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Prime ·-ideals and fuzzy prime ·-ideals in PMV-algebras

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Abstract. In the present paper, by considering the notion of PMV-algebras, we present definition of prime ·-ideals in PMV-algebras and obtain some results on them. In addition, we introduce the notions of fuzzy ·-ideals and fuzzy prime ·-ideals in PMV-algebras. Then by proving some theorems, we state some conditions to obtain fuzzy prime ·-ideals.

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1. Introduction

MV-algebras were defined by C. C. Chang [2, 3] as algebras corresponding to the Lukasiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: CN-algebras, Wajsberg algebras, bounded commutative BCK-algebras and bricks. It is discovered that MV-algebras are naturally related to the Murray-von Neumann order of projections in operator algebras on Hilbert spaces and that they play an interesting role as invariants of approximately finite-dimensional C^*-algebras. They are also naturally related to Ulam's searching games with lies. MV-algebras admit a natural lattice reduct and hence a natural order structure. Many important properties can be derived from the fact, established by Chang that nontrivial MV-algebras are subdirect products of MV-chains, that is, totally ordered MV-algebras. To prove this fundamental result, Chang introduced the notion of prime ideal in an MV-algebra.

A product MV-algebra (or PMV-algebra, for short) is an MV-algebra which has an associative binary operation ".". It satisfies an extra property which will be explained in Preliminaries. PMV-algebras were introduced by A. Di Nola and A. Dvurečenskij in [5]. They also introduced ·-ideals in PMV-algebras. During the last years, PMV-algebras were considered and their equivalence with a certain class of l-rings with strong unit was proved. In 2014, F. Forouzesh, E. Eslami and A. Borumand Saeid defined $\cdot$-prime ideals in PMV-algebras [8].

The concept of fuzzy sets was introduced by Zadeh for the first time [11]. Since then, many studies were performed about this subject and many researchers started working on the fuzzy algebraic structures. Recently, many papers were writed, too. For example, see [1, 10].

In this paper, we introduce the notions of prime $\cdot$-ideals and fuzzy prime $\cdot$-ideals in PMV-algebras and prove some results on them. In fact, we open new fields to anyone that is interested to studying and development of fuzzy ideals in PMV-algebras.

2. Preliminaries

In this section, we review related lemmas and theorems that we use in the next sections.

Definition 2.1 ([4]). An MV-algebra is a structure $M = (M, \oplus, ', 0)$ of type $(2, 1, 0)$ such that:
$(MV1)$ $(M, \oplus, 0)$ is an Abelian monoid,
$(MV2)$ $(a')' = a$,
$(MV3)$ $0' \oplus a = 0'$,
$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$.
If we define the constant $1 = 0'$ and operations $\odot$ and $\ominus$ by $a \odot b = (a' \oplus b')'$, $a \ominus b = a \odot b'$, then
$(MV5)$ $(a \oplus b) = (a' \odot b')'$,
$(MV6)$ $x \oplus 1 = 1$,
$(MV7)$ $(a \ominus b) \oplus b = (b \ominus a) \oplus a$,
$(MV8)$ $a \oplus a' = 1$,
for every $a, b \in A$.

It is clear that $(M, \odot, 1)$ is an abelian monoid. Now, if we define auxiliary operations $\vee$ and $\wedge$ on M by $a \vee b = (a \odot b') \oplus b$ and $a \wedge b = a \odot (a' \oplus b)$, for every $a, b \in M$, then $(M, \vee, \wedge, 0)$ is a bounded distributive lattice.

In MV-algebra M, the following conditions are equivalent: for every $a, b, c \in M$,
(i) $a' \oplus b = 1$.
(ii) $a \odot b' = 0$.
(iii) $b = a \oplus (b \ominus a)$.
(iv) $\exists c \in A$ such that $a \oplus c = b$.
For any two elements a, b of MV-algebra M, $a \leq b$ if and only if a, b satisfy in the above equivalent conditions (i)-(iv).

An ideal of MV-algebra M is a subset I of M, satisfying the following condition: for every $x, y \in I$,
(I1) $0 \in I$,
(I2) $x \leq y$ and $y \in I$ implies that $x \in I$,
(I3) $x \oplus y \in I$.
A proper ideal I of M is a prime ideal if and only if $x \ominus y \in I$ or $y \ominus x \in I$ (or $x \wedge y \in I$ implies that $x \in I$ or $y \in I$), for every $x, y \in M$.

In MV-algebra M, the distance function $d : M \times M \to M$ is defined by $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ which satisfies

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq d(x, y) \oplus d(y, z)$,

(iv) $d(x, y) = d(x', y')$,

(v) $d(x \oplus z, y \oplus t) \leq d(x, y) \oplus d(z, t)$,

for every $x, y, z, t \in M$.

Let I be an ideal of MV-algebra M. Then we denote $x \sim y$ ($x \equiv_I y$) if and only if $d(x, y) \in I$, for every $x, y \in M$. Thus $\sim$ is a congruence relation on M. Denote the equivalence class containing x by $\frac{x}{I}$ and $\frac{M}{I} = \{\frac{x}{I} : x \in M\}$. Then $(\frac{M}{I}, \oplus, ', \frac{0}{I})$ is an MV-algebra, where $(\frac{x}{I})' = \frac{x'}{I}$ and $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$, for all $x, y \in M$.

Let M and K be two MV-algebras. A mapping $f : M \to K$ is called an MV-homomorphism if

(H1) $f(0) = 0$,

(H2) $f(x \oplus y) = f(x) \oplus f(y)$,

(H3) $f(x') = (f(x))'$,

for every $x, y \in M$.

If f is one to one (onto), then f is called an MV-monomorphism (epimorphism) and if f is onto and one to one, then f is called an MV-isomorphism(see [6]).

Proposition 2.2 ([4]). *Let M be an MV-algebra and $W \subseteq M$. Then the principal ideal generated by W is denoted by $\prec W \succ$ and $\prec W \succ = \{x \in M : x \leq w_1 \oplus w_2 \oplus \cdots \oplus w_n,\ for\ some\ w_1, \cdots, w_n \in W\}$. Further, for every ideal J of M, $\prec J \cup \{z\} \succ = \{x \in M : nz \oplus a \geq x,\ for\ some\ n \in \mathbb{N}\ and\ a \in J\}$.*

Lemma 2.3 ([4]). *Let M, N be two MV-algebras and $f : M \to N$ be an MV-homomorphism. Then the following properties hold:*

(i) *$Ker(f)$ is an ideal of M.*

(ii) *If f is an MV-epimorphism, then $\frac{M}{Kerf} \cong N$.*

(iii) *$f(x) \leq f(y)$ iff $x \ominus y \in Ker(f)$.*

(iv) *f is injective iff $Ker(f) = \{0\}$.*

Lemma 2.4 ([4]). *In every MV-algebra M, the natural order "$\leq$" has the following properties: for every $x, x', y, y', z \in M$,*

(i) *$x \leq y$ if and only if $y' \leq x'$.*

(ii) *If $x \leq y$, then $x \oplus z \leq y \oplus z$.*

(iii) *If $x \leq y$, then $x \odot z \leq y \odot z$.*

Proposition 2.5 ([4]). *The following equatoins hold in every MV-algebra:*

(i) *$x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.*

(ii) *$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.*

Definition 2.6 ([6]). In MV-algebra M, a partial addition is defined as following: $x + y$ is defined if and only if $x \leq y'$ and in this case, $x + y = x \oplus y$, for any $x, y \in M$.

Lemma 2.7 ([6]). *Let M be an MV-algebra and let $+$ is the partial addition on M. Then for every $x, y, z \in M$,*

(c) *$x \vee y = x + (x' \odot y)$.*

(d) *If* $x+y$ *and* $(x+y)+z$ *are defined, then* $y+z$ *and* $x+(y+z)$ *are defined and* $(x+y)+z = x+(y+z)$.
(e) $x+y=1$ *if and only if* $y=x'$.
(f) *If* $z+x \leq z+y$*, then* $x \leq y$.
(g) *If* $x+y=z$*, then* $y = x' \odot z$.
(h) *If* $z+x=z+y$*, then* $x=y$.

Definition 2.8 ([5])**.** A product MV-algebra (or PMV-algebra, for short) is a structure $(A, \oplus, ., ', 0)$, where $(A, \oplus, ', 0)$ is an MV-algebra and $\cdot$ is a binary associative operation on A such that the following property is satisfied: if $x+y$ is defined, then $x\cdot z+y\cdot z$ and $z\cdot x+z\cdot y$ are defined and $(x+y)\cdot z = x\cdot z+y\cdot z$, $\quad z\cdot(x+y) = z\cdot x+z\cdot y$, where $+$ is a partial addition on A.

If A is a PMV-algebra, then a unity for the product is an element $e \in A$ such that $e \cdot x = x \cdot e = x$, for every $x \in A$.
A PMV-algebra that has unity for the product will be called unital.

A $\cdot$-ideal of a PMV-algebra A is an ideal I of MV-algebra A such that if $a \in I$ and $b \in A$, then $a \cdot b \in I$ and $b \cdot a \in I$.

Lemma 2.9 ([6])**.** *If* A *is a unital* PMV*-algebra, then*
(i) *The unity for the product is* $e = 1$.
(ii) $x \cdot y \leq x \wedge y$*, for every* $x, y \in A$.

Definition 2.10 ([6])**.** Let X and Y be PMV-algebras. An MV-homomorphism $f : X \to Y$ is called a homomorphism of PMV-algebras (or PMV-homomorphism) if and only if $f(x \cdot y) = f(x) \cdot f(y)$.

Lemma 2.11 ([5])**.** *Let* A *be a* PMV*-algebra. Then* $1.a = a$ *and* $a \leq b$ *implies that* $a.c \leq b.c$ *and* $c.a \leq c.b$*, for every* $a, b, c \in A$.

Definition 2.12 ([6])**.** Let $A = (A, \oplus, ., ', 0)$ be a PMV-algebra, $M = (M, \oplus, ', 0)$ be an MV-algebra and the operation $\Phi : A \times M \longrightarrow M$ be defined by $\Phi(a, m) = am$, which satisfies the following axioms:
(AM1) If $x+y$ is defined in M, then $ax+ay$ is defined in M and $a(x+y) = ax+ay$.
(AM2) If $a+b$ is defined in A, then $ax+bx$ is defined in M and $(a+b)x = ax+bx$.
(AM3) $(a.b)x = a(bx)$, for every $a, b \in A$ and $x, y \in M$.
Then M is called a (left) MV-module over A or briefly an A-module.

We say that M is a unitary MV-module if A has a unity 1_A for the product, i.e.,
(AM4) $1_A x = x$, for every $x \in M$.

Lemma 2.13 ([6])**.** *Let* A *be a* PMV*-algebra and* M *be an* A*-module. Then for every* $a, b \in A$ *and* $x, y \in M$,
(a) $0x = 0$.
(b) $a0 = 0$.
(c) $ax' \leq (ax)'$.
(d) $a'x \leq (ax)'$.
(e) $(ax)' = a'x + (1x)'$.
(f) $x \leq y$ *implies* $ax \leq ay$.
(g) $a \leq b$ *implies* $ax \leq bx$.
(h) $a(x \oplus y) \leq ax \oplus ay$.
(i) $d(ax, ay) \leq ad(x, y)$.

Definition 2.14 ([11]). A fuzzy set in set of A is a mapping $\mu : A \to [0,1]$. Let μ be a fuzzy set in A and $t \in [0,1]$. Then $\mu_t = \{x \in A : \mu(x) \geq t\}$ is called a level subset of μ.

Definition 2.15 ([9]). If A is an MV-algebra, then a fuzzy set μ in A is a fuzzy ideal of A, if it satisfies

(FI1) $\mu(0) \geq \mu(x)$, for all $x \in A$,
(FI2) $\mu(y) \geq \mu(x) \wedge \mu(y \odot x')$, for all $x, y \in A$.

Theorem 2.16 ([9]). *Let μ be a fuzzy ideal in A. Then for every $x, y \in A$,*

(i) $\mu(x \oplus y) = \mu(x) \wedge \mu(y)$.
(ii) $\mu(x \vee y) = \mu(x) \wedge \mu(y)$
(iii) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

Lemma 2.17 ([7]). *Let A be an MV-algebra and $\mu : A \to [0,1]$ be a fuzzy set on A. Then μ is a fuzzy ideal on A if and only if*

(i) $\mu(x) \leq \mu(0)$,
(ii) $\mu(x \oplus y) \geq \mu(x) \wedge \mu(y)$,
(iii) *If $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y \in A$.*

Theorem 2.18 ([7]). *Let μ be a fuzzy set in A. μ is a fuzzy ideal in A if and only if for all $t \in [0,1]$, μ_t is either empty or an ideal of A.*

Corollary 2.19 ([7]). *I is an ideal of A if and only if χ_I is a fuzzy ideal of A, where χ_I is characteristic function of I.*

3. Prime $\cdot$-ideals in PMV-algebras

Note: From now on, in this paper, A is a PMV-algebra.
In this section, we introduce prime $\cdot$-ideals in PMV-algebras and state some conditions to obtain them.

Definition 3.1. Let P be a $\cdot$-ideal of A. If P is a prime ideal of MV-algebra A, then P is called a prime $\cdot$-ideal of A.

Example 3.2. Let $A = \{0, 1, 2, 3\}$ and the operations "$\oplus$" and "." on A be defined as follows:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Consider $0' = 3,\ 1' = 2,\ 2' = 1$ and $3' = 0$. Then it is easy to show that $(A, \oplus, ', ., 0)$ is a PMV-algebra, $J = \{0, 2\}$ and $I = \{0, 1\}$ are prime $\cdot$-ideals of A. Also, $\{0\}$ is not a prime $\cdot$-ideal of A.

Lemma 3.3. *Let A be an MV algebra and $I \subseteq A$. Then I is an ideal of A if and only if*

(i) $0 \in I$,
(ii) *$x \oplus y \in I$, for every $x, y \in I$,*
(iii) *if $x \ominus y$, $y \in I$, then $x \in I$, for any $x, y \in A$.*

Proof. ($\Rightarrow$) Let I be an ideal of A. Then (i) and (ii) are clear. Now, let $x \ominus y$, $y \in I$. Then by (ii) and ($MV7$), $(y \ominus x) \oplus x = (x \ominus y) \oplus y \in I$. Since $x \leq (y \ominus x) \oplus x \in I$, $x \in I$.

($\Leftarrow$) Let (i), (ii) and (iii) be true. If $x \leq y$ and $y \in I$, then $x \ominus y = x \odot y' = 0 \in I$. Thus by (iii), $x \in I$. So I is an ideal of A. □

Theorem 3.4. *Let A, B be PMV-algebras and $f : A \longrightarrow B$ be a PMV-homomorphism. Then*

(i) *If P is a prime $\cdot$-ideal of B, then $f^{-1}(P)$ is a prime $\cdot$-ideal of A.*

(ii) *If f is onto, P is a prime $\cdot$-ideal of A and $Ker(f) \subseteq P$, then $f(P)$ is a prime $\cdot$-ideal of B.*

Proof. (i) The proof is routine.

(ii) The first we show that $f(P)$ is an ideal of B. The proofs of (I_1) and (I_3) are easy. We show that $f(P)$ satisfies in (I_2). Let $a \leq b \in f(P)$, for some $a, b \in B$. Since f is onto, there are $x \in A$ and $y \in P$ such that $a = f(x)$ and $b = f(y)$. Since $f(x) \leq f(y)$, by Lemma 2.3 (iii), $x \ominus y \in P$. Thus by Lemma 3.3, $x \in P$. It means that $a = f(x) \in f(P)$. So $f(P)$ is an ideal of B. It is routine to see that $f(P)$ is a $\cdot$-ideal of B.

Now, let $a \wedge b \in f(P)$, for $a, b \in B$. Then there are $x, y \in A$ such that $a = f(x)$ and $b = f(y)$. We have

$$\begin{aligned} a \wedge b = f(x) \wedge f(y) = f(x \wedge y) \in f(P) \quad &\Rightarrow \quad f(x \wedge y) = f(t),\ t \in P \\ &\Rightarrow \quad x \wedge y \odot t' \in P,\ t \in P \\ &\Rightarrow \quad x \wedge y \in P. \end{aligned}$$

Thus $x \in P$ or $y \in P$. So $a = f(x) \in f(P)$ or $b = f(y) \in f(P)$. Hence $f(P)$ is a prime ideal of B and thus it is a prime $\cdot$-ideal of B. □

Lemma 3.5. *In PMV-algebra A, $(\alpha \oplus \beta)a \leq \alpha a \oplus \beta a$, for every $\alpha, \beta, a \in A$.*

Proof. Since $\beta a \leq (\alpha a)' \oplus \beta a$, by Lemma 2.4 (i), $(\alpha a) \odot (\beta a)' = ((\alpha a)' \oplus \beta a)' \leq (\beta a)'$. Then $(\alpha a) \odot (\beta a)' + \beta a$ is defined, where "$+$" is the partial addition on A. Similarly, $\alpha \odot \beta' + \beta$ is defined, too.

Consider A as A-module, where $ab = a.b$, for every $a, b \in A$. Then by Lemma 2.13 (d) and (g), since $\alpha \odot \beta' \leq \beta'$, $(\alpha \odot \beta')a \leq \beta' a \leq (\beta a)'$. Thus $(\alpha \odot \beta')a + \beta a$ is defined. Now, $\alpha \leq \alpha \vee \beta$ implies that $\alpha a \leq (\alpha \vee \beta)a$ and similarly, $\beta a \leq (\alpha \vee \beta)a$. So $\alpha a \vee \beta a \leq (\alpha \vee \beta)a$. Hence by Lemma 2.7 (c),

$$(\alpha a) \odot (\beta a)' + \beta a = \alpha a \vee \beta a \leq (\alpha \vee \beta)a = (\alpha \odot \beta' \oplus \beta)a = (\alpha \odot \beta' + \beta)a = (\alpha \odot \beta')a + \beta a.$$

Now, by Lemma 2.7(f), $\alpha a \odot (\beta a)' \leq (\alpha \odot \beta')a$. If we set $\alpha \oplus \beta$ instead of α, then by Lemma 2.13 (g), we have $(\alpha \oplus \beta)a \odot (\beta a)' \leq ((\alpha \oplus \beta) \odot \beta')a = (\alpha \wedge \beta')a \leq \alpha a$. Thus

$$(\alpha \oplus \beta)a = (\alpha \oplus \beta)a \vee \beta a = (\alpha \oplus \beta)a \odot (\beta a)' \oplus \beta a \leq \alpha a \oplus \beta a.$$

□

Lemma 3.6. *Let I be an ideal of A and $c \in I$. Then $a.c \in I$, for every $a \in A$.*

Proof. The proof is easy. □

Definition 3.7. $S \subseteq A$ is called a $\cdot$-closed subset of A, if $x.y \in S$, for every $x, y \in S$.

Example 3.8. In Example 3.2, $S = \{1, 3\}$ is a $\cdot$-closed subset of A.

Theorem 3.9. *Let A be unital, I be a $\cdot$-ideal of A and S be a $\cdot$-closed subset of A such that $I \cap S = \varnothing$. Then there exists a prime $\cdot$-ideal P of A such that $I \subseteq P$ and $P \cap S = \varnothing$.*

Proof. Let $T = \{J : J\ is\ a\ \cdot -ideal\ of\ A, I \subseteq J\ and\ J \cap S = \varnothing\}$. Since $I \in T$, $T \neq \varnothing$. By Zorn's Lemma, T has a maximal element P. We show that P is a prime $\cdot$-ideal of A. Let $x \wedge y \in P$ and $x, y \notin P$. Consider $\prec P \cup \{x\} \succ$ and $\prec P \cup \{y\} \succ$. By maximality P, $\prec P \cup \{x\} \succ \cap S \neq \varnothing$ and $\prec P \cup \{y\} \succ \cap S \neq \varnothing$ and so there exist $\alpha \in \prec P \cup \{x\} \succ \cap S$ and $\beta \in \prec P \cup \{y\} \succ \cap S$. Then by Proposition 2.2, there exist $a, b \in P$ and $n, m \in \mathbb{N} \cup \{0\}$ such that $\alpha \leq nx \oplus a$ and $\beta \leq my \oplus b$. By Proposition 2.13 (f),(g), $\alpha.\beta \leq (nx \oplus a).(my \oplus b)$. If we consider A as A-module, where $xy = x.y$, for every $x, y \in A$, then by Lemmas 3.5, 3.6 and 2.13 (h),

$$\begin{aligned}\alpha.\beta &\leq (nx \oplus a).my \oplus (nx \oplus a).b \leq nx.my \oplus a.my \oplus nx.b \oplus a.b \\ &\leq \underbrace{x.y \oplus \cdots \oplus x.y}_{mn\ times} \oplus \underbrace{a.y \oplus \cdots \oplus a.y}_{m\ times} \oplus \underbrace{x.b \oplus \cdots \oplus}_{n\ times} \oplus a.b \in P.\end{aligned}$$

So $\alpha.\beta \in P$. Since S is a $\cdot$-closed subset of A, $\alpha.\beta \in P \cap S$, which is a contradiction. Hence P is a prime $\cdot$-ideal of A. □

Lemma 3.10. *Let A be unital, I be an ideal of A and $r(I)$ be the intersection of all prime $\cdot$-ideals of A containing I. Then*

$$r(I) = \{x \in A : x^n = \underbrace{x.x.\cdots.x}_{n\ times} \in I, for\ some\ n \in \mathbb{N}\}.$$

Proof. Let $T = \{x \in A : x^n = \underbrace{x.x.\cdots.x}_{n\ times} \in I, for\ some\ n \in \mathbb{N}\}$. It is easy to show that $T \subseteq r(I)$. Let $x \in r(I)$. If $x \notin T$, then $x^n \notin I$, for every $n \in \mathbb{N}$. Consider $S = \{x^n \oplus a : n \in \mathbb{N} \cup \{0\}, a \in I\ and\ x^n \leq a'\}$. Let $x^n \oplus a,\ x^m \oplus b \in S$, for $a, b \in I$ and $n, m \in \mathbb{N}$. Since $x^n \leq a'$ and $x^m \leq a'$, $x^n + a$ and $x^m + a$ are defined in A. Thus

$$(x^n \oplus a).(x^m \oplus b) = (x^n + a).(x^m + b) = x^{m+n} + a.x^m + x^n.b + a.b = x^{n+m} \oplus t \in S,$$

where by Lemma 3.6, $t \in I$. It results that S is a $\cdot$-closed subset of A. It is easy to see that $S \cap I = \varnothing$. So by Theorem 3.9, there is a prime $\cdot$-ideal P of A such that $I \subseteq P$ and $S \cap P = \varnothing$. Now, since $x \in r(I)$ and $x = x^1 \oplus 0 \in S$, $x \in P \cap S$, which is a contradiction. Hence $x \in T$. Therefore $T = r(I)$. □

Theorem 3.11. *Let P be a $\cdot$-ideal of A. $P = r(P)$ if and only if $\frac{A}{P}$ has no nilpotent elements.*

Proof. ($\Rightarrow$) Let $P = r(P)$ and $\frac{0}{P} \neq \frac{x}{P}$ be an element of $\frac{A}{P}$. If $(\frac{x}{P})^n = \frac{0}{P}$, for $n \in \mathbb{N}$, then $\frac{x^n}{P} = \frac{0}{P}$. Thus $x^n \in P$. So by Lemma 3.10, $x \in r(P) = P$ and thus $\frac{x}{P} = \frac{0}{P}$, which is a contradiction. Hence $\frac{A}{P}$ has no nilpotent elements.

($\Leftarrow$) Let $\frac{A}{P}$ has no nilpotent elements. It is clear that $P \subseteq r(P)$. Let $x \in r(P)$. Then by Lemma 3.10, $x^n \in P$. Thus $\frac{x^n}{P} = \frac{0}{P}$, for $n \in \mathbb{N}$. Since $\frac{x}{P}$ has no nilpotent elements, $\frac{x}{P} = \frac{0}{P}$.Sand so $x \in P$. Hence $r(P) = P$. □

4. Fuzzy prime $\cdot$-ideals in PMV-algebras

In this section, we introduce fuzzy $\cdot$-ideals, fuzzy prime $\cdot$-ideals in PMV-algebras and obtain some results on them.

Definition 4.1. Let μ be a fuzzy ideal in A. Then

(i) μ is called a fuzzy $\cdot$-ideal in A if and only if $\mu(x.y) \wedge \mu(y.x) \geq \mu(x)$, for every $x, y \in A$.

(ii) μ is called a fuzzy prime $\cdot$-ideal in A if and only if $\mu_t = A$ or μ_t is a prime .-ideal of A.

Example 4.2. (i) Let $A = \{0, 1, 2, 3\}$ and the operations " $\oplus$ " and "." be defined on A as follows:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

.	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

Consider $0' = 3,\ 1' = 2,\ 2' = 1$ and $3' = 0$. Then it is easy to show that $(A, \oplus, ', ., 0)$ is a PMV-algebra Now, let $\mu(0) = 0.8$ and $\mu(2) = \mu(1) = \mu(3) = 0.5$. Then μ is a fuzzy $\cdot$-ideal of A.

(ii) Let $A = \{0, 1\}$ and the operations " $\oplus$ " and "." on A be defined as follows:

$\oplus$	0	1
0	0	1
1	1	1

.	0	1
0	0	0
1	1	1

Then it is easy to show that $(A, \oplus, ', \cdot, 0)$ is a PMV-algebra. Now, let $\mu(0) = 0.8$ and $\mu(1) = 0.5$. Then μ is a fuzzy prime $\cdot$-ideal of A.

Corollary 4.3. *Let μ be a fuzzy $\cdot$-ideal in A such that $\mu(0) \neq \mu(1)$. μ is a fuzzy prime $\cdot$-ideal in A if and only if $\mu_{\mu(0)}$ is a prime $\cdot$-ideal of A.*

Proof. The proof is clear. □

Theorem 4.4. *Let μ be a fuzzy $\cdot$-ideal in A such that $\mu(0) \neq \mu(1)$. μ is a fuzzy prime $\cdot$-ideal in A if and only if $\mu(x) \vee \mu(y) \geq \mu(x \wedge y)$, for every $x, y \in A$.*

Proof. ($\Rightarrow$) Let μ be a fuzzy prime $\cdot$-ideal in A. Then μ_t is a prime $\cdot$-ideal of A. Let $x, y \in A$. Consider $t = \mu(x \wedge y)$. Then $\mu(x \wedge y) \geq t$. Thus $x \wedge y \in \mu_t$. So $x \in \mu_t$ or $y \in \mu_t$ and thus $\mu(x) \geq t$ or $\mu(y) \geq t$. Hence $\mu(x) \vee \mu(y) \geq t$. Therefore $\mu(x) \vee \mu(y) \geq \mu(x \wedge y)$.

($\Leftarrow$) Let $\mu(x) \vee \mu(y) \geq \mu(x \wedge y)$, for every $x, y \in A$ and $\mu_t \neq A$, for $t \in [0, 1]$. The first, we show that μ_t is a $\cdot$-ideal of A. Let $x \in \mu_t$ and $y \in A$. Since μ is a fuzzy $\cdot$-ideal of A, $\mu(x.y) \wedge \mu(y.x) \geq \mu(x) \geq t$. Then $\mu(x.y) \geq t$ and $\mu(y.x) \geq t$. Thus $x.y,\ y.x \in \mu_t$. If $x \wedge y \in \mu_t$, then $\mu(x \wedge y) \geq t$. Thus $\mu(x) \vee \mu(y) \geq t$. So $\mu(x) \geq t$ or $\mu(y) \geq t$. Hence $x \in \mu_t$ or $y \in \mu_t$. □

Corollary 4.5. *Let A be unital and μ be a fuzzy $\cdot$-ideal in A such that $\mu(0) \neq \mu(1)$. μ is a fuzzy prime $\cdot$-ideal in A if and only if $\mu(x) \vee \mu(y) = \mu(x \wedge y)$, for every $x, y \in A$.*

Proof. ($\Rightarrow$) Let μ be a fuzzy prime $\cdot$-ideal in A. By Lemma 2.16 ,(iii), $\alpha.\beta \leq (nx \oplus a).(my \oplus b)$. If we consider A as A-mo, $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$. Then by Theorem 4.4, $\mu(x) \vee \mu(y) = \mu(x \wedge y)$.

($\Leftarrow$) By Theorem 4.4, it is clear. □

Corollary 4.6. *Let I be a $\cdot$-ideal in A. I is a prime $\cdot$-ideal in A if and only if χ_I is a fuzzy prime $\cdot$-ideal of A.*

Proof. ($\Rightarrow$) Let I be a prime $\cdot$-ideal in A. We have $(\chi_I)_t = \{x : \chi_I(x) \geq t\} = I$, for every $t \in [0, 1]$. Then $(\chi_I)_t$ is a prime $\cdot$-ideal in A. Thus χ_I is a fuzzy prime $\cdot$-ideal of A.

($\Leftarrow$) Let χ_I be a fuzzy prime $\cdot$-ideal of A. Then by Corollary 4.3, $I = \{x \in A : \chi(x) \geq 1\} = (\chi_I)_{\chi_I(0)}$ is a prime $\cdot$-ideal of A. □

Theorem 4.7. *Let μ be a fuzzy set of A such that $\mu(0) \neq \mu(1)$. μ is a fuzzy prime $\cdot$-ideal in A if and only if*

(i) *$\mu(x \cdot y) = \mu(y \cdot x) = \mu(0)$, for every $x \in \mu_{\mu(0)}$ and $y \in A$.*

(ii) *$\mu(x \ominus y) = \mu(0)$ or $\mu(y \ominus x) = \mu(0)$, for every $x, y \in A$.*

Proof. ($\Rightarrow$) Let μ be a fuzzy prime $\cdot$-ideal in A. Then by Corollary 4.3, $\mu_{\mu(0)}$ is a prime $\cdot$-ideal of A. Since $\mu_{\mu(0)}$ is a $\cdot$-ideal of A, $x \cdot y,\ y \cdot y \in \mu_{\mu(0)}$. Thus $\mu(x \cdot y) = \mu(y \cdot x) = \mu(0)$, for every $x \in \mu_{\mu(0)}$ and $y \in A$. Since $\mu_{\mu(0)}$ is a prime ideal of A, $x \ominus y \in \mu_{\mu(0)}$ or $y \ominus x \in \mu_{\mu(0)}$). So $\mu(x \ominus y) \geq \mu(0)$ or $\mu(y \ominus x) \geq \mu(0)$. Hence $\mu(x \ominus y) = \mu(0)$ or $\mu(y \ominus x) = \mu(0)$.

($\Leftarrow$) Let (i) and (ii) be true. Then it is easy to see that μ is a fuzzy $\cdot$-ideal and $\mu_{\mu(0)}$ is a prime $\cdot$-ideal of A. Thus by Corollary 4.3, μ is a fuzzy prime $\cdot$-ideal in A. □

Theorem 4.8. *Let μ be a fuzzy $\cdot$-ideal and ν be a $\cdot$-ideal in A such that $\nu(0) \neq \nu(1)$. If $\mu \leq \nu$ and $\mu(0) = \nu(0)$, then ν is a fuzzy prime $\cdot$-ideal in A, too.*

Proof. Since μ is a fuzzy prime $\cdot$-ideal in A, by Theorem 4.7, $\mu(x \ominus y) = \mu(0)$ or $\mu(y \ominus x) = \mu(0)$, for every $x, y \in A$. Let $\mu(x \ominus y) = \mu(0)$. Since $\mu \leq \nu$, $\mu(x \ominus y) \leq \nu(x \ominus y)$. Then $\mu(0) \leq \nu(x \ominus y)$. Since $\mu(0) = \nu(0)$, $\nu(0) \leq \nu(x \ominus y)$. Thus $\nu(0) = \nu(x \ominus y)$. Simillarly, if $\mu(y \ominus x) = \mu(0)$, then $\nu(y \ominus x) = \nu(0)$. Also, it is easy to see that $\nu(x \cdot y) = \nu(y \cdot x) = \nu(0)$, for every for every $x \in \nu_{\nu(0)}$ and $y \in A$. So by Theorem 4.7, ν is a fuzzy prime $\cdot$-ideal in A. □

Theorem 4.9. *If A be unital and μ be a fuzzy $\cdot$-ideal of A such that $\mu(0) \neq \mu(1)$, then the following are equivalent:*

(i) *μ is a fuzzy prime $\cdot$-ideal of A.*

(ii) *$\mu_{\mu(0)}$ is a prime $\cdot$-ideal of A.*

(iii) *$\mu(x) \vee \mu(y) \geq \mu(x \cdot y)$.*

Proof. By Corollary 4.3, (i) and (ii) are equivalent. We prove that (ii) and (iii) are equivalent, too.

(ii) $\Rightarrow$ (iii): Let $\mu_{\mu(0)}$ be a prime $\cdot$-ideal of A. Then $\mu_{\mu(0)}$ is a prime ideal of A. Thus $x \ominus y \in \mu_{\mu(0)}$ or $y \ominus x \in \mu_{\mu(0)}$, for every $x, y \in A$. If $x \ominus y \in \mu_{\mu(0)}$, then since $x.y \leq x \wedge y \leq x \vee y$, by Lemma 2.4 (iii) and Proposition 2.5 (i),

$$x.y \ominus y \leq (x \vee y) \ominus y = (x \vee y) \odot y' = (x \odot y') \vee (y \odot y') = x \ominus y \in \mu_{\mu(0)}.$$

Thus $x.y \ominus y \in \mu_{\mu(0)}$.

Simillarly, if $y \ominus x \in \mu_{\mu(0)}$, then $x.y \ominus x \in \mu_{\mu(0)}$. Thus $\mu(x.y \ominus y) = \mu(0)$ or $\mu(x.y \ominus x) = \mu(0)$. So $\mu(y) \geq \mu(x.y \ominus y) \wedge \mu(x.y) = \mu(x.y)$ or $\mu(x) \geq \mu(x.y \ominus x) \wedge \mu(x.y) = \mu(x.y)$. It results that $\mu(x) \vee \mu(y) \geq \mu(x.y)$.

(iii) $\Rightarrow$ (ii): Let $\mu(x) \vee \mu(y) \geq \mu(x \cdot y)$, for every $x, y \in A$. Then by Lemma 2.9 (ii) and Theorem 2.16 (ii), $\mu(x) \vee \mu(y) \geq \mu(x \wedge y)$. Thus by Theorem 4.4, μ is a fuzzy prime $\cdot$-ideal in A. So μ_t is a prime $\cdot$-ideal of A, for every $t \in [0, 1]$. Hence $\mu_{\mu(0)}$ is a prime $\cdot$-ideal of A. Note that, since $\mu(1) < \mu(0)$, $1 \notin \mu_{\mu(o)}$ and thus $\mu_{\mu(o)}$ is a proper $\cdot$-ideal of A. $\square$

Lemma 4.10. *Let μ be a fuzzy set in A.*

(i) *μ is a fuzzy ideal in A if and only if $(z \ominus y) \ominus x = 0$ implies that $\mu(z) \geq \mu(x) \wedge \mu(y)$.*

(ii) *μ is a fuzzy ideal in A if and only if $(x \ominus y) \leq z$ implies that $\mu(x) \geq \mu(z) \wedge \mu(y)$, for all $x, y, z \in A$.*

Proof. (i) ($\Rightarrow$) Let μ be a fuzzy ideal in A and $(z \ominus y) \ominus x = 0$, for all $x, y, z \in A$. Then $\mu(z) \geq \mu(y) \wedge \mu(z \ominus y)$ and $\mu(z \ominus y) \geq \mu(x) \wedge \mu((z \ominus y) \ominus x) = \mu(x) \wedge \mu(0) = \mu(x)$. Thus $\mu(z) \geq \mu(x) \wedge \mu(y)$, for all $x, y, z \in A$.

($\Leftarrow$) Suppose $(z \ominus y) \ominus x = 0$ implies that $\mu(z) \geq \mu(x) \wedge \mu(y)$, for all $x, y, z \in A$. Since $(0 \ominus x) \ominus x = 0$, $\mu(0) \geq \mu(x) \wedge \mu(x) = \mu(x)$. Also, since $(x \ominus y) \ominus (x \ominus y) = 0$, $\mu(x) \geq \mu(y) \wedge \mu(x \ominus y)$. Thus μ is a fuzzy ideal of A.

(ii) By (i), the proof is clear. $\square$

Theorem 4.11. *Let μ be a fuzzy prime $\cdot$-ideal in A. Then $\mu \oplus \alpha$ is also a fuzzy prime $\cdot$-ideal in A, where $(\mu \oplus \alpha)(x) = \mu(x) \oplus \alpha$ and $\alpha \oplus \beta = Min\{\mu(0), \alpha + \beta\}$, for all $x \in A$ and $\alpha, \beta \in [0, \mu(0))$.*

Proof. The first we show that $\mu \oplus \alpha$ is a fuzzy $\cdot$-ideal of A. Let $(x \ominus y) \leq z$, for $x, y, z \in A$. Since μ is a fuzzy ideal of A, $\mu(x) \geq \mu(z) \wedge \mu(y)$. Consider MV-algebra $[0, 1]$. Then by Proposition 2.5 (ii), $\mu(x) \oplus \alpha \geq (\mu(z) \wedge \mu(y)) \oplus \alpha = (\mu \oplus \alpha)(z) \wedge (\mu \oplus \alpha)(y)$. Thus $(\mu \oplus \alpha)(x) \geq (\mu \oplus \alpha)(z) \wedge (\mu \oplus \alpha)(y)$. So by Lemma 4.10 (ii), $\mu \oplus \alpha$ is a fuzzy ideal of A. Since μ is a fuzzy prime $\cdot$-ideal of A, $\mu(x) \vee \mu(y) \geq \mu(x \wedge y)$. Hence $(\mu(x) \vee \mu(y)) \oplus \alpha \geq \mu(x \wedge y) \oplus \alpha$, for every $x, y \in A$. It results that $(\mu(x) \oplus \alpha) \vee (\mu(y)) \oplus \alpha) \geq \mu(x \wedge y) \oplus \alpha$ and thus $(\mu \oplus \alpha)(x) \vee (\mu \oplus \alpha)(y)) \geq (\mu \oplus \alpha)(x \wedge y)$. Now, since $(\mu \oplus \alpha)(0) = \mu(0) \oplus \alpha \neq \mu(1) \oplus \alpha = (\mu \oplus \alpha)(1)$ and $\mu \leq \mu \oplus \alpha$, by Theorem 4.8, $\mu \oplus \alpha$ is a fuzzy prime $\cdot$-ideal of A. $\square$

Lemma 4.12. *Let A be unital and J be a proper $\cdot$-ideal of A. Then there is a prime $\cdot$-ideal P of A such that $J \subseteq P$.*

Proof. A routine application of Zorn's Lemma shows that there is a proper $\cdot$-ideal I of A which is maximal with respect to the property $J \subseteq I$. We show that I is a prime ideal of A. Let $x \wedge y \in I$ and $x, y \notin I$, for $x, y \in A$. Consider $\prec I \cup \{x\} \succ$ and $\prec I \cup \{y\} \succ$. By maximality I, we have $1 \in \prec I \cup \{x\} \succ$ and $1 \in \prec I \cup \{y\} \succ$. Then by Proposition 2.2, there are $m, n \in \mathbb{N}$ and $a, b \in I$ such that $1 = nx \oplus a$ and $1 = my \oplus b$. Now, let $u = a \oplus b$ and $k = max\{m, n\}$. Then $1 = kx \oplus u$ and $1 = ky \oplus u$. By Lemma 3.5 and Proposition 2.13 (h), we have $1.1 = (kx \oplus u).(ky \oplus u) = (kx \oplus u).ky \oplus (kx \oplus u).u \leq (kx).(ky) \oplus u.ky \oplus (kx).u \oplus u.u \in I$. Thus $1 = 1.1 \in I$, which is a contradiction. So I is a prime ideal of A. Hence I is a prime $\cdot$-idael of A. □

Theorem 4.13. *Let μ be a fuzzy $\cdot$-ideal in A such that $1 \neq \mu(0) \neq \mu(1)$. Then there is a fuzzy prime $\cdot$-ideal ν such that $\mu \leq \nu$.*

Proof. Consider $\mu_{\mu(0)} = \{x \in A : \mu(x) = \mu(0)\}$ that is a proper $\cdot$-ideal of A. By Lemma 4.12, there is a prime $\cdot$-ideal P of A such that $\mu_{\mu(0)} \subseteq P$. By Corollary 4.6, χ_P is a fuzzy prime $\cdot$-ideal of A. Now, let $\nu = \chi_P \oplus \alpha$, where $\alpha = \vee_{x \in A-P}\mu(x)$. Then $\alpha \leq \mu(0) < 1$. Thus by Theorem 4.11, ν is a fuzzy prime $\cdot$-ideal of A. □

Theorem 4.14. *Let A be unital, μ and ν be fuzzy $\cdot$-ideals in A and $\mu \wedge \nu \leq \alpha$, for $\alpha \in [0, \mu(0))$. Then there is a fuzzy prime $\cdot$-ideal h in A such that $\mu \leq h$ and $\nu \wedge h \leq \alpha$.*

Proof. Let $T = \{x \in A : \mu(x) > \alpha\}$ and $S = \{x \in A : \nu(x) > \alpha\}$. If $a, b \in S$, then $\nu(a) > \alpha$ and $\nu(b) > \alpha$. Thus by Lemma 2.9 (ii) and Theorem 2.16 (iii), $\nu(a.b) \geq \nu(a \wedge b) \geq \nu(a) \vee \nu(b) > \alpha$. It results that $a.b \in S$ and thus S is .-closed. We must show that T is a $\cdot$-ideal of A. Since $\alpha \in [0, \mu(0))$, $0 \in T$. Let $x, y \ominus x \in T$, for $x, y \in A$. Then $\mu(x) > \alpha$, $\mu(y \ominus x) > \alpha$. Thus $\mu(x) \wedge \mu(y \ominus x) > \alpha$. Since $\mu(y) \geq \mu(x) \wedge \mu(y \ominus x)$, $\mu(y) > \alpha$. So $y \in T$. Also, since $\mu(x \oplus y) \geq \mu(x)$, $x \oplus y \in T$, for every $x, y \in T$. Hence by Lemma 3.3, T is an ideal of A. Now, let $x \in T$ and $y \in A$. Then $\mu(x) > \alpha$. Since $x.y \leq x \wedge y \leq x$, $\mu(x.y) \geq \mu(x) > \alpha$. Thus $x.y \in T$. So T is a $\cdot$-ideal of A. Let $T \cap S \neq \varnothing$. Then there is $x \in T \cap S$. Thus $\mu(x) > \alpha$ and $\nu(x) > \alpha$. It results that $(\nu \wedge \mu)(x) = \nu(x) \wedge \mu(x) > \alpha$, which is a contradiction. Hence, by Theorem 3.9, there exists a prime $\cdot$-ideal P of A such that $S \cap P = \varnothing$ and $T \subseteq P$. Consider $h = \chi_P \oplus \alpha$. Then by Corollary 4.6 and Theorem 4.11, h is a fuzzy prime $\cdot$-ideal of A. Now, we show that $\mu \leq h$. If $x \in P$, then $\chi_P(x) = 1$ and so $h(x) = 1 \oplus \alpha$. It results that $\mu \leq h$. If $x \notin P$, then $x \notin T$ and so $\mu(x) \leq \alpha$. On the other hand, $h(x) = \chi_P(x) \oplus \alpha = \alpha$. Therefore $\mu(x) \leq h(x)$. It is easy to show that $\nu \wedge h \leq \alpha$. □

5. Conclusions

Prime ideals, $\cdot$-ideals and $\cdot$-prime ideals had been defined in MV-algebras [4, 5, 8]. Lately, ideals in PMV-algebras were considered and some researchers have been interested to them. We defined prime $\cdot$-ideals in PMV-algebras and stated some conditions to have them. They are $\cdot$-ideals that are prime ideals, too. Also, we defined fuzzy prime $\cdot$-ideals in PMV-algebras and obtained some results on them. In fact, we opened new fields to anyone that is interested to studying and development of ideals in MV-algebras.

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