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Intuitionistic fuzzy Zweier I-convergent sequence spaces defined by Orlicz function

Vakeel A. Khan, Ayhan Esi, Yasmeen

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Abstract. In this article we introduce the intuitionistic fuzzy Zweier I-convergent sequence spaces $\mathcal{Z}^I_{(\mu,\nu)}(M)$ and $\mathcal{Z}^I_{0(\mu,\nu)}(M)$ defined by Orlicz function and study the fuzzy topology on the said spaces.

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Corresponding Author: Ayhan Esi (aesi23@hotmail.com)

1. Introduction

After the pioneering work of Zadeh [23], a huge number of research papers have been appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in field of science and engineering. It has a wide range of applications in various fields: population dynamics [3], chaos control [6], computer programming [7], nonlinear dynamical system [8], etc. Fuzzy topology is one of the most important and useful tools and it proves to be very useful for dealing with such situations where the use of classical theories breaks down. K. Hur. H. W. Kang and K. C. Lee[9] introduced the notion of fuzzy equivalence relations and fuzzy partitions. The concept of intuitionistic fuzzy normed space [19] and of intuitionistic fuzzy 2-normed space [16] are the latest developments in fuzzy topology. Quite recently, V. A. Khan, K. Ebadullah and Yasmeen ([10], [11],[12]) studied the notion of I- convergence in Intuitionistic Fuzzy Zweier I-convergent Double Sequence Spaces.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical problems/matrices(double sequences) through the concept of density. The notion of I-convergence, which is a generalization of statistical convergence [5], was introduced by Kostyrko, Salat and Wilczynski [13] by using the idea of I of subsets of the set of natural numbers $\mathbb{N}$ and further studied in [17]. Recently, the notion of statistical convergence of double sequences

$x = (x_{ij})$ has been defined and studied by Mursaleen and Edely [15]; and for fuzzy numbers by Savaş and Mursaleen [20]. Quite recently, Das et al. [4] studied the notion of I and I^*- convergence of double sequences in $\mathbb{R}$.

2. Preliminaries

We recall some notations and basic definitions used in this paper.

Definition 2.1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be continuous t-norm, if it satisfies the following conditions:
(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0,1]$,
(iv) $a * c \leq b * d$ whenever $a \leq b$ and $c \leq d$ for each $a, b, c, d \in [0,1]$.

For example, $a * b = a.b$ is a continuous t-norm.

Definition 2.2. A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be continuous t-conorm, if it satisfies the following conditions:
(i) $\diamond$ is associative and commutative,
(ii) $\diamond$ is continuous,
(iii) $a \diamond 0 = a$ for all $a \in [0,1]$,
(iv) $a \diamond c \leq b \diamond d$ whenever $a \leq b$ and $c \leq d$ for each $a, b, c, d \in [0,1]$.

For example, $a \diamond b = min\{a + b, 1\}$ is a continuous t-conorm.

Definition 2.3. The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space(for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:
(i) $\mu(x,t) + \nu(x,t) \leq 1$,
(ii) $\mu(x,t) > 0$,
(iii) $\mu(x,t) = 1$ if and only if $x = 0$,
(iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(v) $\mu(x,t) * \mu(y,s) \leq \mu(x + y, t + s)$,
(vi) $\mu(x, .) : (0, \infty) \to [0,1]$ is continuous,
(vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,
(viii) $\nu(x,t) < 1$,
(ix) $\nu(x,t) = 0$ if and only if $x = 0$,
(x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(xi) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x + y, t + s)$,
(xii) $\nu(x, .) : (0, \infty) \to [0,1]$ is continuous,
(xiii) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.
In this case (μ, ν) is called an intuitionistic fuzzy norm.

Definition 2.4. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν), if for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu(x_k - L, t) > 1 - \epsilon \text{ and } \nu(x_k - L, t) < \epsilon \text{ for all } k \geq k_0.$$

In this case we write $(\mu, \nu) - \lim x = L$.

Definition 2.5. Let X be a non empty set and let 2^X be the power set of X. Then a family of sets $I \subseteq 2^X$ is said to be an ideal on X, if I is additive, i.e., $A, B \in I \Rightarrow A \cup B \in I$ and hereditary, i.e., $A \in I, B \subseteq A \Rightarrow B \in I$.

Definition 2.6. Let X be a non empty set. Then $\mathcal{F} \subset 2^X$ is said to be a filter on X, if $\phi \notin \mathcal{F}$, for $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

Definition 2.7. Let $I \subset 2^{\mathbb{N}}$ be a non trivial ideal and $(X, \mu, \nu, *, \diamond)$ be an IFNS. A sequence $x = (x_k)$ of elements of X is said to be I-convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\epsilon > 0$ and $t > 0$, the set

$$\{k \in \mathbb{N} : \mu(x_k - L, t) \geq 1 - \epsilon \text{ or } \nu(x_k - L, t) \leq \epsilon\} \in I.$$

In this case L is called the I-limit of the sequence (x_k) with respect to the intuitionistic fuzzy norm (μ, ν) and we write $I_{(\mu,\nu)} - \lim x_k = L$.

Definition 2.8. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Let $r \in (0, 1), t > 0$ and $x \in X$. Then the set

$$B_x(r, t) = \{y \in X : \{k \in \mathbb{N} : \mu(x_k - y_k, t) \leq 1 - r \text{ or } \nu(x_k - y_k, t) \geq r\} \in I\}$$

is called an open ball with centre x and radius r with respect to t.

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar, Mursaleen [1], Malkowsky [14] Ng and Lee [18], and Wang [22]. Şeng*önül* [21] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transformation of the sequence $x = (x_i)$ i.e,

$$y_i = px_i + (1 - p)x_{i-1}$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & \text{if } (i = k), \\ 1 - p, & (i - 1 = k); (i, k \in \mathbb{N}) \\ 0, & \text{otherwise} . \end{cases}$$

Analogous to Başar and Altay [2], Şeng*ön*ül [21] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in \omega : \mathcal{Z}^p x \in c\},$$

$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : \mathcal{Z}^p x \in c_0\}.$$

Definition 2.9. An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If the convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Remark 2.10. If M is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

In this article, we introduce the intuitionistic Zweier I-convergent sequence spaces defined by Orlicz function as follows:

$$\mathcal{Z}^I_{(\mu,\nu)}(M) = \Big\{(x_k) \in \omega : \Big\{k \in \mathbb{N} : M\Big(\frac{\mu(x^{/}_k - L, t)}{\rho}\Big) \leq 1-\epsilon \text{ or } M\Big(\frac{\nu(x^{/}_k - L, t)}{\rho}\Big) \geq \epsilon\Big\} \in I\Big\},$$

$$\mathcal{Z}^I_{0(\mu,\nu)}(M) = \Big\{(x_k) \in \omega : \Big\{k \in \mathbb{N} : M\Big(\frac{\mu(x^{/}_k, t)}{\rho}\Big) \leq 1-\epsilon \text{ or } M\Big(\frac{\nu(x^{/}_k, t)}{\rho}\Big) \geq \epsilon\Big\} \in I\Big\}.$$

3. Main Results

Theorem 3.1. *$\mathcal{Z}^I_{(\mu,\nu)}(M)$ and $\mathcal{Z}^I_{0(\mu,\nu)}(M)$ are linear spaces.*

Proof. We prove the result for $\mathcal{Z}^I_{(\mu,\nu)}(M)$. Similarly the result can be proved for $\mathcal{Z}^I_{0(\mu,\nu)}(M)$. Let $(x^{/}_k), (y^{/}_k) \in \mathcal{Z}^I_{(\mu,\nu)}(M)$ and let α, β be scalars. Then for a given $\epsilon > 0$, we have

$$A_1 = \Big\{k \in \mathbb{N} : M\big(\frac{\mu(x^{/}_k - L_1, \frac{t}{2|\alpha|})}{\rho_1}\big) \leq 1-\epsilon \text{ or } M\big(\frac{\nu(x^{/}_k - L_1, \frac{t}{2|\alpha|})}{\rho_1}\big) \geq \epsilon\Big\} \in I,$$

$$A_2 = \Big\{k \in \mathbb{N} : M\big(\frac{\mu(y^{/}_k - L_2, \frac{t}{2|\beta|})}{\rho_2}\big) \leq 1-\epsilon \text{ or } M\big(\frac{\nu(y^{/}_k - L_2, \frac{t}{2|\beta|})}{\rho_2}\big) \geq \epsilon\Big\} \in I.$$

$$A_1^c = \Big\{k \in \mathbb{N} : M\big(\frac{\mu(x^{/}_k - L_1, \frac{t}{2|\alpha|})}{\rho_1}\big) > 1-\epsilon \text{ or } M\big(\frac{\nu(x^{/}_k - L_1, \frac{t}{2|\alpha|})}{\rho_1}\big) < \epsilon\Big\} \in \mathcal{F}(I),$$

$$A_2^c = \Big\{k \in \mathbb{N} : M\big(\frac{\mu(y^{/}_k - L_2, \frac{t}{2|\beta|})}{\rho_2}\big) > 1-\epsilon \text{ or } M\big(\frac{\nu(y^{/}_k - L_2, \frac{t}{2|\beta|})}{\rho_2}\big) < \epsilon\Big\} \in \mathcal{F}(I).$$

Define the set $A_3 = A_1 \cup A_2$, so that $A_3 \in I$. It follows that A_3^c is a non-empty set in $\mathcal{F}(I)$.

We shall show that for each $(x^{/}_k), (y^{/}_k) \in \mathcal{Z}^I_{(\mu,\nu)}(M)$,

$$A_3^c \subset \Big\{k \in \mathbb{N} : M\Big(\frac{\mu\big((\alpha x^{/}_k + \beta y^{/}_k) - (\alpha L_1 + \beta L_2), t\big)}{\rho}\Big) > 1-\epsilon \text{ or}$$

$$M\Big(\frac{\nu\big((\alpha x^{/}_k + \beta y^{/}_k) - (\alpha L_1 + \beta L_2), t\big)}{\rho}\Big) < \epsilon\Big\}.$$

Let $m \in A_3^c$. In this case

$$M\big(\frac{\mu(x^{/}_m - L_1, \frac{t}{2|\alpha|})}{\rho}\big) > 1-\epsilon \text{ or } M\big(\frac{\nu(x^{/}_m - L_1, \frac{t}{2|\alpha|})}{\rho}\big) < \epsilon$$

and

$$M\big(\frac{\mu(y^{/}_m - L_2, \frac{t}{2|\beta|})}{\rho}\big) > 1-\epsilon \text{ or } M\big(\frac{\nu(y^{/}_m - L_2, \frac{t}{2|\beta|})}{\rho}\big) < \epsilon.$$

Then

$$\begin{aligned}
&M\Big(\frac{\mu\big((\alpha x^{/}_m+\beta y^{/}_m)-(\alpha L_1+\beta L_2),t\big)}{\rho}\Big)\\
&\geq M\Big(\frac{\mu(\alpha x^{/}_m-\alpha L_1,\frac{t}{2})}{\rho}\Big)*M\Big(\frac{\mu(\beta y^{/}_m-\beta L_2,\frac{t}{2})}{\rho}\Big)\\
&=M\Big(\frac{\mu(x^{/}_m-L_1,\frac{t}{2|\alpha|})}{\rho}\Big)*M\Big(\frac{\mu(y^{/}_m-L_2,\frac{t}{2|\beta|})}{\rho}\Big)\\
&>(1-\epsilon)*(1-\epsilon)\\
&=(1-\epsilon).
\end{aligned}$$

and

$$\begin{aligned}
&M\Big(\frac{\nu\big((\alpha x^{/}_m+\beta y^{/}_m)-(\alpha L_1+\beta L_2),t\big)}{\rho}\Big)\\
&\leq M\Big(\frac{\nu(\alpha x^{/}_m-\alpha L_1,\frac{t}{2})}{\rho}\Big)\diamond M\Big(\frac{\nu(\beta y^{/}_m-\beta L_2,\frac{t}{2})}{\rho}\Big)\\
&=M\Big(\frac{\nu(x^{/}_m-L_1,\frac{t}{2|\alpha|})}{\rho}\Big)\diamond M\Big(\frac{\nu(y^{/}_m-L_2,\frac{t}{2|\beta|})}{\rho}\Big)\\
&<\epsilon\diamond\epsilon\\
&=\epsilon.
\end{aligned}$$

So

$$\begin{aligned}
A_3^c\subset\Big\{k\in\mathbb{N}:M\Big(\frac{\mu\big((\alpha x^{/}_k+\beta y^{/}_k)-(\alpha L_1+\beta L_2),t\big)}{\rho}\Big)>1-\epsilon \text{ or}\\
M\Big(\frac{\nu\big((\alpha x^{/}_k+\beta y^{/}_k)-(\alpha L_1+\beta L_2),t\big)}{\rho}\Big)<\epsilon\Big\}.
\end{aligned}$$

Hence $\mathcal{Z}^I_{(\mu,\nu)}(M)$ is a linear space. □

Theorem 3.2. *Every open ball $B_{x^/}(r,t)$ is an open set in $\mathcal{Z}^I_{(\mu,\nu)}(M)$.*

Proof. Let $B_{x^/}(r,t)$ be an open ball with centre $x^/$ and radius r with respect to t. That is

$$B_{x^/}(r,t)=\Big\{y\in X:\{k\in\mathbb{N}:M(\tfrac{\mu(x^{/}_k-y^{/}_k,t)}{\rho})\leq 1-r \text{ or } M(\tfrac{\nu(x^{/}_k-y^{/}_k,t)}{\rho})\geq r\}\in I\Big\}.$$

Let $y^/\in B^c_{x^/}(r,t)$. Then $M(\frac{\mu(x^/-y^/,t)}{\rho})>1-r$ and $M(\frac{\nu(x^/-y^/,t)}{\rho})<r$.
Since $M(\frac{\mu(x^/-y^/,t)}{\rho})>1-r$, there exists $t_0\in(0,1)$ such that $M(\frac{\mu(x^/-y^/,t_0)}{\rho})>1-r$ and $M(\frac{\nu(x^/-y^/,t_0)}{\rho})<r$. Putting $r_0=M(\frac{\mu(x^/-y^/,t)}{\rho})$. Then $r_0>1-r$. Thus there exists $s\in(0,1)$ such that $r_0>1-s>1-r$. For $r_0>1-s$, there exist $r_1,r_2\in(0,1)$ such that $r_0*r_1>1-s$ and $(1-r_0)\diamond(1-r_2)\leq s$. Putting $r_3=max\{r_1,r_2\}$ and consider the ball $B^c_{y^/}(1-r_3,t-t_0)$. We prove that $B^c_{y^/}(1-r_3,t-t_0)\subset B^c_{x^/}(r,t)$.

Let $z^/\in B^c_{y^/}(1-r_3,t-t_0)$. Then

$$M(\tfrac{\mu(y^/-z^/,t-t_0)}{\rho})>r_3 \text{ and } M(\tfrac{\nu(y^/-z^/,t-t_0)}{\rho})<r_3.$$

Thus

$$\begin{aligned}M(\frac{\mu(x^/ - z^/, t)}{\rho}) &\geq M(\frac{\mu(x^/ - y^/, t_0)}{\rho}) * M(\frac{\mu(y^/ - z^/, t - t_0)}{\rho})\\ &\geq (1 - r_0) \diamond (1 - r_3)\\ &\geq (r_0 * r_1)\\ &\geq 1 - s\\ &> 1 - r.\end{aligned}$$

and

$$\begin{aligned}M(\frac{\nu(x^/ - z^/, t)}{\rho}) &\leq M(\frac{\nu(x^/ - y^/, t_0)}{\rho}) \diamond M(\frac{\nu(y^/ - z^/, t - t_0)}{\rho})\\ &\leq (1 - r_0) \diamond (1 - r_3)\\ &\leq (1 - r_0) \diamond (1 - r_2)\\ &\leq s < r.\end{aligned}$$

So $z^/ \in B^c_{x^/}(r, t)$. Hence $B^c_{y^/}(1 - r_3, t - t_0) \subset B^c_{x^/}(r, t)$. □

Define

$\tau_{(\mu,\nu)}(M)$

$= \{A \subset \mathcal{Z}^I_{(\mu,\nu)}(M) :$ for each $x \in A \quad \exists \quad t > 0$ and $r \in (0, 1)$ such that $B_{x^/}(r, t) \subset A\}$.

Then $\tau_{(\mu,\nu)}(M)$ is a topology on $\mathcal{Z}^I_{(\mu,\nu)}(M)$.

Theorem 3.3. *The topology $\tau_{(\mu,\nu)}(M)$ on $\mathcal{Z}^I_{0(\mu,\nu)}(M)$ is first countable.*

Proof. $\{B_{x^/}(\frac{1}{n}, \frac{1}{n}) : n = 1, 2, 3, ...\}$ is a local base at $x^/$, the topology $\tau_{(\mu,\nu)}(M)$ on $\mathcal{Z}^I_{0(\mu,\nu)}(M)$ is first countable. □

Theorem 3.4. *$\mathcal{Z}^I_{(\mu,\nu)}(M)$ and $\mathcal{Z}^I_{0(\mu,\nu)}(M)$ are Housdroff spaces.*

Proof. Let $x^/, y^/ \in \mathcal{Z}^I_{(\mu,\nu)}(M)$ such that $x^/ \neq y^/$. Then $0 < M(\frac{\mu(x^/-y^/,t)}{\rho}) < 1$ and $0 < M(\frac{\nu(x^/-y^/,t)}{\rho}) < 1$. Putting $r_1 = M(\frac{\mu(x^/-y^/,t)}{\rho})$, $r_2 = M(\frac{\nu(x^/-y^/,t)}{\rho})$ and $r = max\{r_1, 1 - r_2\}$. Then for each $r_0 \in (r, 1)$, there exists r_3 and r_4 such that

$$r_3 * r_4 \geq r_0 \text{ and } (1 - r_3) \diamond (1 - r_4) \leq (1 - r_0).$$

Putting $r_5 = max\{r_3, 1 - r_4\}$ and consider the open ball $B^c_{x^/}(1 - r_5, \frac{t}{2})$ and $B^c_{y^/}(1 - r_5, \frac{t}{2})$. Then clearly $B^c_{x^/}(1 - r_5, \frac{t}{2}) \cap B^c_{y^/}(1 - r_5, \frac{t}{2}) = \phi$.

For if there exists $z^/ \in B^c_{x^/}(1 - r_5, \frac{t}{2}) \cap B^c_{y^/}(1 - r_5, \frac{t}{2})$, then

$$\begin{aligned}r_1 = M(\frac{\mu(x^/ - y^/, t)}{\rho}) &\geq M(\frac{\mu(x^/ - z^/, \frac{t}{2})}{\rho}) * M(\frac{\mu(z^/ - y^/, \frac{t}{2})}{\rho})\\ &\geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1\end{aligned}$$

and

$$\begin{aligned} r_2 = M(\frac{\nu(x^/ - y^/, t)}{\rho}) &\leq M(\frac{\nu(x^/ - z^/, \frac{t}{2})}{\rho}) \diamond M(\frac{\nu(z^/ - y^/, \frac{t}{2})}{\rho}) \\ &\leq (1 - r_5) \diamond (1 - r_5) \\ &\leq (1 - r_4) \diamond (1 - r_4) \\ &\leq 1 - r_0 < r. \end{aligned}$$

This is a contradiction. So $\mathcal{Z}^I_{(\mu,\nu)}(M)$ is a Housedroff space.

Similarly we can prove that $\mathcal{Z}^I_{0(\mu,\nu)}(M)$ is a Housedorff space. □

Theorem 3.5. *$\mathcal{Z}^I_{(\mu,\nu)}(M)$ is an IFNS. $\tau_{(\mu,\nu)}(M)$ is a topology on $\mathcal{Z}^I_{(\mu,\nu)}(M)$. Then a sequence $(x^/_x) \in \mathcal{Z}^I_{(\mu,\nu)}(M)$, $x^/_k \to x^/$ if and only if $M(\frac{\mu(x^/_k - x^/, t)}{\rho}) \to 1$ and $M(\frac{\nu(x^/_k - x^/, t)}{\rho}) \to 0$ as $k \to \infty$.*

Proof. Fix $t_0 > 0$. Suppose $x^/_k \to x^/$. Then for $r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $x^/_k \in B_{x^/}(r,t)$ for all $k \geq n_0$. Thus

$$B = \Big\{k \in \mathbb{N} : M(\tfrac{\mu(x^/_k - x^/, t)}{\rho}) \leq 1 - r \text{ or } M(\tfrac{\nu(x^/_k - x^/, t)}{\rho}) \geq r\Big\} \in I$$

such that $B^c \in \mathcal{F}(I)$.
So $1 - M(\frac{\mu(x^/_k - x^/, t)}{\rho}) < r$ and $M(\frac{\nu(x^/_k - x^/, t)}{\rho}) < r$.
Hence $M(\frac{\mu(x^/_k - x^/, t)}{\rho}) \to 1$ and $M(\frac{\nu(x^/_k - x^/, t)}{\rho}) \to 0$ as $k \to \infty$.

Conversely, if for each $t > 0$, $M(\frac{\mu(x^/_k - x^/, t)}{\rho}) \to 1$ and $M(\frac{\nu(x^/_k - x^/, t)}{\rho}) \to 0$ as $k \to \infty$, then for $r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $1 - M(\frac{\mu(x^/_k - x^/, t)}{\rho}) < r$ and $M(\frac{\nu(x^/ - x^/, t)}{\rho}) < r$, for all $k \geq n_0$. It follows that $M(\frac{\mu(x^/_k - x^/, t)}{\rho}) > 1 - r$ and $M(\frac{\nu(x^/_k - x^/, t)}{\rho}) < r$ for all $k \geq n_0$. Thus $x^/_k \in B^c$, for all $k \geq n_0$. So $x^/_k \to x^/$. □

Theorem 3.6. *A sequence $x = (x^/_k) \in \mathcal{Z}^I_{(\mu,\nu)}(M)$ I-converges if and only if for every $\epsilon > 0$ and $t > 0$ there exists a number $N = N(x, \epsilon, t)$ such that*

$$\Big\{k \in \mathbb{N} : M(\tfrac{\mu\big(x^/_k - L, \frac{t}{2}\big)}{\rho}) > 1 - \epsilon \textit{ or } M(\tfrac{\nu\big(x^/_k - L, \frac{t}{2}\big)}{\rho}) < \epsilon\Big\} \in \mathcal{F}(I).$$

Proof. Suppose that $I_{(\mu,\nu)} - x = L$ and let $\epsilon > 0$ and $t > 0$. For a given $\epsilon > 0$, choose $s > 0$ such that $(1 - \epsilon) * (1 - \epsilon) > 1 - s$ and $\epsilon \diamond \epsilon < 0$. Then for each $x \in \mathcal{Z}^I_{(\mu,\nu)}(M)$,

$$A = \Big\{k \in \mathbb{N} : M(\tfrac{\mu(x^/_k - L, \frac{t}{2})}{\rho}) \leq 1 - \epsilon \text{ or } M(\tfrac{\nu(x^/_k - L, \frac{t}{2})}{\rho}) \geq \epsilon\Big\} \in I.$$

Thus

$$A^c = \Big\{k \in \mathbb{N} : M(\tfrac{\mu(x^/_k - L, \frac{t}{2})}{\rho}) > 1 - \epsilon \text{ or } M(\tfrac{\nu(x^/_k - L, \frac{t}{2})}{\rho}) < \epsilon\Big\} \in \mathcal{F}(I).$$

Conversely let us choose $N \in A^c$. Then

$$M\Big(\frac{\mu\big(x^{/}_N-L,\frac{t}{2}\big)}{\rho}\Big)>1-\epsilon \text{ or } M\Big(\frac{\nu\big(x^{/}_N-L,\frac{t}{2}\big)}{\rho}\Big)<\epsilon.$$

Now we want to show that there exists a number $N=N(x^{/}_N,\epsilon,t)$ such that

$$\Big\{k\in\mathbb{N}: M(\tfrac{\mu(x^{/}_k-x^{/}_N,t)}{\rho})\leq 1-s \text{ or } M(\tfrac{\nu(x^{/}_k-x^{/}_N,t)}{\rho})\geq s\Big\}\in I.$$

For this, define for each $x\in\mathcal{Z}^I_{(\mu,\nu)}(M)$,

$$B=\Big\{k\in\mathbb{N}: M(\tfrac{\mu(x^{/}_k-x^{/}_N,t)}{\rho})\leq 1-s \text{ or } M(\tfrac{\nu(x^{/}_k-x^{/}_N,t)}{\rho})\geq s\Big\}\in I.$$

Now we show that $B\subset A$.
Assume that $B\subset A$ does not hold. Then there exists $n\in B$ and $n\notin A$. Thus

$$M(\tfrac{\mu(x^{/}_n-x^{/}_N,t)}{\rho})\leq 1-s \text{ and } M(\tfrac{\mu(x^{/}_k-L,\frac{t}{2})}{\rho})>1-\epsilon.$$

In particular $M(\frac{\mu(x^{/}_N-L,\frac{t}{2})}{\rho})>1-\epsilon$. So

$$\begin{aligned}1-s&\geq M(\frac{\mu(x^{/}_n-x^{/}_N,t)}{\rho})\\&\geq M(\frac{\mu(x^{/}_n-L,\frac{t}{2})}{\rho})*M(\frac{\mu(x^{/}_N-L,\frac{t}{2})}{\rho})\\&\geq(1-\epsilon)*(1-\epsilon)>1-s.\end{aligned}$$

This is not possible.

On the other hand,

$$M(\tfrac{\nu(x^{/}_n-x^{/}_N,t)}{\rho})\geq s \text{ and } M(\tfrac{\nu(x^{/}_k-L,\frac{t}{2})}{\rho})>\epsilon.$$

In particular $M(\frac{\nu(x^{/}_N-L,\frac{t}{2})}{\rho})>\epsilon$. Then

$$\begin{aligned}s&\leq M(\frac{\nu(x^{/}_n-x^{/}_N,t)}{\rho})\\&\leq M(\frac{\nu(x^{/}_n-L,\frac{t}{2})}{\rho})\diamond M(\frac{\nu(x^{/}_N-L,\frac{t}{2})}{\rho})\\&\leq\epsilon\diamond\epsilon<s.\end{aligned}$$

This is not possible. So $B\subset A$. Hence $A\in I$ implies $B\in I$. □

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Vakeel A. Khan (vakhanmaths@gmail.com)
Department of mathematics, Aligarh Muslim University,
Aligarh-202002, India

Ayhan Esi (aesi23@hotmail.com)
Department of mathemathics, University of Adiyaman,
Adiyaman, 02040, Turkey

Yasmeen (yasmeen9828@gmail.com)
Department of mathematics, Aligarh Muslim University,
Aligarh-202002, India