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Generalized intuitionistic fuzzy linear programming problem

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Abstract. In this paper, we obtain characteristic values defined in [16] for non-normal triangular intuitionistic fuzzy numbers. Then we introduce a new ranking function for aggregation intuitionistic fuzzy decisions by using reform index Yager[17], and we solve based on new ranking function, linear programming problems with data as non-normal triangular intuitionistic fuzzy numbers.

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1. Introduction

Fuzzy set (FS) theory has been used to express ambiguous of model parameters in decision making problems particularly in optimization problems. In fuzzy sets, the degree of belonging of an element to a fuzzy set is called membership degree of an element . However, Any particular way is available to express the crisp degree of membership of the elements in FS. Also degree of non-belonging of an element is called non-membership degree of element. Atanassov [2] introduced another extension of Zadeh [19] FS namely the intuitionistic fuzzy set (IFS). IFS assigns to each element of the universe a degree of membership together with the degree of non-membership, which are more or less independent. Das et al. [5] compute criteria in a decision making problem by knowledge measure with intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. The first Angelov [1] use IFS in optimization problems. He formulated an intuitionistic fuzzy optimization (IFO) model by concidering degrees of rejection of objective(s) and constraints together with their degrees of acceptance with the approach of maximizing the degree of acceptance of intuitionistic fuzzy (IF) objective(s) and of constraints and minimizing the degree of rejection of IF objective(s) and constraints. Subsequently he proposed a crisp optimization problem using the Bellman and Zadeh [3] extension principle for aggregating IF decisions. Recently, in [17], Yager pointed out the difficulty in using of the Bellman and ZadehÔÇÖs extension principle for aggregating IF decisions. He also suggested an alternative approach instead of the Bellman and ZadehÔÇÖs extension principle. fuzzy numbers is one of the most important tools for the expression of uncertainty. As regards fuzzy numbers are not linearly ordered, ranking function is one of the fundamental problems of fuzzy arithmetic. This problem is also important in the case of intuitionistic fuzzy numbers (IFNs). Das and Guha study in IFNs. For further study see [6, 7, 8, 9]. Numerous methods have been proposed in literature to rank fuzzy numbers and intuitionistic fuzzy numbers (for example, please see [4, 11, 13, 16]). Grzegorzewski [13] proposed a ranking method for intuitionistic fuzzy numbers based on two families metrics. Nehi [16] introdused two characteristic values for IFNs are defined by the integral of the inverse fuzzy membership and non-membership functions multiplied by the grade with powered parameter and proposed new ranking method for normal trapezoidal intuitionistic fuzzy number. This double-indexed approach is found to be more robust and effective than any single-index approaches for ranking TIFNs. Almost parallel, Li [14] defined two concepts of the value and the ambiguity of a TIFN similar to those for a fuzzy number introduced by Delgado et al. [10]. These are then used to define the value index and the ambiguity index for TIFN. A ratio ranking method is developed for ordering TIFN. Furthermore, the method also takes into consideration a parameter $\lambda \in [0, 1]$ which may reflect the subjective attitude of the decision maker. Dipti and Mehra [12], proposed an approach based on value and ambiguity indexes by Li [14] to the newly defined TIFNs to solve linear programming problems with data as triangular intuitionistic fuzzy numbers. In this paper we obtain characteristic values defined in [16] for non-normal triangular intuitionistic fuzzy numbers.Then we introduce a new ranking function for aggregation intuitionistic fuzzy decisions by using reform index Yager[17] . Our main aim has been to research a meaningful approach to handle linear programming problems (LPPs) with data as non-normal trapezoidal intuitionistic fuzzy numbers.

This paper is organized as follows. Firstly, in section 2, we will express definition of intuitionistic fuzzy sets and non-normal triangular intuitionistic fuzzy numbers and obtain characteristic values in [16] for this Category numbers . In section 3, we express model of Angelov[1] and review intuitionistic fuzzy linear programming models. then, we solve generalized intuitionistic fuzzy linear programming problem. To end with, two sample examples are proposed . The paper is summarized in section 4.

2. Preliminaries

In this section we will review the basic concepts of intuitionistic fuzzy sets and quote a few definitions and properties of triangular intuitionistic fuzzy numbers (TIFNs).

Definition 2.1 ([2])**.** An intuitionistic fuzzy set (IFS)$\tilde{a}$ assigns to each element x of the universe X a membership degree $\mu_{\tilde{a}}(x) \in [0, 1]$ and a non-membership degree

$\nu_{\tilde{a}}(x) \in [0,1]$ such that $0 < \mu_{\tilde{a}}(x) + \nu_{\tilde{a}}(x) \leqslant 1$. IFS $\tilde{a}$ is mathematically represented as:

$$\{(x, \mu_{\tilde{a}}(x), \nu_{\tilde{a}}(x))|x \in X\}.$$

The value $\pi_{\tilde{a}}(x) = 1 - \mu_{\tilde{a}}(x) - \nu_{\tilde{a}}(x)$ is called the degree of hesitancy or the intuitionistic index of x to $\tilde{a}$. In this work, $X = \mathbb{R}$.

Definition 2.2 ([14]). A TIFN $\tilde{a} = \{(\underline{a}^{\mu}, a, \bar{a}^{\mu}; w_{\tilde{a}}), (\underline{a}^{\nu}, a, \bar{a}^{\nu}; u_{\tilde{a}})\}$ is an IFS in $\mathbb{R}$, whose membership and non-membership functions are respectively defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{(x - \underline{a}^{\mu})w_{\tilde{a}}}{a - \underline{a}^{\mu}}, & \underline{a}^{\mu} \leqslant x < a, \\ w_{\tilde{a}}, & x = a, \\ \dfrac{(\bar{a}^{\mu} - x)w_{\tilde{a}}}{\bar{a}^{\mu} - a}, & a < x \leqslant \bar{a}^{\mu}, \\ 0, & otherwise, \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} \dfrac{a - x + (x - \underline{a}^{\nu})u_{\tilde{a}}}{a - \underline{a}^{\nu}}, & \underline{a}^{\nu} \leqslant x < a, \\ u_{\tilde{a}}, & x = a, \\ \dfrac{x - a + (\bar{a}^{\nu} - x)u_{\tilde{a}}}{\bar{a}^{\nu} - a}, & a < x \leqslant \bar{a}^{\nu}, \\ 0, & otherwise. \end{cases}$$

The values $w_{\tilde{a}}, u_{\tilde{a}}$ respectively represent the maximum degree of the membership and the non-membership $0 \leqslant w_{\tilde{a}} \leqslant 1$, $0 \leqslant u_{\tilde{a}} \leqslant 1$ such that $0 \leqslant w_{\tilde{a}} + u_{\tilde{a}} \leqslant 1$.

Note that if $w_{\tilde{a}} = 1, u_{\tilde{a}} = 0$, then we have a normal TIFN.

Definition 2.3. Let $\tilde{a} = \{(\underline{a}^{\mu}, a, \bar{a}^{\mu}; w_{\tilde{a}}), (\underline{a}^{\nu}, a, \bar{a}^{\nu}; u_{\tilde{a}})\}$ and $\tilde{b} = \{(\underline{b}^{\mu}, b, \bar{b}^{\mu}; w_{\tilde{b}}, (\underline{b}^{\nu}, b, \bar{b}^{\nu}; u_{\tilde{b}})\}$ be two TIFNs and k be a real number. Then

$$\tilde{a} + \tilde{b} = \{(\underline{a}^{\mu} + \underline{b}^{\mu}, a + b, \bar{a}^{\mu} + \bar{b}^{\mu}; \min\{w_{\tilde{a}}, w_{\tilde{b}}\}), (\underline{a}^{\nu} + \underline{b}^{\nu}, a + b, \bar{a}^{\nu} + \bar{b}^{\nu}; \max\{u_{\tilde{a}}, u_{\tilde{b}}\})\},$$

$$k\tilde{a} = \begin{cases} \{(k\underline{a}^{\mu}, ka, k\bar{a}^{\mu}; w_{\tilde{a}}), (k\underline{a}^{\nu}, ka, k\bar{a}^{\nu}; u_{\tilde{a}})\}, & k > 0, \\ \{(k\bar{a}^{\mu}, ka, k\underline{a}^{\mu}; w_{\tilde{a}}), (k\bar{a}^{\nu}, ka, k\underline{a}^{\nu}; u_{\tilde{a}})\}, & k < 0. \end{cases}$$

The next consept are taken from [16].

Definition 2.4. Let $A = \{(x, \mu_{\tilde{a}}(x), \nu_{\tilde{a}}(x))|x \in X\}$ be an IF. Let $s(r,k) = \frac{(k+1)r^k}{2}$ a reagular reducing function with positive parameter. Then the characteristic values of membership and non-membership for IF number with parameter denoted by

$C_k^{\mu}(A)$, $C_k^{\nu}(A)$ respectively, are defined by

$$C_{\mu}^{k}(A) = \int_{0}^{1} s(r,k)[f_A^{-1}(r) + g_A^{-1}(r)]dr,$$

$$C_{\nu}^{k}(A) = \int_{0}^{1} s(r,k)[h_A^{-1}(r) + k_A^{-1}(r)]dr,$$

Simple calculation implies that

$$C_{\mu}^{k}(A) = \frac{k+1}{2}\int_{0}^{1} r^k[f_A^{-1}(r) + g_A^{-1}(r)]dr,$$

$$C_{\nu}^{k}(A) = \frac{k+1}{2}\int_{0}^{1} r^k[h_A^{-1}(r) + k_A^{-1}(r)]dr.$$

for $k \in [0, \infty)$.
Note to the definition of the characteristic values of membership and non-membership of a IF number is dependent on the parameter k. Let $k = 0$, Then

$$C_{\mu}^{0}(A) = \frac{1}{2}\int_{0}^{1} [f_A^{-1}(r) + g_A^{-1}(r)]dr,$$

$$C_{\nu}^{0}(A) = \frac{1}{2}\int_{0}^{1} [h_A^{-1}(r) + k_A^{-1}(r)]dr.$$

Let $k = 1$, Then

$$C_{\mu}^{1}(A) = \frac{1}{2}\int_{0}^{1} r[f_A^{-1}(r) + g_A^{-1}(r)]dr,$$

$$C_{\nu}^{1}(A) = \frac{1}{2}\int_{0}^{1} r[h_A^{-1}(r) + k_A^{-1}(r)]dr.$$

Following on the lines, we compute the characteristic values for TIFNs in definition 2. In this case we have:

$$f_{\tilde{a}}(x) = \frac{(x - \underline{a}^{\mu})w_{\tilde{a}}}{a - \underline{a}^{\mu}},$$

$$g_{\tilde{a}}(x) = \frac{(\bar{a}^{\mu} - x)w_{\tilde{a}}}{\bar{a}^{\mu} - a},$$

$$h_{\tilde{a}}(x) = \frac{a - x + (x - \underline{a}^{\nu})u_{\tilde{a}}}{a - \underline{a}^{\nu}},$$

$$k_{\tilde{a}}(x) = \frac{x - a + (\bar{a}^{\nu} - x)u_{\tilde{a}}}{\bar{a}^{\nu} - a}.$$

The inverses for this shape function for each r in [0,1] are:

$$f_{\tilde{a}}^{-1}(x) = \underline{a}^{\mu} + (a - \underline{a}^{\mu})\frac{r}{w_{\tilde{a}}},$$
$$g_{\tilde{a}}^{-1}(x) = \bar{a}^{\mu} + (a - \bar{a}^{\mu})\frac{r}{w_{\tilde{a}}},$$
$$h_{\tilde{a}}^{-1}(x) = \frac{1}{1-u_{\tilde{a}}}[a - \underline{a}^{\nu}u_{\tilde{a}} + (\underline{a}^{\nu} - a)(1-r)],$$
$$k_{\tilde{a}}^{-1}(x) = \frac{1}{1-u_{\tilde{a}}}[a - \bar{a}^{\nu}u_{\tilde{a}} + (\bar{a}^{\nu} - a)(1-r)],$$

Thus

$$C_{\mu}^{k}(\tilde{a}) = \frac{\underline{a}^{\mu} + \bar{a}^{\mu}}{2} + \frac{(2a - \underline{a}^{\mu} - \bar{a}^{\mu})(k+1)}{2w_{\tilde{a}}(k+2)},$$
$$C_{\nu}^{k}(\tilde{a}) = \frac{\underline{a}^{\nu} + \bar{a}^{\nu}}{2} + \frac{(2a - \underline{a}^{\nu} - \bar{a}^{\nu})(k+1)}{2(1-u_{\tilde{a}})(k+2)}.$$

Lemma 2.5. *Let* $\tilde{a} = \{(\underline{a}^{\mu}, a, \bar{a}^{\mu}; w_{\tilde{a}}), (\underline{a}^{\nu}, a, \bar{a}^{\nu}; u_{\tilde{a}})\}$ *and* $\tilde{b} = \{(\underline{b}^{\mu}, b, \bar{b}^{\mu}; w_{\tilde{b}}), (\underline{b}^{\nu}, b, \bar{b}^{\nu}; u_{\tilde{b}})\}$ *be two TIFNÔÇÖs, and* k_1, k_2 *be non negative real numbers. Then*

$$C_{\mu}^{k}(k_1\tilde{a} + k_2\tilde{b}) = \frac{1}{\min\{w_{\tilde{a}}, w_{\tilde{b}}\}}(k_1 w_{\tilde{a}} C_{\mu}^{k}(\tilde{a}) + k_2 w_{\tilde{a}} C_{\nu}^{k}(\tilde{b})$$
$$-\frac{k_1 w_{\tilde{a}}(\underline{a}^{\mu} + \bar{a}^{\mu})}{2} - \frac{k_2 w_{\tilde{b}}(\underline{b}^{\mu} + \bar{b}^{\mu})}{2}) + \frac{k_1(\underline{a}^{\mu} + \bar{a}^{\mu})}{2} + \frac{k_2(\underline{b}^{\mu} + \bar{b}^{\mu})}{2},$$
$$C_{\nu}^{k}(k_1\tilde{a} + k_2\tilde{b}) = \frac{1}{\max\{u_{\tilde{a}}, u_{\tilde{b}}\}}(k_1(1-u_{\tilde{a}}) C_{\nu}^{k}(\tilde{a}) + k_2(1-u_{\tilde{b}}) C_{\nu}^{k}(\tilde{b})$$
$$-\frac{k_1(1-u_{\tilde{a}})(\underline{a}^{\nu} + \bar{a}^{\nu})}{2} - \frac{k_2(1-u_{\tilde{b}})(\underline{b}^{\nu} + \bar{b}^{\nu})}{2}) + \frac{k_1(\underline{a}^{\nu} + \bar{a}^{\nu})}{2} + \frac{k_2(\underline{b}^{\nu} + \bar{b}^{\nu})}{2}.$$

3. Intuitionistic fuzzy linear programming problem

In this section, we review approach of Angelov [1]. Let G_i, $i = 1, ..., p$ denote the p IF goals and C_j, $i = 1, ..., q$ denote the q IF constraints in a space of alternatives X, then by the Bellman and ZadehÔÇÖs extension principle [3], an IF decision D can be viewed as an IFS given by $D = \{(x, \mu_D(x), \nu_D(x))|x \in X\}$ where $\mu_{\tilde{D}}(x) = \min_{i,j}(\mu_{G_i}(x), \mu_{C_j}(x)), \nu_{\tilde{D}}(x) = \max_{i,j}(\nu_{G_i}(x), \nu_{C_j}(x))$. With this principle at the background, Angelov [1] associated a value function with D as $V_D(x) = \mu_D(x) - \nu_D(x), \quad x \in X$, and the optimal solution is taken:

$$\max\left(\min_{i,j}(\mu_{G_i}(x), \mu_{C_j}(x)) - \max_{i,j}(\nu_{G_i}(x), \nu_{C_j}(x))\right).$$

The problem has been transformed by Angelov [1] to the following crisp optimization problem:

$$\begin{array}{ll} \max & \alpha - \beta \\ s.t. & \mu_{G_i}(x) \geqslant \alpha, \quad i = 1, ..., p, \\ & \nu_{G_i}(x) \leqslant \beta, \quad i = 1, ..., p, \\ & \mu_{C_j}(x) \geqslant \alpha, , \quad j = 1, ..., q, \\ & \nu_{C_j}(x) \leqslant \beta, \quad i = 1, ..., q, \\ & \alpha + \beta \leqslant 1, \\ & \alpha \geqslant \beta, \quad \beta \geqslant 0, \quad x \in X. \end{array}$$

The foregoing discussion shows that the approach in [1] is a application of the Bell-manand ZadehÔÇÖs extension principle. However, Yager [17] pointed out certain difficalty of this approach. For instance, consider two alternatives x and y with $\mu_D(x) = 0.51$, $\nu_D(x) = 0.49 \, and \, \mu_D(y) = 0.41$, $\nu_D(y) = 0.38$. Since $\mu_D(x) - \nu_D(x) = 0.02$ *and* $\mu_D(y) - \nu_D(y) = 0.03$, the obvious optimal decision among the two is y. This seems rather strange that despite the membership of acceptance of x being clearly much more than that of y, still y persists to be the decision maker(DM) optimal choice. In [17], Yager suggested an alternative way to view the process involved in using $V_D(x)$. He transformed the value function $V_D(x)$ to the function $F_D(x) = \frac{1}{2}V_D(x) + \frac{1}{2}$ instead. This motivated Yager [17] to suggest a more general possibility of using the function

$$F_D(x) = \mu_D(x) + \lambda\pi_D(x), \quad \lambda \in [0, 1]. \qquad (2)$$

3.1. **Interpretting of the IF linear inequality.** Let us consider a general IF linear inequality $a^T x \underset{\sim IF}{\succ} b$ where $a \in \mathbb{R}^n, x \in \mathbb{R}^n \, and \, b \in \mathbb{R}$. where the membership and the non-membership functions are to be understood in the sense described below. Let $p > 0$ be the tolerance for membership. The membership function $\mu(.)$ is then defined as:

$$\mu(a^T x) = \begin{cases} 0, & a^T x \leqslant b - p, \\ h_1(a^T x), & b - p \leqslant a^T x \leqslant b, \\ 1, & a^T x \geqslant b, \end{cases}$$

where $h_1 : \mathbb{R} \to [0, 1]$ is a continuous non-decreasing function. For constructing the non-membership function $\nu(.)$ we realize that the only relation to be satisfied is $\nu(a^T x) \leqslant 1 - \mu(a^T x)$. Then Dubey et al. [11] considered three cases in the sequel. In first case, They define the non-membership function $\nu(.)$ as follows:

$$\nu(a^T x) = \begin{cases} 1, & a^T x \leqslant b - p - q, \\ h_2(a^T x), & b - p - q \leqslant a^T x \leqslant b, \\ 0, & a^T x \geqslant b, \end{cases}$$

where $h_2 : \mathbb{R} \to [0,1]$ is a continuous non-increasing function. They define the non-membership function for the second case as follows:

$$\nu(a^T x) = \begin{cases} 1, & a^T x \leqslant b - p, \\ h_3(a^T x), & b - p \leqslant a^T x \leqslant b - p + r, \\ 0, & a^T x \geqslant b - p + r, \end{cases}$$

where $h_3 : \mathbb{R} \to [0,1]$ is a continuous non-increasing function. The non-membership function in the third case is defined next :

$$\nu(a^T x) = \begin{cases} 1, & a^T x \leqslant b - p - s, \\ h_4(a^T x), & b - p - s \leqslant a^T x \leqslant b - p - s + w, \\ 0, & a^T x \geqslant b - p - s + w, \end{cases}$$

where $h_4 : \mathbb{R} \to [0,1]$ is a continuous non-increasing function.
Thus, depending on the construction of the non-membership function, the IF inequality $a^T x \underset{IF}{\succsim} b$ has been interpreted in three different ways namely:

1. the optimistic approach,
2. the pessimistic approach,
3. the mixed approach.

Now,we are review to describe the general model of a linear programming problem with IF inequality and IF objective function. Consider the IF linear programming problem

$$\begin{aligned} &\widetilde{\max} \quad \mathbf{C}^T x \\ &s.t. \quad A_i^T x \underset{IF}{\precsim} \mathbf{b}_i, \qquad i = 1, ..., m, \\ &\qquad\quad x \in S. \end{aligned}$$

Dipti et al. [11] associate membership functions with the objective and the constraints IF inequalities respectively , as follows:

$$f_0(\mathbf{C}^T x) = \mu_0(\mathbf{C}^T x) + \frac{1}{2}\pi_0(\mathbf{C}^T x),\ f_i(A_i^T x) = \mu_i(A_i^T x) + \frac{1}{2}\pi_i(A_i^T x),\ \ i = 1, ..., m.$$

Since the non-membership function of the IF inequality can be described in three ways,Dipti et al. [11] propose three models for the IF linear programming problem to capture the same.

3.1.1. *Optimistic approach.* Consider the IF inequality $C^T x \underset{IF}{\succsim} Z_0$. Let $p_0 > 0,\ q_0 > 0$ be the tolerances for the objective function. Following the approach of Yang et al. [18], the equivalent crisp problem is given by:

$$\begin{aligned} &\max \quad \alpha \\ &s.t. \quad f_{01}(\mathbf{C}^T x) + M\delta_0 \geqslant \alpha, \\ &\qquad f_{02}(\mathbf{C}^T x) + M(1 - \delta_0) \geqslant \alpha, \\ &\qquad f_{i1}(A_i^T x) + M\delta_i \geqslant \alpha, \qquad i = 1, ..., m, \\ &\qquad f_{i2}(A_i^T x) + M(1 - \delta_i) \geqslant \alpha, \qquad i = 1, ..., m, \\ &\qquad x \in S, \quad \alpha \in [0,1], \quad \delta_i \in \{0,1\}, \quad i = 0, 1, ..., m, \end{aligned}$$

where M is a large positive real number. On simplification, crisp mixed integer linear programming problem to solve as follow:

$$\begin{aligned}
\max \quad & \alpha \\
s.t. \quad & \mathbf{C}^T x + 2M(p_0+q_0)\delta_0 - 2(p_0+q_0)\alpha \geqslant Z_0 - p_0 - q_0, \\
& (2p_0+q_0)\mathbf{C}^T x - 2Mp_0(p_0+q_0)\delta_0 - 2p_0(p_0+q_0)\alpha \\
& \geqslant Z_0(2p_0+q_0) - 2(1+M)p_0(p_0+q_0), \\
& (2p_i+q_i)A_i^T x - 2Mp_i(p_i+q_i)\delta_i + 2p_i(p_i+q_i)\alpha \\
& \leqslant (2p_i+b_i)(p_i+q_i) + b_i p_i, \qquad i=1,...,m, \\
& A_i^T x + 2M(p_i+q_i)\delta_i + 2(p_i+q_i)\alpha \\
& \leqslant (2M+1)(p_i+q_i) + b_i, \qquad i=1,...,m, \\
& x \in S, \quad \alpha \in [0,1], \quad \delta_i \in \{0,1\}, \quad i=0,1,...,m.
\end{aligned}$$

3.1.2. *Pessimistic approach.* Let p_0, r_0, *with* $0 < r_0 < p_0$, be the tolerances for the IF linear inequality $C^T x \underset{\sim IF}{\succ} Z_0$. Again adopting the method of Yang et al. [18] the resultant optimization model is given by:

$$\begin{aligned}
\max \quad & \alpha \\
s.t. \quad & f_{01}(\mathbf{C}^T x) \geqslant \alpha, \\
& f_{02}(\mathbf{C}^T x) \geqslant \alpha, \\
& f_{i1}(A_i^T x) \geqslant \alpha, \qquad i=1,...,m, \\
& f_{i2}(A_i^T x) \geqslant \alpha, \qquad i=1,...,m, \\
& x \in S, \quad \alpha \in [0,1].
\end{aligned}$$

which on simplification yields the following crisp linear programming problem:

$$\begin{aligned}
\max \quad & \alpha \\
s.t. \quad & (p_0+r_0)\mathbf{C}^T x - 2p_0 r_0 \alpha \geqslant (Z_0 - p_0)(p_0+r_0), \\
& \mathbf{C}^T x - 2p_0\alpha \geqslant Z_0 - 2p_0, \\
& A_i^T x + 2p_i\alpha \leqslant 2p_i + b_i, \qquad i=1,...,m, \\
& (p_i+r_i)A_i^T x + 2p_i r_i\alpha \leqslant (p_i+r_i)(b_i+p_i), \qquad i=1,...,m, \\
& x \in S, \quad \alpha \in [0,1],
\end{aligned}$$

3.1.3. *mixed approach.* Let $p_0 > 0, s_0 > 0, w_0 > 0$, *with* $s_0 < w_0 < s_0 + p_0$, be the tolerances for the IF linear inequality $C^T x \underset{\sim IF}{\succ} Z_0$. Again using the generalized model of Lin [15], the mixed approach of model be transformed to the following equivalent crisp mixed integer linear programming problem:

$$\begin{aligned}
\max \quad & \alpha \\
s.t. \quad & f_{01}(\mathbf{C}^T x) + M\delta_0 \geqslant \alpha, \\
& f_{02}(\mathbf{C}^T x) + M(1-\delta_0) \geqslant \alpha, \\
& f_{03}(\mathbf{C}^T x) + M(1-\delta_0) \geqslant \alpha, \\
& f_{i1}(A_i^T x) + M\delta_i \geqslant \alpha, \qquad i = 1, ..., m, \\
& f_{i2}(A_i^T x) + M(1-\delta_i) \geqslant \alpha, \qquad i = 1, ..., m, \\
& f_{i3}(A_i^T x) + M(1-\delta_i) \geqslant \alpha, \qquad i = 1, ..., m, \\
& x \in S, \quad \alpha \in [0,1], \quad \delta_i \in \{0,1\}, \quad i = 0, 1, ..., m.
\end{aligned}$$

where M is a large positive real number.

Example 3.1. Consider

$$\begin{aligned}
\widetilde{\max} \quad & 2x_1 + x_2 \\
s.t. \quad & x_1 \underset{IF}{\precsim} 3, \\
& 2x_1 + x_2 \underset{IF}{\precsim} 7, \\
& x_1 + x_2 \leqslant 4, \\
& x_1, x_2 \geqslant 0.
\end{aligned}$$

Take $Z_0 = 8$, $p_0 = 3$, $s_0 = 2, w_0 = 3, p_1 = 2, s_1 = 1$, $w_1 = 2$, $p_2 = 3$, $s_2 = 1., w_2 = 2$. Using the mixed approach, the equivalent crisp optimization problem is given by:

$$\begin{aligned}
\max \quad & \alpha \\
s.t. \quad & 2x_1 + x_2 + 6M\delta_0 - 6\alpha \geqslant 3, \\
& -12x_1 - 6x_2 + 18M\delta_0 + 18\alpha \leqslant 18M - 16, \\
& -2x_1 - x_2 + 6M\delta_0 + 6\alpha \leqslant 6M - 2, \\
& x_1 - 4M\delta_1 + 4\alpha \leqslant 7, \\
& 2x_1 - 4M\delta_1 + 4\alpha \leqslant 11, \\
& x_1 + 4M\delta_1 + 4\alpha \leqslant 4M + 5, \\
& 2x_1 + x_2 - 6M\delta_2 + 6\alpha \leqslant -13, \\
& 0.8333x_1 + 0.4166x_2 - M\delta_2 + \alpha \leqslant 4.4166, \\
& x_1 + 0.5x_2 + M\delta_2 + \alpha \leqslant M + 5.5, \\
& x_1 + x_2 \leqslant 4, \\
& x_1, x_2 \leqslant 0, \\
& \alpha \in [0,1], \quad \delta_0, \delta_1, \delta_2 \in \{0,1\}.
\end{aligned}$$

and M is large positive real number. The optimal solution is $x_1^* = 0.34$, $x_2^* = 0.6$, $\alpha = 0.9$, $\delta_0 = 1$, $\delta_1 = 0$, $\delta_2 = 1$.
Hence the optimal solution of the given IF linear program is $x_1^* = 0.34$, $x_2^* = 0.6$ and the DM aspiration level of 8 met with 90.

3.2. **Generalized intuitionistic fuzzy linear programming problem.** The purpose of this section is to study a class of fuzzy linear programming problems in which the data parameters are non-normal TIFNs in definition 2.

Consider the following linear programming problem:

$$\begin{aligned}
&\widetilde{\max} \quad && \sum_{j=1}^{n} \tilde{c}_j x_j \\
&s.t. && \sum_{j=1}^{n} \tilde{a}_{ij} x_j \leqslant \tilde{b}_i, \quad i = 1, ..., m, \\
& && x_j \geqslant 0, \qquad j = 1, ..., n.
\end{aligned}$$

Where

$$\begin{aligned}
\tilde{c}_j &= \{(\underset{j}{\underline{c}}{}^{\mu}, c_j, \bar{c}_j^{\mu}; w_{\tilde{c}_j}), (\underset{j}{\underline{c}}{}^{\nu}, c_j, \bar{c}_j^{\nu}; u_{\tilde{c}_j})\}, \\
\tilde{a}_{ij} &= \{(\underset{ij}{\underline{a}}{}^{\mu}, a_{ij}, \bar{a}_{ij}^{\mu}; w_{\tilde{a}_{ij}}), (\underset{ij}{\underline{a}}{}^{\nu}, a_{ij}, \bar{a}_{ij}^{\nu}; u_{\tilde{a}_{ij}})\} \\
\tilde{b}_i &= \{(\underset{i}{\underline{b}}{}^{\mu}, b_i, \bar{b}_i^{\mu}; w_{\tilde{b}_i}), (\underset{i}{\underline{b}}{}^{\nu}, b_i, \bar{b}_i^{\nu}; u_{\tilde{b}_i})\}
\end{aligned}$$

for $i = 1, ..., m, j = 1, ..., n$ are TIFNs.
Then For any $\lambda \in [0, 1]$, define

$$F(\tilde{a}, \lambda) = (1 - \lambda) C_{\mu}^{k}(\tilde{a}) + \lambda(1 - C_{\nu}^{k}(\tilde{a})) \quad (1)$$

and

$$\tilde{a} \leqslant \tilde{b} \Leftrightarrow F(\tilde{a}, \lambda) \leqslant F(\tilde{b}, \lambda).$$

Using the ranking function F, for a predefined $\lambda \in [0, 1]$ problem is equivalent to the following crisp optimization problem.

$$\begin{aligned}
&(COP)_{\lambda} \ \ \max \quad && F(\sum_{j=1}^{n} \tilde{c}_j x_j, \lambda) \\
&s.t. && F(\sum_{j=1}^{n} \tilde{a}_{ij} x_j, \lambda) \leqslant F(\tilde{b}_i, \lambda), \quad i = 1, ..., m, \\
& && x_j \geqslant 0, \qquad j = 1, ..., n.
\end{aligned}$$

Further, using new ranking function F and Proposition 1, this problem is equivalent to the following linear programming problem.

$$
\begin{aligned}
(CLP)_\lambda \max \quad & \frac{1-\lambda}{\min\{w_{\tilde{c}_j}\}}\Big(\sum_{j=1}^{n} C_\mu^k(\tilde{c}_j) w_{\tilde{c}_j} x_j - \sum_{j=1}^{n} \frac{\bar{c}_j^\mu + \underline{c}_j^\mu}{2} w_{\tilde{c}_j} x_j\Big) \\
& + (1-\lambda)\sum_{j=1}^{n} \frac{\bar{c}_j^\mu + \underline{c}_j^\mu}{2} x_j + \lambda\Big[1 - \frac{1}{\max\{u_{\tilde{c}_j}\}}\Big(\sum_{j=1}^{n} C_\nu^k(\tilde{c}_j)(1-u_{\tilde{c}_j})x_j \\
& - \sum_{j=1}^{n} \frac{\bar{c}_j^\nu + \underline{c}_j^\nu}{2}(1-u_{\tilde{c}_j})x_j\Big)\Big] - \lambda \sum_{j=1}^{n} \frac{\bar{c}_j^\nu + \underline{c}_j^\nu}{2} x_j \\
\textit{subject to} \quad & \\
& \frac{1-\lambda}{\min\{w_{\tilde{a}_{ij}}\}}\Big(\sum_{j=1}^{n} C_\mu^k(\tilde{a}_{ij}) w_{\tilde{a}_{ij}} x_j - \sum_{j=1}^{n} \frac{\bar{c}_j^\mu + \underline{c}_j^\mu}{2} w_{\tilde{a}_{ij}} x_j\Big) \\
& + (1-\lambda)\sum_{j=1}^{n} \frac{\bar{c}_j^\mu + \underline{c}_j^\mu}{2} x_j + \lambda\Big[1 - \frac{1}{\max\{u_{\tilde{a}_{ij}}\}}\Big(\sum_{j=1}^{n} C_\nu^k(\tilde{a}_{ij})(1-u_{\tilde{a}_{ij}})x_j \\
& - \sum_{j=1}^{n} \frac{\bar{c}_j^\nu + \underline{c}_j^\nu}{2}(1-u_{\tilde{a}_{ij}})x_j\Big)\Big] - \lambda \sum_{j=1}^{n} \frac{\bar{c}_j^\nu + \underline{c}_j^\nu}{2} x_j \\
& \leqslant (1-\lambda)C_\mu^k(\tilde{b}_i) + \lambda(1 - C_\nu^k(\tilde{b}_i)), \qquad i = 1, ..., m, \\
& \quad x_j \geqslant 0, \qquad j = 1, ..., n.
\end{aligned}
$$

We assume that the DM is rational enough to provide the intuitionistic fuzzy data such that problem $(CLP)_\lambda$ remains bounded and feasible for at least one choice of λ. For $\lambda = 1$, $(CLP)_\lambda$ reduce

$$
\begin{aligned}
\max \quad & 1 - \frac{1}{\max\{u_{\tilde{c}_j}\}}\Big[\sum_{j=1}^{n} C_\nu^k(\tilde{c}_j)(1-u_{\tilde{c}_j})x_j - \sum_{j=1}^{n} \frac{\bar{c}_j^\nu + \underline{c}_j^\nu}{2}(1-u_{\tilde{c}_j})x_j\Big] - \sum_{j=1}^{n} \frac{\bar{c}_j^\nu + \underline{c}_j^\nu}{2} x_j \\
\textit{subject to} \quad & \\
& 1 - \frac{1}{\max\{u_{\tilde{a}_{ij}}\}}\Big[\sum_{j=1}^{n} C_\nu^k(\tilde{a}_{ij})(1-u_{\tilde{a}_{ij}})x_j - \sum_{j=1}^{n} \frac{\bar{c}_j^\nu + \underline{c}_j^\nu}{2}(1-u_{\tilde{a}_{ij}})x_j\Big] \\
& - \sum_{j=1}^{n} \frac{\bar{c}_j^\nu + \underline{c}_j^\nu}{2} x_j \leqslant 1 - C_\nu^k(\tilde{b}_i), \qquad i = 1, ..., m, \\
& x_j \geqslant 0, \qquad j = 1, ..., n.
\end{aligned}
$$

If we take $\lambda = 0$, $(CLP)_\lambda$, we get the following optimization problem:

$$\max \quad \frac{1}{\min\{w_{\tilde{c}_j}\}}(\sum_{j=1}^{n} C_\mu^k(\tilde{c}_j) w_{\tilde{c}_j} x_j - \sum_{j=1}^{n} \frac{\bar{c}_j^\mu + \underline{c}_j^\mu}{2} w_{\tilde{c}_j} x_j) + \sum_{j=1}^{n} \frac{\bar{c}_j^\mu + \underline{c}_j^\mu}{2} x_j$$

subject to

$$\begin{aligned} &\frac{1}{\min\{w_{\tilde{a}_{ij}}\}}(\sum_{j=1}^{n} C_\mu^k(\tilde{a}_{ij}) w_{\tilde{a}_{ij}} x_j - \sum_{j=1}^{n} \frac{\bar{c}_j^\mu + \underline{c}_j^\mu}{2} w_{\tilde{a}_{ij}} x_j) \\ &+ \sum_{j=1}^{n} \frac{\bar{c}_j^\mu + \underline{c}_j^\mu}{2} x_j \leqslant C_\mu^k(\tilde{b}_i), \qquad i = 1, ..., m, \\ &\quad x_j \geqslant 0, \qquad j = 1, ..., n. \end{aligned}$$

For the sake of observation, consider a particular situation when only $C_j,\ j = 1, ..., n$ are TIFNs and the rest of data parameters are crisp numbers in a linea r program:

$$\begin{aligned} (IFOLP)\ \ \widetilde{\max}\ \ & \sum_{j=1}^{n} \tilde{c}_j x_j \\ s.t. \qquad & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \quad i = 1, ..., m, \\ & x_j \geqslant 0, \qquad j = 1, ..., n. \end{aligned}$$

Using the proposed ranking function, (IFOLP) is equivalent to the following crisp problem.

$$\begin{aligned} (IFOLP)\ \ \max \quad & F(\sum_{j=1}^{n} \tilde{c}_j x_j, \lambda) \\ s.t. \qquad & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \ \ i = 1, ..., m, \\ & x_j \geqslant 0, \qquad j = 1, ..., n. \end{aligned}$$

It is worth to notice that if $x_0^*, x_{\lambda_1}^*$ and x_1^* are optimal solution of $(COLP)_0, (COLP)_{\lambda_1}$, $0 < \lambda_1 < 1$ and $(COLP)_1$, respectively, then
$F(\sum\limits_{j=1}^{n} \tilde{c}_j x_{0j}^*, 0) < F(\sum\limits_{j=1}^{n} \tilde{c}_j x_{\lambda_1 j}^*, \lambda_1) < F(\sum\limits_{j=1}^{n} \tilde{c}_j x_{1j}^*, 1)$.

3.3. **Numerical illustration.** We present example to depict the working of the proposed ranking technique for linear programming problem wherein the data is non-normal TIFNs.

Example 3.2. Consider the intuitionistic fuzzy linear program in [12]:

$$\begin{aligned}
&\widetilde{\max} \quad \tilde{5}x_1 + \tilde{3}x_2 \\
&s.t. \quad \tilde{4}x_1 + \tilde{3}x_2 \leqslant \widetilde{12}, \\
&\qquad \tilde{1}x_1 + \tilde{3}x_2 \leqslant \tilde{6}, \\
&\qquad x_1, x_2 \geqslant 0, \\
&Where \\
&c_1 = \tilde{5} = \{(4, 5, 6; \frac{3}{4}), (4, 5, 6.1; \frac{1}{4})\}, \\
&c_2 = \tilde{3} = \{(2.5, 3, 3.2; \frac{1}{2}), (2, 3, 3.5; \frac{1}{4})\}, \\
&a_{11} = \tilde{4} = \{(3.5, 4, 4.1; 1), (3, 4, 5; 0)\}, \\
&a_{12} = \tilde{3} = \{(2.5, 3, 3.5; \frac{3}{4}), (2.4, 3, 3.6; \frac{1}{5})\}, \\
&a_{21} = \tilde{1} = \{(0, 1, 2; 1), (0, 1, 2; 0)\}, \\
&a_{22} = \tilde{3} = \{(2.8, 3, 3.2; \frac{3}{4}), (2.5, 3, 3.2; \frac{1}{6})\}, \\
&b_1 = \widetilde{12} = \{(11, 12, 13; 1), (11, 12, 14; 0)\}, \\
&b_2 = \tilde{6} = \{(5.5, 6, 7.5; \frac{3}{4}), (5, 6, 8.1; \frac{1}{4})\}.
\end{aligned}$$

For $\lambda = 1$, using the method described in earlier section, the equivalent crisp formulation is

$$\begin{aligned}
&\max \quad 5x_1 + 3x_2 \\
&subject\, to \quad 5.2x_1 + 3x_2 \leqslant 12, \\
&\qquad x_1 + 3x_2 \leqslant 6.1667, \\
&\qquad x_1, x_2 \geqslant 0.
\end{aligned}$$

The optimal solution of the problem is $x_1^* = 1.3888$, $x_2^* = 1.5926$, with optimal objective value 11.7222. while The optimal solution of the problem in [12] is $x_1^* = 3.8795$, $x_2^* = 0$, with optimal objective value 11.6872
We next solve the given program for $\lambda = 0$. Following the directions specified in earlier section, we formulate the equivalent crisp model as follows.

$$\begin{aligned}
&\max \quad 1 - 5x_1 + 2.56x_2 \\
&subject\, to \quad 4x_1 + 3x_2 \geqslant 12.25, \\
&\qquad x_1 + 2.54x_2 \geqslant 5.1833, \\
&\qquad x_1, x_2 \geqslant 0.
\end{aligned}$$

The optimal solution of the problem is $x_1^* = 0$, $x_2^* = 4.5915$, with optimal objective value, 9.0111. while The optimal solution of the problem in [12] is $x_1^* = 4.6554$, $x_2^* = 0$, with optimal objective value 9.1946. The second example in [12] has been of linear program in which the technology and resource coefficients are non-normal TIFNs while the objective function coefficients are crisp numbers.

Example 3.3. Consider the following intuitionistic fuzzy linear program

$$\begin{aligned}
\widetilde{\max} \quad & \widetilde{25}x_1 + \widetilde{48}x_2 \\
s.t. \quad & 15x_1 + 30x_2 \leqslant 45000, \\
& 24x_1 + 6x_2 \leqslant 24000, \\
& 21x_1 + 14x_2 \leqslant 28000, \\
& x_1, x_2 \geqslant 0, \\
& Where \\
& c_1 = \widetilde{25} = \{(19, 25, 33; 0.9), (18, 25, 34; \frac{1}{4})\}, \\
& c_2 = \widetilde{48} = \{(44, 48, 54; 0.9), (43, 48, 56; \frac{1}{4})\}.
\end{aligned}$$

Applying the ranking function F, the crisp linear programs for $\lambda = 1$, $\lambda = 0$ are respectively given as follow:

$$\begin{aligned}
\max \quad & 1 + 32x_1 + 58.5x_2 \\
s.t. \quad & 15x_1 + 30x_2 \leqslant 45000, \\
& 24x_1 + 6x_2 \leqslant 24000, \\
& 21x_1 + 14x_2 \leqslant 28000, \\
& x_1, x_2 \geqslant 0,
\end{aligned}$$

The optimal solution of the problem is $x_1^* = 500,\ x_2^* = 1250$, with optimal objective value 89126. while The optimal solution of the problem in [12] is $x_1^* = 0,\ x_2^* = 1500$, with optimal objective value 70500.

$$\begin{aligned}
\max \quad & 25.42x_1 + 48.40x_2 \\
s.t. \quad & 15x_1 + 30x_2 \leqslant 45000, \\
& 24x_1 + 6x_2 \leqslant 24000, \\
& 21x_1 + 14x_2 \leqslant 28000, \\
& x_1, x_2 \geqslant 0,
\end{aligned}$$

The optimal solution of the problem is $x_1^* = 500,\ x_2^* = 1250$, with optimal objective value 73210. while The optimal solution of the problem in [12] is $x_1^* = 0,\ x_2^* = 1500$, with optimal objective value 62000.

4. Conclusions

We studied the generalized model for linear programming set up in the intuitionistic fuzzy scenario. The characteristic values defined in [16] have been computed for non-normal triangular intuitionistic fuzzy numbers. Thereafter, a ranking function has been proposed by note to reform index than the ones existing in literature. Then we solve a class of linear programming problems in which the data parameters are non-normal TIFNs. The solution methodology for such a class of linear programs is illustrated through examples. The task of developing a more effective ranking method for a class of non-normal TIFNs which can also be effectively applied to solve linear programming problems with intuitionistic fuzzy parameters is still an open research issue.

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