Theorem 3.3. *The strength of the line graph of a strong fuzzy bull graph* [4] *is* 2.

Proof. The line graph of a strong fuzzy bull graph is a strong fuzzy butterfly graph (A bull graph G and its line graph are shown in Figure 2). Then by Theorem 2.10, the strength of the line graph of a strong fuzzy bull graph is 2.

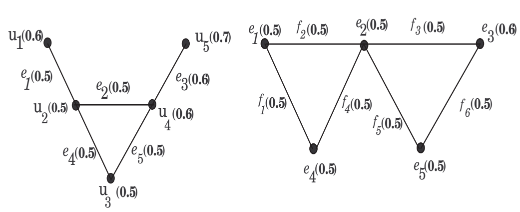

FIGURE 2. A strong fuzzy Bull graph G (left) and its line graph L(G)(right)

□

Theorem 3.4. *The strength of line graph of a strong fuzzy diamond graph* [4] *is* 2.

Proof. The line graph of a strong fuzzy diamond graph is a strong fuzzy wheel graph on 5 vertices as shown in figure 6. Then by Theorems 2.6, 2.7, 2.8 strength of line graph of a strong fuzzy diamond graph is 2.

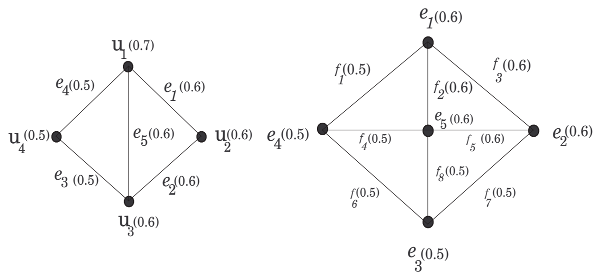

FIGURE 3. A strong fuzzy diamond graph G (left) and its line graph L(G)(right)

□

Proposition 3.5. *In a strong fuzzy cycle of length n suppose there are $l = [\frac{n}{2}] - 1$ weakest edges which do not altogether form a subpath. Let s denote the maximum length of a subpath which does not contain any weakest edge. Then*

$$\mathscr{S}(G) = \begin{cases} [\frac{n}{2}] & \text{if } \; s \leq [\frac{n}{2}], \\ s & \text{if } \; s > [\frac{n}{2}]. \end{cases}$$

Proof. Let u, v be two non-adjacent vertices of G. Then in G there are two paths joining u and v. If both the paths contain a weakest vertex then the extra strong path joining u and v is the shortest path joining u and v in its underlying crisp graph, which is of length $\leq [\frac{n}{2}]$. If u and v are the end vertices of a path having length $[\frac{n}{2}]$ then the extra strong path joining u and v is of length $= [\frac{n}{2}]$.

Otherwise, there is a $u-v$ path P having no weakest vertices. Then P is an extra strong path joining u and v. The length of P, by hypothesis, is $\leq s$. If u and v are the end vertices of the maximal subpath which does not contain any weakest edge in G then the length of P is s. Hence the proof is complete. □

Now we consider the case of the strength of line graph of a fuzzy cycle. To determine this we introduce the following definitions:

Definition 3.6. Two paths P_1 and P_2 of a fuzzy cycle C are said to be vertex disjoint or simply disjoint, if $V(P_1) \cap V(P_2) = \phi$ and edge disjoint, if $E(P_1) \cap E(P_2) = \phi$.

Definition 3.7. Suppose P_1 and P_2 are two disjoint paths of a fuzzy cycle C with respective end points u_1, v_1 and u_2, v_2. Then, $< (V(C) \setminus (V(P_1 \cup P_2)) \cup \{u_1, u_2, v_1, v_2\}) >$ is a union of two disjoint paths of C, called complementary paths relative to the paths P_1 and P_2.

Definition 3.8. Let $G(V, \mu, \sigma)$ be a fuzzy graph. A path P in G with all its edges have weight equal to w where $w = \min\{\sigma(uv) : \sigma(uv) > 0\}$ is called a weakest path. A weakest path which is not a proper subpath of any other weakest path in the fuzzy graph is called a maximal weakest path in G.

Hereafter we denote the weight of weakest paths of any fuzzy graph G by w.

Note 3.9. *A graph may have more than one maximal weakest paths. For example, in the strong fuzzy cycle G in Figure 1 $u_2u_3u_4u_5u_6u_7u_8$ and $u_8u_9u_{10}u_{11}u_{12}$ are maximal weakest paths of G.*

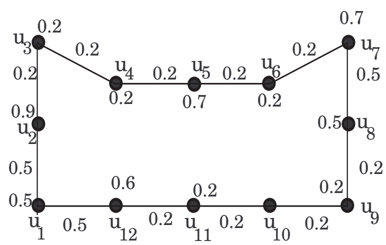

FIGURE 4. A strong fuzzy cycle G

Definition 3.10. Two paths of the collection P of pairwise disjoint paths in a fuzzy cycle C are said to be consecutive, if one of the complementary paths relative to them contains all other paths of P.

Definition 3.11. A collection P of pairwise disjoint paths in a fuzzy cycle C is said to form a chain, if its members can be arranged in a sequence $P_1, P_2, \dots, P_n$ such that $(P_1, P_2), (P_2, P_3), \dots, (P_{n-1}, P_n)$ and (P_1, P_n) are consecutive.

Proposition 3.12. *Let G be a strong fuzzy path (or a strong fuzzy cycle), then its fuzzy line graph $L(G)$ is also a strong fuzzy path (strong fuzzy cycle).*

Proof. Let G be a strong fuzzy path. Let G^* be its crisp graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{e_1, e_2, \dots, e_n\}$, where $e_i = v_iv_{i+1}, i = 1, 2, \dots, n-1$. Since for $1 < i < n-1$ the edge e_i in G^* is adjacent only to the edge e_{i-1} and e_{i+1}, the vertex e_i of the crisp graph $L^*(G)$ of $L(G)$ is adjacent only to the vertices e_{i-1} and e_{i+1} of $L^*(G)$. Since the edge e_1 of G^* is adjacent only to the edge e_2 of G^* and the edge e_n of G^* is adjacent only to the edge e_{n-1} of G^*, the vertices e_1 and e_n of $L^*(G)$ are adjacent only to its vertices e_2 and e_{n-1} respectively. Thus $L^*(G)$ is a path with vertices $e_1, e_2, \dots, e_n$ and edges $e_1e_2, e_2e_3, \dots, e_{n-1}e_n$. The lemma now follows from the definition of $L(G)$. □

Similar is the case of a fuzzy cycle.

Proposition 3.13. *If P is a weakest path of length k in a strong fuzzy graph G, then in the fuzzy line graph $L(G)$ of G the path P' corresponding to the path P of G with vertex set as edge set of P is a weakest path in $L(G)$ of length $k-1$.*

Theorem 3.14. *Let G be a strong fuzzy cycle of length n. Suppose there are l weakest edges which form m maximal weakest paths in G, where $2 \leq m \leq [\frac{l}{2}]$. Then in the line graph $L(G)$ of G there are $l+m$ weakest edges.*

Proof. By Proposition 3.13, for a weakest path P of G with strength w and length l, the path P' of $L(G)$ with vertex set as edge set of P is a path of length $(l-1)$ with weight w. Note that the end vertices of u and v of P' are also have weight w. Then the edges of the complementary paths 3.10 incident with u and v in $L(G)$ are also have weight w. Thus each maximal weakest path P in G of length l gives a weakest path in $L(G)$ of length $l+1$.So for m such weakest paths, there are $(l+m)$ weakest edges in $L(G)$.

Also if P_1' and P_2' are two paths of $L(G)$ obtained as explained above, corresponding to two distinct maximal paths P_1 and P_2 of G, then they are edge disjoint. [Note that the path P' of L(G) thus obtained need not be maximal. See Figure 2]. □

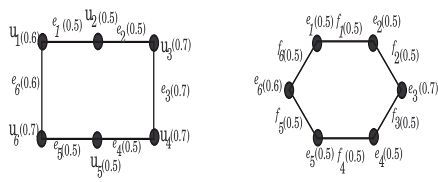

FIGURE 5. A fuzzy cycle G of length 6 with 4 weakest edges (Left) and its line graph $L(G)$ (Right).

Proposition 3.15. *Suppose P_1 and P_2 are two disjoint weakest paths of lengths n_1 and n_2 respectively in the fuzzy cycle C. Suppose one of the complementary paths relative to these paths is of length one, then there exist a weakest path of length (n_1+n_2) in $L(G)$ with edges of P_1, P_2 and P as vertex set.*

Theorem 3.16. *Let G be a strong fuzzy cycle of length n. Suppose G contains exactly one maximal weakest path of P. Let its length be l. Then the strength $\mathscr{S}(L(G))$ of the line graph $L(G)$ of G is*

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) - 1 & \text{if } \; l \leq [\frac{n-1}{2}], \\ \mathscr{S}(G) & \text{if } \; l > [\frac{n-1}{2}]. \end{cases}$$

Proof. Since P is a path of length l in G by Proposition 3.13, the path P' of $L(G)$ with vertex set as edge set of P is a weakest path of $L(G)$ of length $l-1$. If $l = n-1$, then all edges in G but one is weakest. In this case, all the edges of $L(G)$ are weakest. Thus by 2.3, $\mathscr{S}(L(G)) = [\frac{n}{2}] = \mathscr{S}(G)$. Let us suppose that $l < n-1$. Since all the vertices of P' are weakest, the edges incident to the vertices of P' are also weakest edges and all the other edges are non-weakest. So there are $l+1$ edges incident with the vertices of P' by Theorem 3.14. Hence the maximal weakest path of $L(G)$ is of length $l+1$. Therefore by Theorem 2.2,

$$\mathscr{S}(L(G)) = \begin{cases} n-(l+1) & \text{if } \; l+1 \leq [\frac{n+1}{2}], \\ [\frac{n}{2}] & \text{if } \; l+1 > [\frac{n+1}{2}], \end{cases}$$

$$= \begin{cases} \mathscr{S}(G) - 1 & \text{if } \; l \leq [\frac{n-1}{2}], \\ \mathscr{S}(G) & \text{if } \; l > [\frac{n-1}{2}]. \end{cases}$$

□

Theorem 3.17. *Let G be a strong fuzzy cycle of length n. Let there be m maximal weakest paths $P_1, P_2, \ldots, P_m$ in G, where $m \geq 1$. Suppose, for $i = 1, 2, \ldots, m-1$ one of the complementary paths Q_i between P_i and P_{i+1} is of length one such that $P_1 Q_1 P_2 Q_2 \ldots P_{m-1} Q_{m-1} P_m$ is a path of length $l+m$. Then the strength $\mathscr{S}(L(G))$ of $L(G)$ is*

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) - m & \text{if } \ l \leq [\frac{n+1}{2}] - m, \\ [\frac{n}{2}] & \text{if } \ l > [\frac{n+1}{2}] - m. \end{cases}$$

Proof. Let Q be the complementary path between P_m and P_1 which does not contain any of the paths $P_1, P_2, P_3, \ldots, P_m$. If Q is of length one then in $L(G)$ either both ends of each edge are weakest vertices or one of the ends is a weakest vertex. Thus every edge in $L(G)$ in this case is a weakest edge. Hence $\mathscr{S}(L(G)) = [\frac{n}{2}] = \mathscr{S}(G)$ by Theorem 2.3 . Now suppose that the length of Q is not one. Note that the vertices of $L(G)$ corresponding to the edges of $P_1, P_2, \ldots, P_m$ are weakest. Though the vertices of $L(G)$ corresponding to the edges in $Q_1 \cup Q_2 \cup \ldots \cup Q_{m-1}$ are not weakest, the edges incident with them have weakest vertices on the other end. Thus all edges of the path in $L(G)$ with vertex set as edge set of the path $P_1 Q_1 P_2 Q_2 \ldots P_{m-1} Q_{m-1} P_m$ together form a weakest path P of length $l+m-1$. Since there are more than one edge in Q, the edge e_1 of Q incident with P_1 and the edge e_2 of Q incident with the path P_m are different in $L(G)$. The vertex of $L(G)$ corresponding to the edge e_1 of G is adjacent to one end vertex of P by a weakest edge and the vertex of $L(G)$ corresponding to the edge e_2 is adjacent to the other end of P by a weakest edge. All other edges of $L(G)$ are of non weakest. Hence $L(G)$ contains only one maximal weakest path of length $l+m$. Therefore the strength $\mathscr{S}(L(G))$ of $L(G)$ is

$$\mathscr{S}(L(G)) = \begin{cases} n - (l+m) & \text{if } \ l \leq [\frac{n+1}{2}] - m, \\ [\frac{n}{2}] & \text{if } \ l > [\frac{n+1}{2}] - m. \end{cases}$$

$$= \begin{cases} (n-l) - m & \text{if } \ l \leq [\frac{n+1}{2}] - m, \\ [\frac{n}{2}] & \text{if } \ l > [\frac{n+1}{2}] - m. \end{cases}$$

$$= \begin{cases} \mathscr{S}(G) - m & \text{if } \ l \leq [\frac{n+1}{2}] - m, \\ [\frac{n}{2}] & \text{if } \ l > [\frac{n+1}{2}] - m. \end{cases}$$

Hence the proof. □

Theorem 3.18. *Let G be a strong fuzzy cycle of length n. Suppose there are l weakest edges in G which do not altogether form a subpath in G. Let $P_1, P_2, \ldots, P_m$ be the chain of all m maximal weakest paths in G. Suppose there exists atleast two indices $i < j$ such that the complementary paths between P_i, P_{i+1} and P_j, P_{j+1} which do not contain any of the P_k's are of length greater than one (when $j = m, P_{j+1} = P_1$). Let s denote the maximum length of the subpath which does not contain any weakest edge of G. Then if $l < [\frac{n}{2}] - (m+1)$ the strength $\mathscr{S}((L(G))$ of the line graph $L(G)$ of G is*

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) & \text{if } \ s \leq [\frac{n}{2}] \ \text{ or } \ s > [\frac{n}{2}] + 1 \\ & \text{or } \ s = [\frac{n}{2}] + 1 \ \text{ and } n \text{ even}, \\ \mathscr{S}(G) - 1 & \text{if } \ s = [\frac{n}{2}] + 1 \ \text{ and } n \text{ odd}. \end{cases}$$

Proof. Since the l weakest edges of G are distributed to form m maximal weakest paths in G, then there are $l+m$ weakest edges in $L(G)$. Also the maximum length of paths in $L(G)$ which do not contain any weakest edge is clearly $s-1$. By applying the Theorem 2.5 the strength $\mathscr{S}(L(G))$ of $L(G)$, when $l+m < [\frac{n}{2}]-1$

$$\mathscr{S}(L(G)) = \begin{cases} [\frac{n}{2}] & \text{if } s \leq [\frac{n}{2}]+1, \\ s-1 & \text{if } s > [\frac{n}{2}]+1. \end{cases}$$

Consider the case $s \leq [\frac{n}{2}]+1$. Then either $s \leq [\frac{n}{2}]$ or $s = [\frac{n}{2}]+1$. Also $l+m < [\frac{n}{2}]-1$ implies that $l < [\frac{n}{2}]-1$. So when $s \leq [\frac{n}{2}]$, $\mathscr{S}(G) = [\frac{n}{2}] = \mathscr{S}(L(G))$.

When $s = [\frac{n}{2}]+1$,

$$\mathscr{S}(G) = [\frac{n+1}{2}] = \begin{cases} [\frac{n}{2}] & \text{if } n \text{ even}, \\ [\frac{n}{2}]+1 & \text{if } n \text{ odd}. \end{cases}$$

where as $\mathscr{S}(L(G)) = [\frac{n}{2}]$ which implies

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) & \text{if } n \text{ even}, \\ \mathscr{S}(G)-1 & \text{if } n \text{ odd}. \end{cases}$$

When $s > [\frac{n}{2}]+1$, $s > \frac{n}{2}$. Therefore $\mathscr{S}(G) = \mathscr{S}(L(G))$. Therefore

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) & \text{if } s \leq [\frac{n}{2}] \text{ or} \\ & s > [\frac{n}{2}]+1 \text{ or } s = [\frac{n}{2}]+1 \text{ and } n \text{ even}, \\ \mathscr{S}(G)-1 & \text{if } s = [\frac{n}{2}]+1 \text{ and } n \text{ odd}. \end{cases}$$

Hence the proof. □

Theorem 3.19. *Let G be a strong fuzzy cycle of length n. Suppose there are l weakest edges in G which do not altogether form a subpath in G. Let us suppose that these weakest edges form a chain of paths $P_1, P_2, \ldots, P_n$. Also suppose there exist atleast two indices $i < j$ such that the complementary paths between P_i, P_{i+1} and P_j, P_{j+1} which do not contain any one of the P_ks are of length greater than one (when $j = m, P_{j+1} = P_1$ in G). Let s denote the maximum length of the subpaths which do not contain any weakest edge in G. Then if $l > [\frac{n}{2}]-(m+1)$ the strength $\mathscr{S}(L(G))$ of the line graph $L(G)$ of G is*

$$\mathscr{S}(L(G) = \begin{cases} \mathscr{S}(G) & \text{if } l > [\frac{n}{2}]-1, \text{ or if } l \leq [\frac{n}{2}]-1 \text{ and } s \leq [\frac{n}{2}], \\ \mathscr{S}(G)-1 & \text{if } l \leq [\frac{n}{2}]-1, \text{ and } s > [\frac{n}{2}]. \end{cases}$$

If $l = [\frac{n}{2}]-(m+1)$ then

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) & \text{if } l < [\frac{n}{2}]-1, \ s \leq [\frac{n}{2}] \text{ and } n \text{ is odd}, \\ \mathscr{S}(G)+1 & \text{if } l < [\frac{n}{2}]-1, \ s \leq [\frac{n}{2}] \text{ and } n \text{ even}, \\ \mathscr{S}(G)-1 & \text{if } l < [\frac{n}{2}]-1, \ s > [\frac{n}{2}]. \end{cases}$$

Proof. For $l > [\frac{n}{2}]-(m+1)$, consider the following cases.

Case 3.20. $l > [\frac{n}{2}]-1$.

Here, by applying the Theorem 2.4 we get $\mathscr{S}(L(G)) = [\frac{n}{2}]$ which is equal to $\mathscr{S}(G)$.

Case 3.21. $l \leq [\frac{n}{2}] - 1 < l + m$

Then by Lemma 3.5 and by Theorem 2.5

$$\mathscr{S}(G) = \begin{cases} [\frac{n}{2}] & \textit{if} \;\; s \leq [\frac{n}{2}], \\ s & \textit{if} \;\; s > [\frac{n}{2}]. \end{cases}$$

That is if $s \leq [\frac{n}{2}]$ then $s - 1 \leq [\frac{n}{2}]$ which gives $\mathscr{S}((L(G)) = [\frac{n}{2}] = \mathscr{S}(G)$.(See Figure 6 with $n = 12, l = 4, m = 2$ and Figure 7 with $n = 13, l = 5, m = 2$). *If* $s > [\frac{n}{2}]$ then $s - 1 = [\frac{n}{2}]$. So $\mathscr{S}(L(G)) = [\frac{n}{2}] = s - 1 = \mathscr{S}(G) - 1$. (See Figure 8 with $n = 13, l = 4, m = 2$.)

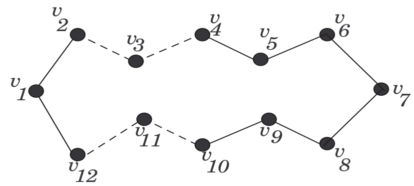

Figure 6. A fuzzy graph G with 12 vertices and 4 nonconsecutive weakest edges.

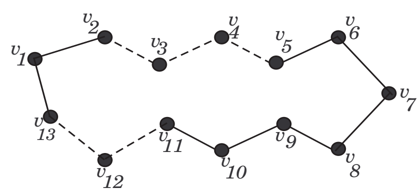

Figure 7. A fuzzy graph G with 13 vertices and 5 nonconsecutive weakest edges.

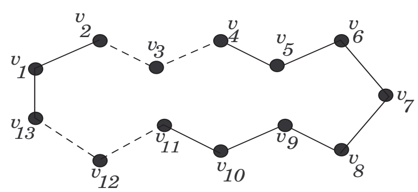

Figure 8. A fuzzy graph G with 13 vertices and 4 nonconsecutive weakest edges.

Consider the case $l = [\frac{n}{2}] - (m+1)$ then $\mathscr{S}(L(G)) = [\frac{n+1}{2}]$. Since $m \geq 2$, $l < [\frac{n}{2}] - 1$. By applying the Theorem [11] $\mathscr{S}(G) = [\frac{n}{2}]$ if $s \leq [\frac{n}{2}]$. So

$$\mathscr{S}(L(G)) = [\frac{n+1}{2}] = \begin{cases} [\frac{n}{2}] & \textit{if } n \textit{ even,} \\ [\frac{n}{2}] + 1 & \textit{if } n \textit{ odd.} \end{cases}$$

Therefore

$$\mathscr{S}(L(G)) = \begin{cases} \mathscr{S}(G) & \textit{if } n \textit{ even,} \\ \mathscr{S}(G) + 1 & \textit{if } n \textit{ odd.} \end{cases}$$

If $l < [\frac{n}{2}] - 1$ then if $s > [\frac{n}{2}]$, $\mathscr{S}(G) = s$. So $\mathscr{S}(L(G)) = s - 1 = \mathscr{S}(G) - 1$. Hence the proof. □

4. Applications

In many real world problems, we get partial information about that problem. So there is a vagueness in the description of the objects or in its relationships or in both. The fuzzy graph seems to be a relevant mathematical model for solving it. The theorems discussed in this paper have vital applications in the following areas:

- Mobile transmission network.
- Decision making theory.
- Geographical Information Systems (GIS) based road network analysis, GIS based water supply network etc.

5. Conclusions

In this paper we determined the strength of the line graphs of strong fuzzy butterfly graph, strong fuzzy star graph, strong fuzzy bull graph, strong fuzzy diamond graph, strong fuzzy path and strong fuzzy cycle.

Acknowledgements. The first author is grateful to University of Calicut for the fellowship under which research was conducted.

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