Prime \( \cdot \)-ideals and fuzzy prime \( \cdot \)-ideals in \( PMV \)-algebras

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ABSTRACT. In the present paper, by considering the notion of \( PMV \)-algebras, we present definition of prime \( \cdot \)-ideals in \( PMV \)-algebras and obtain some results on them. In addition, we introduce the notions of fuzzy \( \cdot \)-ideals and fuzzy prime \( \cdot \)-ideals in \( PMV \)-algebras. Then by proving some theorems, we state some conditions to obtain fuzzy prime \( \cdot \)-ideals.

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1. Introduction

\( MV \)-algebras were defined by C. C. Chang [2, 3] as algebras corresponding to the Lukásiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: \( CN \)-algebras, Wajsberg algebras, bounded commutative \( BCK \)-algebras and bricks. It is discovered that \( MV \)-algebras are naturally related to the Murray-von Neumann order of projections in operator algebras on Hilbert spaces and that they play an interesting role as invariants of approximately finite-dimensional \( C^* \)-algebras. They are also naturally related to Ulam’s searching games with lies. \( MV \)-algebras admit a natural lattice reduct and hence a natural order structure. Many important properties can be derived from the fact, established by Chang that nontrivial \( MV \)-algebras are subdirect products of \( MV \)-chains, that is, totally ordered \( MV \)-algebras.

To prove this fundamental result, Chang introduced the notion of prime ideal in an \( MV \)-algebra.

A product \( MV \)-algebra (or \( PMV \)-algebra, for short) is an \( MV \)-algebra which has an associative binary operation “\( \cdot \)”. It satisfies an extra property which will be explained in Preliminaries. \( PMV \)-algebras were introduced by A. Di Nola and A. Dvurečenskij in [5]. They also introduced \( \cdot \)-ideals in \( PMV \)-algebras. During the last years, \( PMV \)-algebras were considered and their equivalence with a certain class
of \(l\)-rings with strong unit was proved. In 2014, F. Forouzesh, E. Eslami and A. Borumand Saeid defined \(-prime ideals in PMV\)-algebras [8].

The concept of fuzzy sets was introduced by Zadeh for the first time [11]. Since then, many studies were performed about this subject and many researchers started working on the fuzzy algebraic structures. Recently, many papers were written, too. For example, see [1, 10].

In this paper, we introduce the notions of prime \(-ideals and fuzzy prime \(-ideals in PMV\)-algebras and prove some results on them. In fact, we open new fields to anyone that is interested to studying and development of fuzzy ideals in PMV-algebras.

2. Preliminaries

In this section, we review related lemmas and theorems that we use in the next sections.

Definition 2.1 ([4]). An MV-algebra is a structure \(M = (M, \oplus, ', 0)\) of type \((2, 1, 0)\) such that:

\[(MV1) \ (M, \oplus, 0) \text{ is an Abelian monoid,}\]
\[(MV2) \ (a')' = a,\]
\[(MV3) \ 0' \oplus a = 0',\]
\[(MV4) \ (a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a.\]

If we define the constant \(1 = 0'\) and operations \(\odot\) and \(\ominus\) by \(a \odot b = (a' \oplus b)'\), \(a \ominus b = a \odot b'\), then

\[(MV5) \ (a \odot b) = (a' \odot b')',\]
\[(MV6) \ a \odot 1 = 1,\]
\[(MV7) \ (a \odot b) \odot b = (b \odot a) \odot a,\]
\[(MV8) \ a \odot a' = 1,\]

for every \(a, b \in A\).

It is clear that \((M, \odot, 1)\) is an abelian monoid. Now, if we define auxiliary operations \(\lor\) and \(\land\) on \(M\) by \(a \lor b = (a \odot b') \oplus b\) and \(a \land b = a \odot (a' \oplus b)\), for every \(a, b \in M\), then \((M, \lor, \land, 0)\) is a bounded distributive lattice.

In MV-algebra \(M\), the following conditions are equivalent: for every \(a, b, c \in M\),

\[\text{(i) } a' \ominus b = 1,\]
\[\text{(ii) } a \odot b' = 0,\]
\[\text{(iii) } b = a \oplus (b \odot a).\]

For any two elements \(a, b\) of MV-algebra \(M\), \(a \leq b\) if and only if \(a, b\) satisfy in the above equivalent conditions (i)-(iv).

An ideal of MV-algebra \(M\) is a subset \(I\) of \(M\), satisfying the following condition:
for every \(x, y \in I\),

\[(11) \ 0 \in I,\]
\[(12) \ x \leq y \text{ and } y \in I \text{ implies that } x \in I,\]
\[(13) \ x \oplus y \in I.\]

A proper ideal \(I\) of \(M\) is a prime ideal if and only if \(x \oplus y \in I\) or \(y \ominus x \in I\) (or \(x \land y \in I\) implies that \(x \in I\) or \(y \in I\)), for every \(x, y \in M\).
In MV-algebra $M$, the distance function $d : M \times M \to M$ is defined by $d(x, y) = (x \odot y) \oplus (y \odot x)$ which satisfies

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq d(x, y) \oplus d(y, z)$,
(iv) $d(x, y) \leq d(x', y')$,
(v) $d(x \oplus z, y \oplus t) \leq d(x, y) \oplus d(z, t)$,

for every $x, y, z, t \in M$.

Let $I$ be an ideal of MV-algebra $M$. Then we denote $x \sim y (x \equiv_I y)$ if and only if $d(x, y) \in I$, for every $x, y \in M$. Thus $\sim$ is a congruence relation on $M$. Denote the equivalence class containing $x$ by $\bar{x}$ and $\bar{M} = \{ \bar{x} : x \in M \}$. Then $(\bar{M}, \bar{\odot}, \bar{\oplus}, \bar{\ominus}, \bar{\otimes})$ is an MV-algebra, where $\bar{x}' = \frac{x}{I}$ and $\frac{x \oplus y}{I} = \frac{x \oplus y}{I}$, for all $x, y \in M$.

Let $M$ and $K$ be two MV-algebras. A mapping $f : M \to K$ is called an MV-homomorphism if

(H1) $f(0) = 0$,
(H2) $f(x \odot y) = f(x) \odot f(y)$,
(H3) $f(x') = (f(x))'$

for every $x, y \in M$.

If $f$ is one to one (onto), then $f$ is called an MV-monomorphism (epimorphism) and if $f$ is onto and one to one, then $f$ is called an MV-isomorphism (see [6]).

**Proposition 2.2** ([4]). Let $M$ be an MV-algebra and $W \subseteq M$. Then the principal ideal generated by $W$ is denoted by $\langle W \rangle$ and $\langle W \rangle = \{ x \in M : x \leq w_1 \oplus w_2 \oplus \cdots \oplus w_n, \text{ for some } w_1, \ldots, w_n \in W \}$. Further, for every ideal $J$ of $M$, $\langle J \cup \{ 0 \} \rangle = \{ x \in M : n \ominus a \geq x, \text{ for some } n \in \mathbb{N} \text{ and } a \in J \}$.

**Lemma 2.3** ([4]). Let $M, N$ be two MV-algebras and $f : M \to N$ be an MV-homomorphism. Then the following properties hold:

(i) $\text{Ker}(f)$ is an ideal of $M$.
(ii) If $f$ is an MV-epimorphism, then $\frac{M}{\text{Ker}(f)} \cong N$.
(iii) $f(x) \leq f(y)$ iff $x \odot y \in \text{Ker}(f)$.
(iv) $f$ is injective iff $\text{Ker}(f) = \{ 0 \}$.

**Lemma 2.4** ([4]). In every MV-algebra $M$, the natural order $\leq$ has the following properties: for every $x, x', y, y', z \in M$,

(i) $x \leq y$ if and only if $y' \leq x'$.
(ii) If $x \leq y$, then $x \odot z \leq y \odot z$.
(iii) If $x \leq y$, then $x \odot z \leq y \odot z$.

**Proposition 2.5** ([4]). The following equations hold in every MV-algebra:

(i) $x \odot (y \lor z) = (x \odot y) \lor (x \odot z)$.
(ii) $x \odot (y \land z) = (x \odot y) \land (x \odot z)$.

**Definition 2.6** ([6]). In MV-algebra $M$, a partial addition is defined as following: $x + y$ is defined if and only if $x \leq y'$ and in this case, $x + y = x \oplus y$, for any $x, y \in M$.

**Lemma 2.7** ([6]). Let $M$ be an MV-algebra and let $+$ be the partial addition on $M$. Then for every $x, y, z \in M$,

(c) $x \oplus y = x + (x' \odot y)$.
(d) If \(x + y\) and \((x + y) + z\) are defined, then \(y + z\) and \(x + (y + z)\) are defined and 
\[(x + y) + z = x + (y + z).
\]
(e) \(x + y = 1\) if and only if \(y = x'\).
(f) \(z + x \leq z + y\), then \(x \leq y\).

**Definition 2.13** (5). A product \(MV\)-algebra (or \(PMV\)-algebra, for short) is a structure \((A, \oplus, '.', 0)\), where \((A, \oplus', 0)\) is an \(MV\)-algebra and \(\cdot\) is a binary associative operation on \(A\) such that the following property is satisfied: if \(x + y\) is defined, then 
\[x' \cdot y = y',\]
and \(z' \cdot y' = y'\) are defined and \((x + y) \cdot z = x' \cdot y'\).

**Lemma 2.11** (5). \(\mathcal{A} \cdot x = a\). for every \(a \in \mathcal{A}\).

**Definition 2.8** (5). A product \(MV\)-algebra (or \(PMV\)-algebra, for short) is a structure \((A, \oplus, '.', 0)\), where \((A, \oplus', 0)\) is an \(MV\)-algebra and \(\cdot\) is a binary associative operation on \(A\) such that the following property is satisfied: if \(x + y\) is defined, then 
\[x' \cdot y = y',\]
and \(z' \cdot y' = y'\) are defined and \((x + y) \cdot z = x' \cdot y'\).

**Lemma 2.9** (6). If \(A\) is a unital \(PMV\)-algebra, then 

(i) The unity for the product is \(e = 1\).
(ii) \(x \cdot y \leq x \land y\), for every \(x, y \in A\).

**Definition 2.10** (6). Let \(X, Y\) be \(PMV\)-algebras. An \(MV\)-homomorphism \(f : X \to Y\) is called a homomorphism of \(PMV\)-algebras (or \(PMV\)-homomorphism) if and only if \(f(x + y) = f(x) \cdot f(y)\).

**Lemma 2.11** (5). Let \(A\) be a \(PMV\)-algebra. Then \(1.a = a\) and \(a \leq b\) implies that 
\[a.c \leq b.c \text{ and } c.a \leq c.b, \text{ for every } a, b, c \in A.
\]

**Definition 2.12** (6). Let \(A = (A, \oplus, '.', 0)\) be a \(PMV\)-algebra, \(M = (M, \oplus', 0)\) be an \(MV\)-algebra and the operation \(\Phi : A \times M \to M\) be defined by \(\Phi(a, m) = am\), which satisfies the following axioms:

- (AM1) If \(x + y\) is defined in \(M\), then \(ax + ay\) is defined in \(M\) and \(a(x + y) = ax + ay\).
- (AM2) If \(a + b\) is defined in \(A\), then \(ax + bx\) is defined in \(M\) and \((a + b)x = ax + bx\).
- (AM3) \((a.b)x = a(bx)\), for every \(a, b \in A\) and \(x, y \in M\).

Then \(M\) is called a (left) \(MV\)-module over \(A\) or briefly an \(A\)-module.

We say that \(M\) is a unitary \(MV\)-module if \(A\) has a unity \(1_A\) for the product, i.e.,

- (AM4) \(1_Ax = x\), for every \(x \in M\).

**Lemma 2.13** (6). Let \(A\) be a \(PMV\)-algebra and \(M\) be an \(A\)-module. Then for every \(a, b \in A\) and \(x, y \in M\),

- (a) \(0x = 0\).
- (b) \(a0 = 0\).
- (c) \(ax' \leq (ax)'\).
- (d) \(a' x \leq (ax)'\).
- (e) \((ax)' = a'x + (1x)'\).
- (f) \(x \leq y\) implies \(ax \leq ay\).
- (g) \(a \leq b\) implies \(ax \leq bx\).
- (h) \(a(x \oplus y) \leq ax \oplus ay\).
- (i) \(d(ax, ay) \leq ad(x, y)\).
Definition 2.14 ([11]). A fuzzy set in set of A is a mapping \( \mu : A \rightarrow [0, 1] \). Let \( \mu \) be a fuzzy set in A and \( t \in [0, 1] \). Then \( \mu_t = \{ x \in A : \mu(x) \geq t \} \) is called a level subset of \( \mu \).

Definition 2.15 ([9]). If A is an \( MV \)-algebra, then a fuzzy set \( \mu \) in A is a fuzzy ideal of A, if it satisfies

\begin{enumerate}[(FI1)]
  \item \( \mu(0) \geq \mu(x) \), for all \( x \in A \),
  \item \( \mu(y) \geq \mu(x) \wedge \mu(y \odot x') \), for all \( x, y \in A \).
\end{enumerate}

Theorem 2.16 ([9]). Let \( \mu \) be a fuzzy ideal in A. Then for every \( x, y \in A \),
\begin{enumerate}[(i)]
  \item \( \mu(x \oplus y) = \mu(x) \wedge \mu(y) \).
  \item \( \mu(x \odot y) = \mu(x) \wedge \mu(y) \).
  \item \( \mu(x \wedge y) \geq \mu(x) \vee \mu(y) \).
\end{enumerate}

Lemma 2.17 ([7]). Let A be an \( MV \)-algebra and \( \mu : A \rightarrow [0, 1] \) be a fuzzy set on A. Then \( \mu \) is a fuzzy ideal on A if and only if
\begin{enumerate}[(i)]
  \item \( \mu(x) \leq \mu(0) \),
  \item \( \mu(x \odot y) \geq \mu(x) \wedge \mu(y) \),
  \item \( \mu(x \wedge y) \geq \mu(x) \vee \mu(y) \), for all \( x, y \in A \).
\end{enumerate}

Theorem 2.18 ([7]). Let \( \mu \) be a fuzzy set in A. \( \mu \) is a fuzzy ideal in A if and only if for all \( t \in [0, 1] \), \( \mu_t \) is either empty or an ideal of A.

Corollary 2.19 ([7]). I is an ideal of A if and only if \( \chi_I \) is a fuzzy ideal of A, where \( \chi_I \) is characteristic function of I.

3. Prime \~{}-ideals in \( PMV \)-algebras

Note: From now on, in this paper, A is a \( PMV \)-algebra.

In this section, we introduce prime \~{}-ideals in \( PMV \)-algebras and state some conditions to obtain them.

Definition 3.1. Let \( P \) be a \~{}-ideal of A. If \( P \) is a prime ideal of \( MV \)-algebra A, then \( P \) is called a prime \~{}-ideal of A.

Example 3.2. Let \( A = \{0, 1, 2, 3\} \) and the operations “\( \oplus \)” and “\( \cdot \)” on \( A \) be defined as follows:

\[
\begin{array}{c|cccc}
\oplus & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 1 & 3 & 3 \\
2 & 2 & 3 & 2 & 3 \\
3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
2 & 0 & 0 & 2 & 2 \\
3 & 0 & 1 & 2 & 3 \\
\end{array}
\]

Consider \( 0' = 3 \), \( 1' = 2 \), \( 2' = 1 \) and \( 3' = 0 \). Then it is easy to show that \((A, \oplus', \cdot', 0)\) is a \( PMV \)-algebra. \( J = \{0, 2\} \) and \( I = \{0, 1\} \) are prime \~{}-ideals of A. Also, \( \{0\} \) is not a prime \~{}-ideal of A.

Lemma 3.3. Let \( A \) be an \( MV \)-algebra and \( I \subseteq A \). Then \( I \) is an ideal of \( A \) if and only if
\begin{enumerate}[(i)]
  \item \( 0 \in I \),
  \item \( x \oplus y \in I \), for every \( x, y \in I \),
  \item \( x \ominus y, y \in I \), then \( x \in I \), for any \( x, y \in A \).
\end{enumerate}
Proof. \((\Rightarrow)\) Let \(I\) be an ideal of \(A\). Then (i) and (ii) are clear. Now, let \(x \oplus y, y \in I\). Then by (iii) and (MV7), \((y \oplus x) \oplus x = (x \oplus y) \oplus y \in I\). Since \(x \leq (y \oplus x) \oplus x \in I, x \in I\).

\((\Leftarrow)\) Let (i), (ii) and (iii) be true. If \(x \leq y\) and \(y \in I\), then \(x \oplus y = x \oplus y' = 0 \in I\). Thus by (iii), \(x \in I\). So \(I\) is an ideal of \(A\). \(\Box\)

**Theorem 3.4.** Let \(A, B\) be PMV-algebras and \(f : A \to B\) be a PMV-homomorphism. Then

(i) If \(P\) is a prime \(-\)-ideal of \(B\), then \(f^{-1}(P)\) is a prime \(-\)-ideal of \(A\).

(ii) If \(f\) is onto, \(P\) is a prime \(-\)-ideal of \(A\) and \(\text{Ker}(f) \subseteq P\), then \(f(P)\) is a prime \(-\)-ideal of \(B\).

**Proof.** (i) The proof is routine.

(ii) The first we show that \(f(P)\) is an ideal of \(B\). The proofs of \((I_1)\) and \((I_3)\) are easy. We show that \(f(P)\) satisfies \((I_2)\). Let \(a \leq b \in f(P)\), for some \(a, b \in B\). Since \(f\) is onto, there are \(x \in A\) and \(y \in P\) such that \(a = f(x)\) and \(b = f(y)\). Since \(f(x) \leq f(y)\), by Lemma 2.3 (iii), \(x \oplus y \in P\). Thus by Lemma 3.3, \(x, y \in P\). It means that \(a = f(x) \in f(P)\). So \(f(P)\) is an ideal of \(B\). It is routine to see that \(f(P)\) is a \(-\)-ideal of \(B\).

Now, let \(a \cap b \in f(P)\), for \(a, b \in B\). Then there are \(x, y \in A\) such that \(a = f(x)\) and \(b = f(y)\). We have

\[
\begin{align*}
a \cap b &= f(x) \cap f(y) = f(x \cap y) \in f(P) \Rightarrow f(x \cap y) = f(t), t \in P \\
&\Rightarrow x \cap y \cap t' \in P, t \in P \\
&\Rightarrow x \cap y \in P.
\end{align*}
\]

Thus \(x \in P\) or \(y \in P\). So \(a = f(x) \in f(P)\) or \(b = f(y) \in f(P)\). Hence \(f(P)\) is a prime ideal of \(B\) and thus it is a prime \(-\)-ideal of \(B\). \(\Box\)

**Lemma 3.5.** In PMV-algebra \(A\), \((\alpha \oplus \beta) a \leq \alpha a \oplus \beta a\), for every \(\alpha, \beta, a \in A\).

**Proof.** Since \(\beta a \leq (\alpha a) \oplus \beta a\), by Lemma 2.4 (i), \((\alpha a) \oplus (\beta a) = ((\alpha a) \oplus \beta a) \leq (\beta a)\). Then \((\alpha a) \oplus (\beta a) \leq \beta a\) is defined, where \(\oplus\) is the partial addition on \(A\). Similarly, \((\alpha \oplus \beta)' + \beta\) is defined, too.

Consider \(A\) as an \(A\)-module, where \(ab = a.b\), for every \(a, b \in A\). Then by Lemma 2.13 (d) and (g), since \(\alpha \oplus \beta = (\alpha \oplus \beta)' \leq (\beta a)'\), \((\alpha \oplus \beta)' a \leq \beta a \leq (\beta a)'\). Thus \((\alpha \oplus \beta)' a + \beta a\) is defined. Now, \(\alpha \leq \alpha \cup \beta\) implies that \(\alpha a \leq (\alpha \cup \beta)a\) and similarly, \(\beta a \leq (\alpha \cup \beta)a\). So \(\alpha a \cup \beta a \leq (\alpha \cup \beta)a\). Hence by Lemma 2.7 (c),

\[
(\alpha a) \oplus (\beta a) = \alpha a \cup \beta a \leq (\alpha \cup \beta) a = (\alpha \cup \beta)' a = (\alpha \cup \beta)' a + \beta a.
\]

Now, by Lemma 2.7(f), \((\alpha a) \oplus (\beta a) \leq (\alpha \cup \beta)' a\). If we set \(\alpha \oplus \beta\) instead of \(\alpha\), then by Lemma 2.13 (g), we have \((\alpha \oplus \beta) a \oplus (\beta a)' \leq ((\alpha \oplus \beta) \cup \beta)' a = (\alpha \cup \beta)' a \leq \alpha a\). Thus

\[
(\alpha \oplus \beta) a = (\alpha \oplus \beta) a \cup \beta a = (\alpha \oplus \beta) a \oplus (\beta a)' \oplus \beta a \leq \alpha a \oplus \beta a.
\]

\(\Box\)

**Lemma 3.6.** Let \(I\) be an ideal of \(A\) and \(c \in I\). Then \(a.c \in I\), for every \(a \in A\).

**Proof.** The proof is easy. \(\Box\)

**Definition 3.7.** \(S \subseteq A\) is called a \(-\)-closed subset of \(A\), if \(x, y \in S\), for every \(x, y \in S\).
Example 3.8. In Example 3.2, $S = \{1, 3\}$ is a $\langle -$closed subset of $A$.

Theorem 3.9. Let $A$ be unital, $I$ be a $\langle -$ideal of $A$ and $S$ be a $\langle -$closed subset of $A$ such that $I \cap S = \emptyset$. Then there exists a prime $\langle -$ideal $P$ of $A$ such that $I \subseteq P$ and $P \cap S = \emptyset$.

Proof. Let $T = \{ J : \text{J is a} \cdot \langle -$ideal of $A, I \subseteq J \text{ and } J \cap S = \emptyset \}$. Since $I \in T$, $T \neq \emptyset$. By Zorn’s Lemma, $T$ has a maximal element $P$. We show that $P$ is a prime $\langle -$ideal of $A$. Let $x \land y \in P$ and $x, y \notin P$. Consider $\langle P \cup \{x\} \rangle$ and $\langle P \cup \{y\} \rangle$.

By maximality $P$, $\langle P \cup \{x\} \rangle \cap S \neq \emptyset$ and $\langle P \cup \{y\} \rangle \cap S \neq \emptyset$ and so there exist $\alpha \in \langle P \cup \{x\} \rangle \cap S$ and $\beta \in \langle P \cup \{y\} \rangle \cap S$. Then by Proposition 2.2, there exist $a, b \in P$ and $n, m \in \mathbb{N} \cup \{0\}$ such that $\alpha \leq nx \oplus a$ and $\beta \leq my \oplus b$. By Proposition 2.13 (f), (g), $\alpha, \beta \leq (nx \oplus a)(my \oplus b)$. If we consider $A$ as $A$-module, where $xy = x.y$, for every $x, y \in A$, then by Lemmas 3.5, 3.6 and 2.13 (h),

$$\alpha \cdot \beta \leq (nx \oplus a).my \oplus (nx \oplus a).b \leq nx.my \oplus a.my \oplus nx.b \oplus a.b \in P.$$ 

So $\alpha, \beta \in P$. Since $S$ is a $\langle -$closed subset of $A$, $\alpha, \beta \in P \cap S$, which is a contradiction. Hence $P$ is a prime $\langle -$ideal of $A$. □

Lemma 3.10. Let $A$ be unital, $I$ be an ideal of $A$ and $r(I)$ be the intersection of all prime $\langle -$ideals of $A$ containing $I$. Then

$$r(I) = \{ x \in A : x^n = x.x \cdots x \in I, \text{for some } n \in \mathbb{N} \}.$$ 

Proof. Let $T = \{ x \in A : x^n = x.x \cdots x \in I, \text{for some } n \in \mathbb{N} \}$. It is easy to show that $T \subseteq r(I)$. Let $x \in r(I)$. If $x \notin T$, then $x^n \notin I$, for every $n \in \mathbb{N}$. Consider $S = \{ x^n \oplus a : n \in \mathbb{N} \cup \{0\}, a \in I \text{ and } x^n \leq a' \}$. Let $x^n \oplus a, x^m \oplus b \in S$, for $a, b \in I$ and $n, m \in \mathbb{N}$. Since $x^n \leq a'$ and $x^m \leq a'$, $x^n + a$ and $x^m + a$ are defined in $A$. Thus $(x^n \oplus a)(x^m \oplus b) = (x^n + a)(x^m + b) = x^{n+m} + a.x^m + x^n.b + a.b = x^{n+m} + t \in S$, where by Lemma 3.6, $t \in I$. It results that $S$ is a $\langle -$closed subset of $A$. It is easy to see that $S \cap I = \emptyset$. So by Theorem 3.9, there is a prime $\langle -$ideal $P$ of $A$ such that $I \subseteq P$ and $S \cap P = \emptyset$. Now, since $x \in r(I)$ and $x = x^1 \oplus 0 \in S$, $x \in P \cap S$, which is a contradiction. Hence $x \in T$. Therefore $T = r(I)$. □

Theorem 3.11. Let $P$ be a $\langle -$ideal of $A$. $P = r(P)$ if and only if $\frac{A}{P}$ has no nilpotent elements.

Proof. $(\Rightarrow)$ Let $P = r(P)$ and $\frac{0}{P} \neq \frac{A}{P}$. be an element of $\frac{A}{P}$. If $(\frac{A}{P})^n = \frac{0}{P}$, for $n \in \mathbb{N}$, then $\frac{A}{P} = \frac{0}{P}$. Thus $x^n \in P$. So by Lemma 3.10, $x \in r(P)$ and thus $\frac{x}{P} = \frac{0}{P}$, which is a contradiction. Hence $\frac{A}{P}$ has no nilpotent elements.

$(\Leftarrow)$ Let $\frac{A}{P}$ has no nilpotent elements. It is clear that $P \subseteq r(P)$. Let $x \in r(P)$. Then by Lemma 3.10, $x^n \in P$. Thus $\frac{x^n}{P} = \frac{0}{P}$, for $n \in \mathbb{N}$. Since $\frac{x^n}{P}$ has no nilpotent elements, $\frac{x}{P} = \frac{0}{P}$ and so $x \in P$. Hence $r(P) = P$. □
4. FUZZY PRIME \(-\)IDEALS IN PMV-ALGEBRAS

In this section, we introduce fuzzy \(-\)ideals, fuzzy prime \(-\)ideals in PMV-algebras and obtain some results on them.

**Definition 4.1.** Let \( \mu \) be a fuzzy ideal in \( A \). Then

(i) \( \mu \) is called a fuzzy \(-\)ideal in \( A \) if and only if \( \mu(x, y) \land \mu(y, x) \geq \mu(x) \), for every \( x, y \in A \).

(ii) \( \mu \) is called a fuzzy prime \(-\)ideal in \( A \) if and only if \( \mu_t = A \) or \( \mu_t \) is a prime \(-\)ideal of \( A \).

**Example 4.2.** (i) Let \( A = \{0, 1, 2, 3\} \) and the operations "\( \oplus \)" and "\( .\)" be defined on \( A \) as follows:

\[
\begin{array}{c|cccc}
\oplus & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 1 & 2 & 3 \\
2 & 2 & 2 & 2 & 3 \\
3 & 3 & 3 & 3 & 3 \\
\end{array}
\quad \begin{array}{c|cccc}
. & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 \\
2 & 0 & 1 & 2 & 2 \\
3 & 0 & 1 & 2 & 3 \\
\end{array}
\]

Consider \( 0' = 3, 1' = 2, 2' = 1 \) and \( 3' = 0 \). Then it is easy to show that \( (A, \oplus, ., 0) \) is a PMV-algebra. Now, let \( \mu(0) = 0.8 \) and \( \mu(1) = \mu(2) = 0.5 \). Then \( \mu \) is a fuzzy \(-\)ideal of \( A \).

(ii) Let \( A = \{0, 1\} \) and the operations "\( \oplus \)" and "\( .\)" on \( A \) be defined as follows:

\[
\begin{array}{c|cc}
\oplus & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\quad \begin{array}{c|c}
. & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 1 & 1 \\
\end{array}
\]

Then it is easy to show that \( (A, \oplus, ., 0) \) is a PMV-algebra. Now, let \( \mu(0) = 0.8 \) and \( \mu(1) = 0.5 \). Then \( \mu \) is a fuzzy prime \(-\)ideal of \( A \).

**Corollary 4.3.** Let \( \mu \) be a fuzzy \(-\)ideal in \( A \) such that \( \mu(0) \neq \mu(1) \). \( \mu \) is a fuzzy prime \(-\)ideal in \( A \) if and only if \( \mu(0) \) is a prime \(-\)ideal of \( A \).

**Proof.** The proof is clear. \( \square \)

**Theorem 4.4.** Let \( \mu \) be a fuzzy \(-\)ideal in \( A \) such that \( \mu(0) \neq \mu(1) \). \( \mu \) is a fuzzy prime \(-\)ideal in \( A \) if and only if \( \mu(x) \lor \mu(y) \geq \mu(x \land y) \), for every \( x, y \in A \).

**Proof.** (\( \Rightarrow \)) Let \( \mu \) be a fuzzy prime \(-\)ideal in \( A \). Then \( \mu_t \) is a prime \(-\)ideal of \( A \). Let \( x, y \in A \). Consider \( t = \mu(x \land y) \). Then \( \mu(x \land y) \geq t \). Thus \( x \land y \in \mu_t \). So \( x \in \mu_t \) or \( y \in \mu_t \) and thus \( \mu(x) \geq t \) or \( \mu(y) \geq t \). Hence \( \mu(x) \lor \mu(y) \geq t \). Therefore \( \mu(x) \lor \mu(y) \geq \mu(x \land y) \).

(\( \Leftarrow \)) Let \( \mu(x) \lor \mu(y) \geq \mu(x \land y) \), for every \( x, y \in A \) and \( \mu_t \neq A \), for \( t \in [0, 1] \). The first, we show that \( \mu_t \) is a \(-\)ideal of \( A \). Let \( x \in \mu_t \) and \( y \in A \). Since \( \mu \) is a fuzzy \(-\)ideal of \( A \), \( \mu(x, y) \land \mu(x, y) \geq \mu(x) \geq t \). Then \( \mu(x, y) \geq t \) and \( \mu(y, x) \geq t \). Thus \( x, y \in \mu_t \). If \( x \land y \in \mu_t \), then \( \mu(x \land y) \geq t \). Thus \( \mu(x) \lor \mu(y) \geq t \). So \( \mu(x) \geq t \) or \( \mu(y) \geq t \). Hence \( x \in \mu_t \) or \( y \in \mu_t \). \( \square \)

**Corollary 4.5.** Let \( A \) be unital and \( \mu \) be a fuzzy \(-\)ideal in \( A \) such that \( \mu(0) \neq \mu(1) \). \( \mu \) is a fuzzy prime \(-\)ideal in \( A \) if and only if \( \mu(x) \lor \mu(y) = \mu(x \land y) \), for every \( x, y \in A \).
Proof. \((\Rightarrow)\) Let \(\mu\) be a fuzzy prime \(-\)ideal in \(A\). By Lemma 2.16 , (iii), \(\alpha \beta \leq (\alpha x \lor a) \cdot (\beta y \lor b)\). If we consider \(A\) as \(A\)-mo, \(\mu(x \land y) \geq \mu(x) \lor \mu(y)\). Then by Theorem 4.4, \(\mu(x) \lor \mu(y) = \mu(x \land y)\).

\((\Leftarrow)\) By Theorem 4.4, it is clear. \(\square\)

**Corollary 4.6.** Let \(I\) be a \(-\)ideal in \(A\). \(I\) is a prime \(-\)ideal in \(A\) if and only if \(\chi_I\) is a fuzzy prime \(-\)ideal of \(A\).

Proof. \((\Rightarrow)\) Let \(I\) be a prime \(-\)ideal in \(A\). We have \(\{x : \chi_I(x) \geq t\} = I\) for every \(t \in [0,1]\). Then \(\chi_I\) is a prime \(-\)ideal in \(A\). Thus \(\chi_I\) is a fuzzy prime \(-\)ideal of \(A\).

\((\Leftarrow)\) Let \(\chi_I\) be a fuzzy prime \(-\)ideal of \(A\). Then by Corollary 4.3, \(I = \{x \in A : \chi(x) \geq 1\} = \chi_I(0)\) is a prime \(-\)ideal of \(A\). \(\square\)

**Theorem 4.7.** Let \(\mu\) be a fuzzy set of \(A\) such that \(\mu(0) \neq \mu(1)\). \(\mu\) is a fuzzy prime \(-\)ideal in \(A\) if and only if

(i) \(\mu(x \cdot y) = \mu(y \cdot x) = \mu(0)\), for every \(x \in \mu(0)\) and \(y \in A\).

(ii) \(\mu(x \circ y) = \mu(y \circ x) = \mu(0)\), for every \(x, y \in A\).

Proof. \((\Rightarrow)\) Let \(\mu\) be a fuzzy prime \(-\)ideal in \(A\). Then by Corollary 4.3, \(\mu(0)\) is a prime \(-\)ideal of \(A\). Since \(\mu(0)\) is a \(-\)ideal of \(A\), \(x \cdot y \in \mu(0)\) for every \(x \in \mu(0)\) and \(y \in A\). Since \(\mu(0)\) is a prime ideal of \(A\), \(x \circ y \in \mu(0)\) for every \(x \in \mu(0)\) and \(y \in A\). So \(\mu(x \circ y) \geq \mu(0)\) or \(\mu(y \circ x) \geq \mu(0)\). Hence \(\mu(x \circ y) = \mu(0)\) or \(\mu(y \circ x) = \mu(0)\).

\((\Leftarrow)\) Let (i) and (ii) be true. Then it is easy to see that \(\mu\) is a fuzzy \(-\)ideal and \(\mu(0)\) is a prime \(-\)ideal of \(A\). Thus by Corollary 4.3, \(\mu\) is a fuzzy prime \(-\)ideal in \(A\). \(\square\)

**Theorem 4.8.** Let \(\mu\) be a fuzzy \(-\)ideal and \(\nu\) be a \(-\)ideal in \(A\) such that \(\nu(0) \neq \nu(1)\). If \(\mu \leq \nu\) and \(\mu(0) = \nu(0)\), then \(\nu\) is a fuzzy prime \(-\)ideal in \(A\), too.

Proof. Since \(\mu\) is a fuzzy prime \(-\)ideal in \(A\), by Theorem 4.7, \(\mu(x \circ y) = \mu(0)\) or \(\mu(y \circ x) = \mu(0)\), for every \(x, y \in A\). Let \(x \circ y = \mu(0)\). Since \(\mu \leq \nu\), \(\mu(x \circ y) \leq \nu(x \circ y)\). Then \(\nu(0) \leq \nu(x \circ y)\). Since \(\mu(0) = \nu(0)\), \(\nu(0) \leq \nu(x \circ y)\). Thus \(\nu(0) = \nu(x \circ y)\). Similiarly, if \(\mu(y \circ x) = \mu(0)\), then \(\nu(y \circ x) = \nu(0)\). Also, it is easy to see that \(\nu(x \cdot y) = \nu(y \cdot x) = \nu(0)\), for every for every \(x \in \nu(0)\) and \(y \in A\). So by Theorem 4.7, \(\nu\) is a fuzzy prime \(-\)ideal in \(A\). \(\square\)

**Theorem 4.9.** If \(A\) is unital and \(\mu\) be a fuzzy \(-\)ideal of \(A\) such that \(\mu(0) \neq \mu(1)\), then the following are equivalent:

(i) \(\mu\) is a fuzzy prime \(-\)ideal of \(A\).

(ii) \(\mu(0)\) is a prime \(-\)ideal of \(A\).

(iii) \(\mu(x) \lor \mu(y) \geq \mu(x \cdot y)\).

Proof. By Corollary 4.3, (i) and (ii) are equivalent. We prove that (ii) and (iii) are equivalent, too.

\((\Rightarrow)\) Let \(\mu(0)\) be a prime \(-\)ideal of \(A\). Then \(\mu(0)\) is a prime ideal of \(A\). Thus \(x \circ y \in \mu(0)\) or \(y \circ x \in \mu(0)\), for every \(x, y \in A\). If \(x \circ y \in \mu(0)\), then since \(x, y \circ \mu(0) \leq x \circ y \lor y \circ x\), by Lemma 2.4 (iii) and Proposition 2.5 (i),

\[x \cdot y \leq (x \lor y) \circ y = (x \lor y) \circ (y \lor y') = (x \lor y) \circ (y \lor y') = x \lor y \in \mu(0)\].

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Thus \( x \cdot y \subseteq y \in \mu_{\mu(0)} \).

Similarly, if \( y \cdot x \in \mu_{\mu(0)} \), then \( x \cdot y \subseteq x \in \mu_{\mu(0)} \). Thus \( \mu(x \cdot y \subseteq y) = \mu(0) \) or \( \mu(x \cdot y \subseteq x) = \mu(0) \). So \( \mu(y) \geq \mu(x \cdot y \subseteq y) \land \mu(x \cdot y) \) or \( \mu(x) \geq \mu(x \cdot y \subseteq x) \land \mu(x \cdot y) = \mu(x \cdot y) \). It results that \( \mu(x) \lor \mu(y) \geq \mu(x \cdot y) \).

(iii) \( \Rightarrow \) (ii): Let \( \mu(x) \lor \mu(y) \geq \mu(x \cdot y) \), for every \( x, y \in A \). Then by Lemma 2.9 (ii) and Theorem 2.16 (ii), \( \mu(x) \lor \mu(y) \geq \mu(x \land y) \). Thus by Theorem 4.4, \( \mu \) is a fuzzy prime ---ideal in \( A \). So \( \mu_{\mu(0)} \) is a prime ---ideal of \( A \). Note that, since \( \mu(1) < \mu(0) \), \( 1 \notin \mu_{\mu(0)} \) and thus \( \mu_{\mu(0)} \) is a proper ---ideal of \( A \).

\[ \square \]

**Lemma 4.10.** Let \( \mu \) be a fuzzy set in \( A \).

(i) \( \mu \) is a fuzzy ideal in \( A \) if and only if \( (z \subseteq y) \subseteq x = 0 \) implies that \( \mu(z) \geq \mu(x) \land \mu(y) \).

(ii) \( \mu \) is a fuzzy ideal in \( A \) if and only if \( (x \subseteq y) \subseteq z \) implies that \( \mu(x) \geq \mu(z) \land \mu(y) \), for all \( x, y, z \in A \).

**Proof.** (i) \( (\Rightarrow) \) Let \( \mu \) be a fuzzy ideal in \( A \) and \( (z \subseteq y) \subseteq x = 0 \), for all \( x, y, z \in A \). Then \( \mu(z) \geq \mu(y) \land \mu(z \subseteq y) \) and \( \mu(z \subseteq y) \geq \mu(x) \land \mu((z \subseteq y) \subseteq x) = \mu(x) \land \mu(0) = \mu(x) \). Thus \( \mu(z) \geq \mu(x) \land \mu(y) \), for all \( x, y, z \in A \).

\( (\Leftarrow) \) Suppose \( (z \subseteq y) \subseteq x = 0 \) implies that \( \mu(z) \geq \mu(x) \land \mu(y) \), for all \( x, y, z \in A \). Since \( (0 \subseteq y) \subseteq x = 0, \mu(0) \geq \mu(x) \land \mu(x) = \mu(x) \). Also, since \( (x \subseteq y) \subseteq (x \subseteq y) = 0, \mu(x) \geq \mu(y) \land \mu(x \subseteq y) \). Thus \( \mu \) is a fuzzy ideal of \( A \).

(ii) By (i), the proof is clear. \( \square \)

**Theorem 4.11.** Let \( \mu \) be a fuzzy prime ---ideal in \( A \). Then \( \mu \oplus \alpha \) is also a fuzzy prime ---ideal in \( A \), where \( (\mu \oplus \alpha)(x) = \mu(x) \oplus \alpha \) and \( \alpha \oplus \beta = \text{Min}(\mu(0), \alpha + \beta) \), for all \( \alpha, \beta \in [0, \mu(0)] \).

**Proof.** The first we show that \( \mu \oplus \alpha \) is a fuzzy ---ideal of \( A \). Let \( (x \subseteq y) \subseteq z \), for \( x, y, z \in A \). Since \( \mu \) is a fuzzy ideal of \( A \), \( \mu(x) \geq \mu(z) \land \mu(y) \). Consider \( M^V \)-algebra \( [0, 1] \). Then by Proposition 2.5 (ii), \( \mu(x) \oplus \alpha \geq \mu(z) \land \mu(y) \). Thus \( \mu \oplus \alpha \) is also a fuzzy ---ideal of \( A \). Since \( \mu \) is a fuzzy prime ---ideal of \( A \), \( \mu(x) \lor \mu(y) \geq \mu(x \lor y) \). Hence \( (\mu(x) \lor \mu(y)) \lor \alpha \geq \mu(x \lor y) \land \alpha \), for every \( x, y \in A \). It results that \( (\mu(x) \lor \alpha) \lor (\mu(y)) \lor \alpha \geq \mu(x \lor y) \land \alpha \) and thus \( (\mu \oplus \alpha)(x) \lor (\mu \oplus \alpha)(y) \geq (\mu \oplus \alpha)(x \lor y) \). Now, since \( (\mu \oplus \alpha)(0) = \mu(0) \lor \alpha \neq \mu(1) \lor \alpha = (\mu \lor \alpha)(1) \) and \( \mu \leq \mu + \alpha \), by Theorem 4.8, \( \mu \oplus \alpha \) is a fuzzy prime ---ideal of \( A \).

\[ \square \]

**Lemma 4.12.** Let \( A \) be unital and \( J \) be a proper ---ideal of \( A \). Then there is a prime ---ideal \( P \) of \( A \) such that \( J \subseteq P \).

**Proof.** A routine application of Zorn’s Lemma shows that there is a proper ---ideal \( I \) of \( A \) which is maximal with respect to the property \( J \subseteq I \). We show that \( I \) is a prime ideal of \( A \). Let \( x \land y \in I \) and \( x, y \notin I \), for \( x, y \in A \). Consider \( \preceq I \cup \{x\} \supseteq \land \preceq I \cup \{y\} \). By maximality \( I \), we have 1 \( \in \preceq I \cup \{x\} \supseteq \land \preceq I \cup \{y\} \). Then by Proposition 2.2, there are \( m, n \in \mathbb{N} \) and \( a, b \in I \) such that \( 1 = mx \oplus a \) and \( 1 = my \oplus b \). Now, let \( u = a \oplus b \) and \( k = \max\{m, n\} \). Then \( 1 = kx \oplus u \) and \( 1 = ky \oplus u \). By Lemma 3.5 and Proposition 2.13 (h), we have \( 1.1 = (kx \oplus u).((ky \oplus u) = (kx \oplus u).ky \ominus (kx \oplus u).u \leq (kx).(ky) \ominus (kx) \ominus u.u \in I \). Thus \( 1 = 1.1 \in I \),
which is a contradiction. So \( I \) is a prime ideal of \( A \). Hence \( I \) is a prime \( \mu \)-ideal of \( A \).

**Theorem 4.13.** Let \( \mu \) be a fuzzy \( \mu \)-ideal in \( A \) such that \( 1 \neq \mu(0) \neq \mu(1) \). Then there is a fuzzy prime \( \mu \)-ideal \( \nu \) such that \( \mu \leq \nu \).

**Proof.** Consider \( \mu(0) = \{x \in A : \mu(x) = \mu(0)\} \) that is a proper \( \mu \)-ideal of \( A \). By Lemma 4.12, there is a prime \( \mu \)-ideal \( P \) of \( A \) such that \( \mu(0) \subseteq P \). By Corollary 4.6, \( \chi_P \) is a fuzzy prime \( \mu \)-ideal of \( A \). Now, let \( \nu = \chi_P \oplus \alpha \), where \( \alpha = \bigvee_{x \in A} \mu(x) \). Then \( \alpha \leq \mu(0) < 1 \). Thus by Theorem 4.11, \( \nu \) is a fuzzy prime \( \mu \)-ideal of \( A \).

**Theorem 4.14.** Let \( A \) be unital, \( \mu \) and \( \nu \) be fuzzy \( \mu \)-ideals in \( A \) and \( \mu \land \nu \leq \alpha \), for \( \alpha \in [0, \mu(0)) \). Then there is a fuzzy prime \( \mu \)-ideal \( \chi \) in \( A \) such that \( \mu \leq \chi \) and \( \nu \lor h \leq \alpha \).

**Proof.** Let \( T = \{x \in A : \mu(x) > \alpha\} \) and \( S = \{x \in A : \nu(x) > \alpha\} \). If \( a, b \in S \), then \( \nu(a) > \alpha \) and \( \nu(b) > \alpha \). Thus by Lemma 2.9 (ii) and Theorem 2.16 (iii), \( \nu(a \land b) \geq \nu(a) \land \nu(b) > \alpha \). It results that \( a \lor b \in S \) and thus \( S \) is \( \nu \)-closed. We must show that \( T \) is a \( \nu \)-ideal of \( A \). Since \( \alpha \in [0, \mu(0)) \), \( 0 \in T \). Let \( x, y \in T \), for \( x, y \in A \). Then \( \mu(x) > \alpha \), \( \mu(y \lor x) > \alpha \). Thus \( \mu(x) \land \mu(y \lor x) > \alpha \). Since \( \nu(y) \geq \mu(x) \land \mu(y \lor x) \), \( \mu(y) > \alpha \). So \( y \in T \). Also, since \( \mu(x \lor y) \geq \mu(x) \lor \mu(y) \), \( x \lor y \in T \), for every \( x, y \in T \). Hence by Lemma 3.3, \( T \) is an ideal of \( A \). Now, let \( x \in T \) and \( y \in A \). Then \( \mu(x) > \alpha \). Since \( x \lor y \leq x \land y \leq x \), \( \mu(x \land y) \geq \mu(x) > \alpha \). Thus \( x \lor y \in T \). So \( T \) is a \( \mu \)-ideal of \( A \). Let \( T \cap S \neq \emptyset \). Then there is \( x \in T \cap S \). Thus \( \mu(x) > \alpha \) and \( \nu(x) > \alpha \). It results that \( \nu \land \mu(x) \geq \nu(x) \land \mu(x) > \alpha \), which is a contradiction. Hence, by Theorem 3.9, there exists a prime \( \mu \)-ideal \( P \) of \( A \) such that \( S \cap P = \emptyset \) and \( T \subseteq P \). Consider \( h = \chi_P \oplus \alpha \). Then by Corollary 4.6 and Theorem 4.11, \( h \) is a fuzzy prime \( \mu \)-ideal of \( A \). Now, we show that \( \mu \leq h \). If \( x \in P \), then \( \chi_P(x) = 1 \) and \( \nu(h(x)) = 1 \oplus \alpha \). It results that \( \mu \leq h \). If \( x \notin P \), then \( x \notin T \) and so \( \mu(x) \leq \alpha \). On the other hand, \( h(x) = \chi_P(x) \oplus \alpha = \alpha \). Therefore \( \mu(x) \leq h(x) \). It is easy to show that \( \nu \land h \leq \alpha \). \( \square \)

5. Conclusions

Prime ideals, \( \mu \)-ideals and \( \mu \)-prime ideals had been defined in \( MV \)-algebras [4, 5, 8]. Lately, ideals in \( PMV \)-algebras were considered and some researchers have been interested to them. We defined prime \( \mu \)-ideals in \( PMV \)-algebras and stated some conditions to have them. They are \( \mu \)-ideals that are prime ideals, too. Also, we defined fuzzy prime \( \mu \)-ideals in \( PMV \)-algebras and obtained some results on them. In fact, we opened new fields to anyone that is interested to studying and development of ideals in \( MV \)-algebras.

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