Annals of Fuzzy Mathematics and Informatics
Volume 12, No. 4, (October 2016), pp. 539–546
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
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# On connectedness and disconnectedness in $Q$-TOP

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Received 5 February 2016; Revised 23 March 2016; Accepted 21 April 2016

Abstract. In this paper, we have briefly studied the concepts of connectednesses and disconnectednesses for $Q$-topological spaces with respect to a class of $Q$-topological spaces, motivated by the works of Preuss, Arhangel'skii and Wiegandt and few others. Also, we have given some characterization(s) of connectednesses and disconnectednesses for $Q$-topological spaces. We have also showed that within the class of stratified $Q$-topological spaces, the subclass of $T_0$-$Q$-topological spaces is a disconnectedness.

2010 AMS Classification: 54A40, 18A05

Keywords: $\Omega$-algebra, $Q$-topological space, $Q$-Sierpinski space, $T_0$-$Q$-topological space, $Q$-connectedness, $Q$-disconnectedness.

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## 1. Introduction

In 2008, Solovyov [9] introduced the notion of $Q$-topology, where $Q$ is any fixed member of a variety of $\Omega$-algebras, which provides a common framework for fixed-basis approach to lattice-valued topology (which in turn gives rise to the category $Q$-**TOP** of such spaces). In addition to this, he also introduced (in [9]) $Q$-Sierpinski space and $T_0$-ness for $Q$-topological spaces. In topology (and also in fuzzy topology), $T_0$-ness has been shown to be a kind of 'disconnectedness' in the sense of G. Preuss [4, 5, 6] (a notion, studied also by Arhangel'skii and Wiegandt [2] in considerable detail). In this paper, we have shown an analogous connection for $T_0$-$Q$-topological spaces, in addition to giving some characterization(s) of connectednesses and disconnectednesses for $Q$-topological spaces. In the later part of the paper, we restrict to stratified $Q$-topological spaces to obtain the analogues of some of the above mentioned connections, wherein the $Q$-Sierpinski space will also play a role.

## 2. Preliminaries

For all undefined category-theoretic notions used in this paper, [1] may be referred. All subcategories used here are assumed to be full.

We begin by recalling the notions of $\Omega$-algebras and their homomorphisms (most of the definitions in the preliminaries are given in [7, 8] also; we recall these here for the sake of completeness); for details, cf. [3, 9].

**Definition 2.1.** Let $\Omega = (n_\lambda)_{\lambda\in I}$ be a class of cardinal numbers.

- An $\Omega$-**algebra** is a pair $(A, (\omega_\lambda^A)_{\lambda\in I})$ consisting of a set $A$ and a family of maps $\omega_\lambda^A : A^{n_\lambda} \to A$. $B \subseteq A$ is called a **subalgebra** of $(A, (\omega_\lambda^A)_{\lambda\in I})$ if $\omega_\lambda^A((b_i)_{i\in n_\lambda}) \in B$, for every $\lambda \in I$ and every $(b_i)_{i\in n_\lambda} \in B^{n_\lambda}$. Given $S \subseteq A$, $\langle S\rangle$ denotes the subalgebra of $(A, (\omega_\lambda^A)_{\lambda\in I})$ 'generated by $S$', i.e., $\langle S\rangle$ is the intersection of all subalgebras of $(A, (\omega_\lambda^A)_{\lambda\in I})$ containing $S$.
- Given $\Omega$-algebras $(A, (\omega_\lambda^A)_{\lambda\in I})$ and $(B, (\omega_\lambda^B)_{\lambda\in I})$, a map $f : A \to B$ is called an $\Omega$-**algebra homomorphism** provided that for every $\lambda \in I$, the following diagram

$$\begin{tikzcd} A^{n_\lambda} \arrow[r, "f^{n_\lambda}"] \arrow[d, "\omega_\lambda^A"'] & B^{n_\lambda} \arrow[d, "\omega_\lambda^B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

Let $\mathbf{Alg}(\Omega)$ denote the category of $\Omega$-algebras and $\Omega$-algebra homomorphisms (this category has products).

- A **variety** of $\Omega$-algebras is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, subalgebras, and homomorphic images.

Throughout this paper, $\Omega = (n_\lambda)_{\lambda\in I}$ denotes a fixed class of cardinal numbers, $\mathbf{V}$ denotes a fixed variety of $\Omega$-algebras and $Q$ denotes a fixed member of $\mathbf{V}$.

Each function $f : X \to Y$ between sets $X$ and $Y$ provides two functions $f^{\leftarrow} : 2^Y \to 2^X$ and $f^{\rightarrow} : 2^X \to 2^Y$, given by $f^{\leftarrow}(B) = \{x \in X \mid f(x) \in B\}$ and $f^{\rightarrow}(A) = \{f(x) \mid x \in A\}$, and also a function $f_Q^{\leftarrow} : Q^Y \to Q^X$, given by $f_Q^{\leftarrow}(\alpha) = \alpha \circ f$.

- Given a set $X$, a subset $\tau$ of $Q^X$ is called a $Q$-**topology** on $X$ if $\tau$ is a subalgebra of $Q^X$, in which case the pair $(X, \tau)$ is called a $Q$-**topological space**.
- Given two $Q$-topological spaces $(X, \tau)$ and $(Y, \eta)$, a $Q$-**continuous function** from $(X, \tau)$ to $(Y, \eta)$ is a function $f : X \to Y$ such that $f_Q^{\leftarrow}(\alpha) \in \tau$ for every $\alpha \in \eta$.
- Given a $Q$-topological space $(X, \tau)$ and $Y \subseteq X$, $(i_Q^{\leftarrow})^{\rightarrow}(\tau)$ $(= \{p \circ i \mid p \in \tau\})$ is called the $Q$-**subspace topology** on $Y$, where $i : Y \to X$ is the inclusion map. We shall denote the $Q$-subspace topology on $Y$ as $\tau_Y$.

- A $Q$-topological space $(X, \tau)$ is called $\mathbf{T}_0$ if for every distinct $x, y \in X$, there exists $p \in \tau$ such that $p(x) \neq p(y)$.

The meanings of homeomorphisms, embeddings, and products, etc. for $Q$-topological spaces are on expected lines.

Let $Q$-**TOP** denote the category of $Q$-topological spaces and $Q$-continuous maps between them.

## 3. $Q$-connectedness and $Q$-disconnectedness

In what follows, we will frequently deal with the classes of $Q$-topological spaces. We assume that all such considered classes are closed under $Q$-homeomorphisms.

If $\mathcal{A}$ is any class of $Q$-topological spaces, then we shall refer to its members as "$\mathcal{A}$-spaces".

A class $\mathcal{A}$ of $Q$-topological spaces is called

(i) hereditary, if each $Q$-subspace of each $\mathcal{A}$-space is in $\mathcal{A}$,

(ii) continuously closed, if each $Q$-continuous image of any $\mathcal{A}$-space is in $\mathcal{A}$.

Further, we define the classes $D\mathcal{A}$ and $C\mathcal{A}$ of $Q$-topological spaces as follows.

$D\mathcal{A} = \{X \in obQ\text{-}\mathbf{TOP} \mid$ every $Q$-continuous map $f : Y \to X$ is constant, for every $Y \in \mathcal{A}\}$ and

$C\mathcal{A} = \{X \in obQ\text{-}\mathbf{TOP} \mid$ every $Q$-continuous map $f : X \to Y$, $Y \in \mathcal{A}$ is constant$\}$.

**Definition 3.1.** The class $D\mathcal{A}$ will be called a $Q$-disconnectedness and the class $C\mathcal{A}$ will be called a $Q$-connectedness. The operators $D$ and $C$ are referred to as $Q$-disconnectedness and $Q$-connectedness operators respectively.

It can be easily observed that every $Q$-disconnectedness and $Q$-connectedness contains all trivial $Q$-topological spaces.

**Remark 3.2.** In the above definition, without any loss of generality, $\mathcal{A}$ can be chosen to a continuously closed/hereditary class of $Q$-topological spaces.

**Proposition 3.3.** *Let $\mathcal{A}$ be a class of $Q$-topological spaces. Then*

(1) *If $\mathcal{A}$ is continuously closed, then*

$$D\mathcal{A} = \{X \in obQ\text{-}\mathbf{TOP} \mid \textit{no non-trivial } \mathcal{A}\textit{-space is a } Q\textit{-subspace of } X\}.$$

(2) *If $\mathcal{A}$ is hereditary, then*

$$C\mathcal{A} = \{X \in obQ\text{-}\mathbf{TOP} \mid X \textit{ cannot be mapped } Q\textit{-continuously onto a non-trivial } \mathcal{A}\textit{-space}\}.$$

*Proof.* For convenience, we denote $\{X \in obQ\text{-}\mathbf{TOP} \mid$ no non-trivial $\mathcal{A}$-space is a $Q$-subspace of $X\}$ by $\mathcal{U}$ and $\{X \in obQ\text{-}\mathbf{TOP} \mid X$ cannot be mapped $Q$-continuously onto a non-trivial $\mathcal{A}$-space$\}$ by $\mathcal{V}$.

(1) Let $X \in D\mathcal{A}$ and $Y$ be a $Q$-subspace of $X$. If $Y$ is an $\mathcal{A}$-space, then the inclusion map $i : Y \to X$, being $Q$-continuous, must be constant but then $Y$ comes out to a trivial $\mathcal{A}$-space. Thus $X \in \mathcal{U}$. So $D\mathcal{A} \subseteq \mathcal{U}$. Now if $X \notin D\mathcal{A}$, then there is an $\mathcal{A}$-space $Y$ and a non-constant $Q$-continuous map $f : Y \to X$. As $\mathcal{A}$ is continuously closed, $f(Y)$ is a non-trivial $\mathcal{A}$-space and also it is a $Q$-subspace of $X$. Thus $X \notin \mathcal{U}$. So $\mathcal{U} \subseteq D\mathcal{A}$. Hence $D\mathcal{A} = \mathcal{U}$. This proves (1).

(2) Let $X \in C\mathcal{A}$ and $f : X \to Y \in \mathcal{A}$ be an onto $Q$-continuous map. Then $f$ must be constant, whereby $Y$ turns out to a trivial $\mathcal{A}$-space. Thus $X \in \mathcal{V}$. So

$C\mathcal{A} \subseteq \mathcal{V}$. Now suppose that $X \notin C\mathcal{A}$. Then there is a non-constant $Q$-continuous map $f : X \to Y \in \mathcal{A}$, whereby $f(X)$ is a non-trivial $Q$-subspace of $Y$. As $\mathcal{A}$ is hereditary, $f(X) \in \mathcal{A}$, showing that $f$ maps $X$ $Q$-continuously onto a non-trivial $\mathcal{A}$-space, namely, $f(X)$. Hence $X \notin \mathcal{V}$, and thus, $\mathcal{V} \subseteq C\mathcal{A}$. Therefore $C\mathcal{A} = \mathcal{V}$. □

**Proposition 3.4.** *Let $\mathcal{A}_1, \mathcal{A}_2$ be any classes of $Q$-topological spaces. Then*
(1) $\mathcal{A}_1 \subseteq \mathcal{A}_2 \Rightarrow D\mathcal{A}_1 \supseteq D\mathcal{A}_2$ *and* $C\mathcal{A}_1 \supseteq C\mathcal{A}_2$.
(2) $\mathcal{A}_1 \subseteq DC\mathcal{A}_1$ *and* $\mathcal{A}_1 \subseteq CD\mathcal{A}_1$.
(3) $DCD\mathcal{A}_1 = D\mathcal{A}_1$ *and* $CDC\mathcal{A}_1 = C\mathcal{A}_1$.

*Proof.* (1) Let $\mathcal{A}_1 \subseteq \mathcal{A}_2$. If $X \in D\mathcal{A}_2$, then every $Q$-continuous map from each $\mathcal{A}_2$-space to $X$ is constant. Thus, in particular, every $Q$-continuous map from each $\mathcal{A}_1$-space to $X$ is constant (as $\mathcal{A}_1 \subseteq \mathcal{A}_2$). So $X \in D\mathcal{A}_1$. Thus, $D\mathcal{A}_1 \supseteq D\mathcal{A}_2$. Similarly, it can be shown that $C\mathcal{A}_1 \supseteq C\mathcal{A}_2$.

(2) If $X \notin DC\mathcal{A}_1$, then there is a non-trivial $Y \in C\mathcal{A}_1$ and a non-constant $Q$-continuous map $f : Y \to X$, which in turn implies that $X \notin \mathcal{A}_1$. Thus $\mathcal{A}_1 \subseteq DC\mathcal{A}_1$. Similarly, it can be shown that $\mathcal{A}_1 \subseteq CD\mathcal{A}_1$.

(3) Suppose that $X \notin DCD\mathcal{A}_1$. Then there is some non-trivial $Y \in CD\mathcal{A}_1$ and a non-constant $Q$-continuous map $f : Y \to X$, which in turn implies that $X \notin D\mathcal{A}_1$. Thus $D\mathcal{A}_1 \subseteq DCD\mathcal{A}_1$. Next, let $X \notin D\mathcal{A}_1$. Then there is some non-trivial $Y \in \mathcal{A}_1$ and a non-constant $Q$-continuous map $f : Y \to X$, whereby we are getting that $X \notin DCD\mathcal{A}_1$ (since $\mathcal{A}_1 \subseteq CD\mathcal{A}_1$, using (2)). So $DCD\mathcal{A}_1 \subseteq D\mathcal{A}_1$. Hence $DCD\mathcal{A}_1 = D\mathcal{A}_1$. Similarly, it can be shown that $CDC\mathcal{A}_1 = C\mathcal{A}_1$. □

**Corollary 3.5.** *For any classes $\mathcal{A}, \mathcal{A}_1$ and $\mathcal{A}_2$ of $Q$-topological spaces, we have :*
(1) *If $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then $CD\mathcal{A}_1 \subseteq CD\mathcal{A}_2$.*
(2) *If $\mathcal{A}$ is a $Q$-disconnectedness, then $DC\mathcal{A} = \mathcal{A}$, and if $\mathcal{A}$ is a $Q$-connectedness, then $CD\mathcal{A} = \mathcal{A}$.*

**Proposition 3.6.** *For any class $\mathcal{A}$ of $Q$-topological spaces, $D\mathcal{A}$ is hereditary and $C\mathcal{A}$ is continuously closed.*

*Proof.* Let $X \in D\mathcal{A}$ and $Y \subseteq X$. If possible, suppose that $Y \notin D\mathcal{A}$ ($Y$ has $Q$-subspace topology). Then there is some $\mathcal{A}$-space $Z$ and a non-constant $Q$-continuous function $f : Z \to Y$, but then we have a non-constant $Q$-continuous function, namely, $i \circ f : Z \to X$ ($i : Y \to X$ is the inclusion function), showing that $X \notin D\mathcal{A}$, a contradiction. Thus $Y \in D\mathcal{A}$.

Next, let $X \in C\mathcal{A}$ and $Y$ be a non-trivial $Q$-**TOP**-object which is $Q$-continuous image of $X$, i.e., there is an onto (clearly it is non-constant also) $Q$-continuous function $f : X \to Y$. If possible, suppose that $Y \notin C\mathcal{A}$. Then there is some $\mathcal{A}$-space $Z$ and a non-constant $Q$-continuous function $g : Y \to Z$, but then we have a non-constant $Q$-continuous function, namely $g \circ f : X \to Z$, whereby $X \notin C\mathcal{A}$, a contradiction. So $Y \in C\mathcal{A}$. □

**Proposition 3.7.** *A class $\mathcal{B}$ of $Q$-topological spaces is a $Q$-disconnectedness if and only if $\mathcal{B}$ satisfies the condition that $X \in \mathcal{B}$ if and only if each non-trivial $Q$-subspace of $X$ can be mapped $Q$-continuously onto a non-trivial $\mathcal{B}$-space.*

*Proof.* Let the class $\mathcal{B}$ be a $Q$-disconnectedness. Then $\mathcal{B} = D\mathcal{A}$, for some continuously closed class $\mathcal{A}$ of $Q$-topological spaces. We will show that if $X \notin D\mathcal{A}$, then there is some non-trivial $Q$-subspace $Y$ of $X$ which cannot be mapped $Q$-continuously onto any non-trivial $\mathcal{B}$-space. Now as $X \notin D\mathcal{A}$, there exists a non-trivial $\mathcal{A}$-space $Y$ and a non-constant $Q$-continuous map $f : Y \to X$. Note that $f(Y)$ is also an $\mathcal{A}$-space and it is a non-trivial $Q$-subspace of $X$. If possible, suppose that $f(Y)$ is mapped $Q$-continuously onto a non-trivial $\mathcal{B}(= D\mathcal{A})$-space $Z$, i.e., there is some non-constant $Q$-continuous map $g : f(Y) \to Z$, showing that $Z \notin D\mathcal{A}$, a contradiction. Next, let $X \in \mathcal{B}(= D\mathcal{A})$ and $Y$ be any non-trivial $Q$-subspace of $X$. Then $Y \in \mathcal{B}(= D\mathcal{A})$ (as $D\mathcal{A}$ is hereditary). Note that $id : Y \to Y$ is an onto $Q$-continuous map. In other words, $Y$ can be mapped $Q$-continuously onto a non-trivial $\mathcal{B}$-space, viz., $Y$ itself.

Conversely, suppose that the class $\mathcal{B}$ of $Q$-topological spaces satisfies the given condition. We show that $\mathcal{B}$ is $Q$-disconnectedness by showing that $\mathcal{B} = DC\mathcal{B}$. For this, it is sufficient to show that $DC\mathcal{B} \subseteq \mathcal{B}$ (as $\mathcal{B} \subseteq DC\mathcal{B}$ by Proposition 3.4(2)). If $X \notin \mathcal{B}$, then by hypothesis, there is some non-trivial $Q$-subspace $Y$ of $X$ which cannot be mapped $Q$-continuously onto any non-trivial $\mathcal{B}(= D\mathcal{A})$-space, whereby $Y \in C\mathcal{B}$ (by Proposition 3.3), which in turn implies that $X \notin DC\mathcal{B}$ (again by Proposition 3.3). Thus $DC\mathcal{B} \subseteq \mathcal{B}$. □

**Remark 3.8.** For a hereditary class $\mathcal{B}$ of $Q$-topological spaces, following conditions are equivalent:

(1) Every non-trivial $Q$-subspace $Y$ of $X \in \mathcal{B}$ can be mapped $Q$-continuously onto a non-trivial $\mathcal{B}$-space.

(2) For every non-trivial $Q$-subspace $Y$ of $X \in \mathcal{B}$, there is some non-constant $Q$-continuous map from $Y$ to a non-trivial $\mathcal{B}$-space.

**Proposition 3.9.** *A class $\mathcal{B}$ of $Q$-topological spaces is a $Q$-disconnectedness if and only if $\mathcal{B}$ satisfies the condition that $X \in \mathcal{B}$ if and only if for every non-trivial $Q$-subspace $Y$ of $X \in \mathcal{B}$, there is some non-constant $Q$-continuous map from $Y$ to a non-trivial $\mathcal{B}$-space.*

*Proof.* A proof of the necessary part follows from Proposition 3.7 and Remark 3.8, and the fact that the class $\mathcal{B}$ is hereditary, being a $Q$-disconnectedness.

A proof of the sufficient part is given as follows:
Suppose that the class $\mathcal{B}$ of $Q$-topological spaces satisfies the given condition. We show that $\mathcal{B}$ is $Q$-disconnectedness by showing that $\mathcal{B} = DC\mathcal{B}$. For this, we will only show that $DC\mathcal{B} \subseteq \mathcal{B}$. If $X \notin \mathcal{B}$, then by hypothesis, there is some non-trivial $Q$-subspace $Y$ of $X$ such that every $Q$-continuous map $f : Y \to Z \in \mathcal{B}$ is constant, whereby $Y \in C\mathcal{B}$. By using Proposition 3.3, we get that $X \notin DC\mathcal{B}$. Thus $DC\mathcal{B} \subseteq \mathcal{B}$. □

**Proposition 3.10.** *Every $Q$-disconnectedness $\mathcal{B}$ is productive.*

*Proof.* Let $\mathcal{B}$ be $Q$-disconnectedness. Then $\mathcal{B} = D\mathcal{A}$, for some continuously closed class $\mathcal{A}$ of $Q$-topological spaces. Let $X_j \in \mathcal{B}(= D\mathcal{A})$, for every $j \in J$ ($J$ is some index set). Let $Y \in \mathcal{A}$ and $f : Y \to X$ is a $Q$-continuous map, where $X$ is the product $Q$-topology of $Q$-topological spaces $X_j, j \in J$. Let $p_j : X \to X_j$ be the $j^{th}$-projection map. Then $p_j \circ f : Y \to X_j$ is $Q$-continuous, for every $j \in J$. Since for every $j \in J$, $X_j \in \mathcal{B}$, all $p_j \circ f$ are constant maps, whereby it follows that $f$ is a constant map. Thus we get that every $Q$-continuous map $f : Y \to X$ must be constant, for every $Y \in \mathcal{A}$. So $X \in \mathcal{B}$. □

**Proposition 3.11.** *A class $\mathcal{A}$ of $Q$-topological spaces is a $Q$-connectedness if and only if $\mathcal{A}$ satisfies the condition that $X \in \mathcal{A}$ if and only if every non-trivial $Q$-continuous image of $X$ has a non-trivial $Q$-subspace which is an $\mathcal{A}$-space.*

*Proof.* First, let $\mathcal{A}$ be $Q$-connectedness. Then $\mathcal{A} = C\mathcal{B}$, for some hereditary class $\mathcal{B}$ of $Q$-topological spaces. Note that $C\mathcal{B}$ is continuously closed. If $X \notin \mathcal{A}(= C\mathcal{B})$, then there is a non-trivial $Y \in \mathcal{B}$ and a non-constant $Q$-continuous map $f : X \to Y$. Thus, $f(X)$ (i.e., a $Q$-continuous image of $X$) is a non-trivial $Q$-subspace of $Y$. So $f(X) \in \mathcal{B}$. We claim that no non-trivial $\mathcal{A}$-space is a $Q$-subspace of $f(X)$. To prove it, suppose that there is a non-trivial $\mathcal{A}$-space $Z$ which is a $Q$-subspace of $f(X)$, then the inclusion map $i : Z \to f(X)$ is a non-constant $Q$-continuous map, which implies that $Z \notin \mathcal{A}$, a contradiction.

Next, let $X \in \mathcal{A}(= C\mathcal{B})$ and $Y$ be a non-trivial $Q$-continuous image of $X$, i.e., there is an onto $Q$-continuous map $g : X \to Y$, so $Y \in \mathcal{A}(= C\mathcal{B})$ (as $C\mathcal{B}$ is continuously closed) and we say that $Y$ has a non-trivial $\mathcal{A}$-space which is a $Q$-subspace of $Y$, viz., $Y$ itself.

Conversely, suppose that the class $\mathcal{A}$ satisfies the given condition. We will show that $\mathcal{A}$ is $Q$-connectedness by showing that $\mathcal{A} = CD\mathcal{A}$. Clearly, $\mathcal{A} \subseteq CD\mathcal{A}$. We only requires to show that $CD\mathcal{A} \subseteq \mathcal{A}$. If $X \notin \mathcal{A}$, then there exists a non-trivial $Q$-continuous image $Y$ of $X$ such that no non-trivial $\mathcal{A}$-space is a $Q$-subspace of $Y$, i.e., there is an onto (and hence non-constant) $Q$-continuous map $f : X \to Y$ with the property that no non-trivial $\mathcal{A}$-space is a $Q$-subspace of $Y$, whereby $Y \in D\mathcal{A}$ (by Proposition 3.3), but then $X \notin CD\mathcal{A}$. So $CD\mathcal{A} \subseteq \mathcal{A}$. □

## 4. Some special $Q$-disconnectedness and $Q$-connectedness classes

From now onward in this paper, we shall be dealing only with the stratified $Q$-topological spaces (as defined below), so that we can talk about indiscrete $Q$-topological spaces and ensure that constant maps are $Q$-continuous.

**Definition 4.1** ([9]). A $Q$-topological space $(X, \tau)$ is said to be stratified if $\bar{q} \in \tau$, for each $q \in Q$, where $\bar{q} : X \to Q$ is $q$-valued constant map.

By $D_A$ we shall denote a fixed two-point indiscrete $Q$-topological space (i.e., the one whose $Q$-topology consists of all constant maps only).

Let **Str-$Q$-TOP** and **Str-$Q$-TOP$_0$** respectively denote the categories of all stratified $Q$-topological spaces and stratified $T_0$-$Q$-topological spaces.

The counterpart of the $Q$-Sierpinski space in the category **Str-$Q$-TOP** turns out to be $(Q, \langle\{id\} \cup \{\bar{q} \mid q \in Q\}\rangle)$, where $id : Q \to Q$ is the identity map. We shall denote it by $Q_S$. It can be then noticed that this $Q_S$ also has the property similar to the general $Q$-Sierpinski space (introduced in [9], see also [7]), viz., for any $(X, \tau) \in ob$**Str-$Q$-TOP**:

$$\mu \in \tau \text{ if and only if } \mu : (X, \tau) \to Q_S \text{ is } Q\text{-continuous.}$$

**Proposition 4.2.** *If $\mathcal{B}$ is a $Q$-disconnectedness with $D_A \in \mathcal{B}$, then $\mathcal{B} = \mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}$.*

*Proof.* As $\mathcal{B}$ is a $Q$-disconnectedness, $\mathcal{B} = D\mathcal{A}$, for some continuously closed class $\mathcal{A}$ of $Q$-topological spaces. To show $\mathcal{B} = \mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}$, we will only show that $\mathbf{Str}\text{-}Q\text{-}\mathbf{TOP} \subseteq \mathcal{B}$. Let $X \in ob\mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}$. As $D_A \in \mathcal{B}$, for every $Y \in \mathcal{A}$, each $Q$-continuous function $f : Y \to D_A$ must be constant. But then $Y$ consists of only one element, whereby it follows that any $Q$-continuous function from $Y$ to $X$ must be constant. So $X \in \mathcal{B}$. Hence $\mathcal{B} = \mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}$. □

**Proposition 4.3.** *If $C_A$ denotes the class of all indiscrete $Q$-topological spaces, then $\mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0 = DC_A$.*

*Proof.* Let $(X, \tau) \notin ob\mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0$. Then there exists $x, y \in X$, with $x \neq y$, such that $\mu(x) = \mu(y)$, for every $\mu \in \tau$. Let $Y = \{x, y\}$. Then $(Y, \tau_Y)$ is an indiscrete $Q$-subspace of $X$. Clearly, the inclusion function $i : Y \to X$ is non-constant and $Q$-continuous. Thus $(X, \tau) \notin DC_A$. So, $DC_A \subseteq \mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0$.

Next, let $(X, \tau) \notin DC_A$. Then there exists some $Y \in C_A$ and a non-constant $Q$-continuous function $f : Y \to X$. But we have $y_1, y_2 \in Y$, with $y_1 \neq y_2$ such that $f(y_1) \neq f(y_2)$. Let $f(y_1) = x_1$ and $f(y_2) = x_2$, so $x_1 \neq x_2$. Note that for any $\mu \in \tau$, as $Y \in C_A$, $f_Q^{\leftarrow}(\mu) = \mu \circ f$ must be constant. Hence $\mu(f(y_1)) = \mu(f(y_2))$, i.e., $\mu(x_1) = \mu(x_2)$, for every $\mu \in \tau$. Thus $(X, \tau) \notin ob\mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0$. So, $\mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0 \subseteq DC_A$. Hence, we have $\mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0 = DC_A$. □

**Proposition 4.4.** *If $\mathcal{B}$ is a $Q$-disconnectedness with $D_A \notin \mathcal{B}$ and $Q_S \in \mathcal{B}$, then $\mathcal{B} = \mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0$.*

*Proof.* As $\mathcal{B}$ is a $Q$-disconnectedness, $\mathcal{B} = D\mathcal{A}$, for some continuously closed class $\mathcal{A}$ of $Q$-topological spaces.

Let $X \in \mathcal{B}$. To show $X \in ob\mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0$, it will suffices to show that there is no non-trivial $Q$-subspace of $X$ which is indiscrete. Let $Y$ be a non-trivial $Q$-subspace of $X$. Then, clearly $Y \in \mathcal{B}$ by Proposition 3.6. If $Y$ is indiscrete, then the two-point indiscrete $Q$-topological space $D_A$ is $Q$-homeomorphic to a $Q$-subspace of $Y$. Thus $D_A \in \mathcal{B}$, a contradiction. So $Y$ cannot be indiscrete. Hence $\mathcal{B} \subseteq \mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0$.

Next, let $X \in ob\mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0$. If possible, suppose that there is some $Y \in \mathcal{A}$ and a non-constant $Q$-continuous function $f : Y \to X$. Then $f(Y)$ is a non-trivial $\mathcal{A}$-space. Also, $f(Y)$ is a non-trivial $Q$-subspace of $X$. Thus $f(Y)$ cannot be indiscrete. So, there is a non-constant $Q$-continuous function $\mu : f(Y) \to Q_S$. But $Q_S \notin \mathcal{B} = D\mathcal{A}$, a contradiction. Hence $X \in \mathcal{B}$ and thus, $\mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0 \subseteq \mathcal{B}$. Therefore, $\mathcal{B} = \mathbf{Str}\text{-}Q\text{-}\mathbf{TOP}_0$. □

**Proposition 4.5.** *The class $C_A$ of all indiscrete $Q$-topological spaces is a $Q$-connectedness. In fact, $C_A = C(Q_S)$.*

*Proof.* Let $(X, \tau) \in C_A$ and $(Y, \delta)$ be any non-trivial $Q$-continuous image of $(X, \tau)$, i.e., there is an onto $Q$-continuous map $g : (X, \tau) \to (Y, \delta)$. Since for every $\nu \in \delta$, $\nu : (Y, \delta) \to Q_S$ is $Q$-continuous, $\nu \circ g : (X, \tau) \to Q_S$ is $Q$-continuous. Then $\nu \circ g \in \tau$, but then $\nu \circ g$ is constant, whereby it follows that $\nu$ is constant. Thus $\nu$ is the indiscrete $Q$-topology, i.e., $(Y, \delta) \in C_A$. Here $(Y, \delta)$ itself is a $Q$-subspace of $(Y, \delta)$. So, $C_A$ is a $Q$-connectedness by Proposition 3.11.

Let $(X, \tau) \in C_A$ and $f : (X, \tau) \to Q_S$ be any $Q$-continuous map. Then $f \in \tau$. Thus, $f$ must be constant, whereby $X \in C(Q_S)$. So $C_A \subseteq C(Q_S)$. Next, let $(X, \tau) \in C(Q_S)$. Then every $Q$-continuous map from $(X, \tau)$ to $Q_S$ must be constant. Thus, for every $\mu \in \tau$, since $\mu : (X, \tau) \to Q_S$ is $Q$-continuous, $\mu$ must be constant, i.e., $\tau$ contains only constant maps. Hence $\tau$ is the indiscrete $Q$-topology, i.e., $(X, \tau) \in C_A$. So $C(Q_S) \subseteq C_A$. Hence $C_A = C(Q_S)$. □

## 5. Conclusion

In this paper, we have briefly studied the concepts of connectednesses and disconnectednesses for $Q$-topological spaces with respect to a class of $Q$-topological spaces, motivated by the works of G. Preuss, Arhangel'skii and Wiegandt and some others (cf. e.g., [2, 4, 5, 6, 10]) for topological spaces, uniform spaces, and fuzzy topological spaces. We have also shown that for $Q$-topological spaces, $T_0$-ness is a disconnectedness in the sense of Preuss, in addition to giving some characterization(s) of connectednesses and disconnectednesses for $Q$-topological spaces.

**Acknowledgements.** The authors are grateful to the referees for making suggestions for improving the paper.

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