Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 12, No. 3, (September 2016), pp. 449–468
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Solving fourth order fuzzy differential equation by fuzzy Laplace transform

Latif Ahmad, Muhammad Farooq, Saleem Abdullah

Received 25 November 2015; Accepted 8 February 2016

Abstract. In this paper, we use the fuzzy Laplace transformation (FLT) for the *nth* derivative of a fuzzy-valued function named as *nth* derivative theorem and under the strongly generalized differentiability concept, we use it in an analytical solution method for the solution of a fourth order fuzzy initial value problem (FIVP). We have also generalized the r-level set representation of fuzzy-valued functions. The related theorems and properties are proved. The method is illustrated with the help of some examples and the results are plotted.

2010 AMS Classification: 34A07, 34K36, 26E50

Keywords: Fuzzy valued function, Fuzzy derivative and fuzzy differential equation, Fuzzy Laplace transform, Fuzzy generalized differentiability, Generalized Hukuhara differentiability.

Corresponding Author: Saleem Abdullah (saleemabdullah81@yahoo.com)

1. Introduction

The term fuzzy derivative has been introduced in 1972 by Chang and Zadeh [1], while the the term fuzzy differential equation (FDE) was first formulated in 1978 by Kaleva [2] and Seikala [3]. The theory of fuzzy differential equations (FDEs) has two branches, the one in which the Hukuhara derivative is the main tool and the other one is inclusion. This paper is based on the concept of generalized Hukuhara differentiability. The solution of FDE has a wide range of applications in the dynamic system of uncertainty. Moreover the field of FDEs is becoming a necessary part of science and real word problems [2, 3]. In the last few decades, many method have been applied for the solution of FIVP as discussed in [4, 8, 19, 20] but every method has advantages and disadvantages. In the near past Allahviranloo, Kaini and Barkhordari [5] introduced an approach toward the existence and uniqueness to solve a second order FIVP. Here we will adopt FLT method in order to find an analytical solution of FIVP. Recently Salahshour and Allahviranloo [6] has found the analytical solution of second order FIVP, then the third order FIVP has been solved by Hawrra and Amal [7]. In order to solve a FIVP Allahviranloo, Kiani and Barkhordari [5] stated under what condition FLT can be applied to solve a FIVP. In [5], they proposed two conditions for the existence of solution of FIVP using FLT and its inverse, and gave some useful results in the form of first order and second order derivative theorem such as linearity, continuity, uniformity and convergency under the new definition of absolute value of fuzzy-valued functions etc. They also proposed two types of absolute value of fuzzy valued function which define the convergence and exponential order of a fuzzy-valued function to find an appropriate condition. In addition they have also proved that a large class of fuzzy-valued function can be solved with the help of FLT. In this paper, we generalize FLT for nth order FIVP.

The original contributions of the paper are as follows:

- proof of nth derivative theorem [22] for FLT following the approach of [18] which is different from [22] as they have proved the same theorem by induction,
- generalization of the r-level set of the fuzzy valued functions in the form of (i) and (ii)-differentiable for nth derivative,
- generalization of the system of differential equations to nth order in the form of differential operators, which are (i) and (ii)-differentiable,
- constructing solution of nth order FIVP by FLT.

The paper is arranged as follows:
In section 2, we recall some basics definitions and theorems. In section 3, fuzzy Laplace Transform is defined. Then we prove nth derivative theorem, which provides us a base for the generalization of the FLT to solve nth order FIVP. In section 4, we solve FDEs by FLT. To illustrate the method, several examples are given in section 5. Conclusion is given in section 6.

2. Preliminaries

In this section we will recall some basics definitions and theorems needed throughout the paper such as fuzzy number, fuzzy-valued function and the derivative of the fuzzy-valued functions as presented in [2, 3, 8, 9].

Definition 2.1. A fuzzy number is defined as the mapping such that $u : \mathbb{R} \to [0, 1]$, which satisfies the following four properties:

(i) u is upper semi-continuous.

(ii) u is fuzzy convex that is $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}.x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$.

(iii) u is normal that is $\exists\, x_0 \in \mathbb{R}$, where $u(x_0) = 1$.

(iv) $A = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, where $\overline{A}$ is closure of A.

Definition 2.2. A fuzzy number in parametric form is an order pair of the form $u = (\underline{u}(r), \overline{u}(r))$, where $0 \leq r \leq 1$ satisfying the following conditions:

(i) $\underline{u}(r)$ is a bounded left continuous increasing function in the interval $[0, 1]$.

(ii) $\overline{u}(r)$ is a bounded left continuous decreasing function in the interval $[0, 1]$.

(iii) $\underline{u}(r) \leq \overline{u}(r)$.
If $\underline{u}(r) = \overline{u}(r) = r$, then r is called crisp number.

Now we recall a triangular fuzzy number which must be in the form of $u = (l, c, r)$ where $l, c, r \in \mathbb{R}$ and $l \leq c \leq r$, then $\underline{u}(\alpha) = l + (c - r)\alpha$ and $\overline{u}(\alpha) = r - (r - c)\alpha$ are the end points of the α level set. Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number if

$$\widetilde{y}(t) = \begin{cases} 1, & if \;\; y = t, \\ 0, & if \;\; y \neq t. \end{cases}$$

For arbitrary fuzzy numbers $u = (\underline{u}(\alpha), \overline{u}(\alpha))$ and $v = (\underline{v}(\alpha), \overline{v}(\alpha))$ and an arbitrary crisp number j, we define addition and scalar multiplication as:
(i) $(\underline{u+v})(\alpha) = (\underline{u}(\alpha) + \underline{v}(\alpha))$.
(ii) $(\overline{u+v})(\alpha) = (\overline{u}(\alpha) + \overline{v}(\alpha))$.
(iii) $(j\underline{u})(\alpha) = j\underline{u}(\alpha)$, $(j\overline{u})(\alpha) = j\overline{u}(\alpha) \quad j \geq 0$.
(iv) $(j\underline{u})(\alpha) = j\overline{u}(\alpha)\alpha, (j\overline{u})(\alpha) = j\underline{u}(\alpha)\alpha$, $j < 0$.

Definition 2.3. Let x, y $\in E$. If $\exists\ z \in E$ such that $x = y + z$, then z is called the H-difference of x and y and is given by $x \ominus y$.

Remark 2.4 ([10])**.** Let X be a cartesian product of the universes, X_1, $X_1, \cdots, X_n$, that is $X = X_1 \times X_2 \times \cdots \times X_n$ and $A_1, \cdots, A_n$ be n fuzzy numbers in $X_1, \cdots, X_n$ respectively then f is a mapping from X to a universe Y, and $y = f(x_1, x_2, \cdots, x_n)$, then the Zadeh extension principle allows us to define a fuzzy set B in Y as;

$$B = \{(y, u_B(y)) | y = f(x_1, \cdots, x_n), (x_1, \cdots, x_n) \in X\},$$

where

$$u_B(y) = \begin{cases} \sup_{(x_1, \cdots, x_n) \in f^{-1}(y)} \min\{u_{A_1}(x_1), \cdots u_{A_n}(x_n)\}, & if \quad f^{-1}(y) \neq 0, \\ 0, \qquad otherwise, \end{cases}$$

where f^{-1} is the inverse of f.

The extension principle reduces in the case if $n = 1$ and is given as follows: $B = \{(y, u_B(y) | y = f(x),\ x \in X)\}$, where

$$u_B(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \min\{u_A(x)\}, \text{ if } f^{-1}(y) \neq 0, \\ 0, \quad otherwise. \end{cases}$$

By Zadeh extension principle the approximation of addition of E is defined by $(u \oplus v)(x) = \sup_{y \in R} \min(u(y), v(x - y))$, $x \in \mathbb{R}$ and scalar multiplication of a fuzzy number is defined by

$$(k \odot u)(x) = \begin{cases} u(\frac{x}{k}), & k > 0, \\ 0 & \text{otherwise} \;, \end{cases}$$

where $\widetilde{0} \in E$.

The Housdorff distance between the fuzzy numbers [6, 11] is defined by

$$d : E \times E \longrightarrow \mathbb{R}^+ \cup \{0\},$$

$$d(u, v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\},$$

where $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r)) \subset \mathbb{R}$.

We know that if d is a metric in E, then it will satisfies the following properties, introduced by Puri and Ralescu [12]:

(1) $d(u + w, v + w) = d(u, v)$, $\forall$ u, v, w $\in$ E.
(2) $(k \odot u, k \odot v) = |k| d(u, v)$, $\forall k \in \mathbb{R}$, and u, v $\in$ E.
(3) $d(u \oplus v, w \oplus e) \leq d(u, w) + d(v, e)$, $\forall$ u, v, w, e $\in$ E.

Definition 2.5 ([13])**.** If $f : \mathbb{R} \times E \longrightarrow E$, then f is continuous at point $(t_0, x_0) \in \mathbb{R} \times E$ provided that for any fixed number $r \in [0, 1]$ and any $\epsilon > 0$, $\exists\ \delta(\epsilon, r)$ such that

$$d([f(t, x)]^r, [f(t_0, x_0)]^r) < \epsilon,$$

whenever $|t - t_0| < \delta(\epsilon, r)$ and $d([x]^r, [x_0]^r) < \delta(\epsilon, r)$ $\forall\ t \in \mathbb{R}$, $x \in E$.

Theorem 2.6 ([14])**.** *Let f be a fuzzy-valued function on $[a, \infty)$ given in the parametric form as $(\underline{f}(x, r), \overline{f}(x, r))$ for any constant number $r \in [0, 1]$. Here we assume that $\underline{f}(x, r)$ and $\overline{f}(x, r)$ are Reman-Integral on $[a, b]$ for every $b \geq a$. Also we assume that $\underline{M}(r)$ and $\overline{M}(r)$ are two positive functions, such that*

$$\int_a^b |\underline{f}(x, r)| dx \leq \underline{M}(r)$$

and

$$\int_a^b |\overline{f}(x, r)| dx \leq \overline{M}(r),$$

for every $b \geq a$. Then $f(x)$ is improper integral on $[a, \infty)$. Thus an improper integral will always be a fuzzy number.
In short

$$\int_a^r f(x) dx = (\int_a^b |\underline{f}(x, r)| dx, \int_a^b |\overline{f}(x, r)| dx).$$

It is well known that Hukuhare differentiability for fuzzy function was introduced by Puri and Ralescu in 1983.

Definition 2.7 ([15])**.** Let $f : (a, b) \to E$ where $x_0 \in (a, b)$, then we say that f is strongly generalized differentiable at x_0 (Beds and Gal differentiability). If $\exists$ an element $f'(x_0) \in E$ such that

(i) $\forall h > 0$ sufficiently small, $\exists\ f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$, then the following limits hold (in the metric d):

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

or

(ii) $\forall h > 0$ sufficiently small, $\exists\ f(x_0) \ominus f(x_0 + h)$, $f(x_0 - h) \ominus f(x_0)$, then the following limits hold (in the metric d):

$$\lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0),$$

or

(iii) $\forall h > 0$ sufficiently small, $\exists\ f(x_0+h)\ominus f(x_0)$, $f(x_0-h)\ominus f(x_0)$, then following limits hold (in metric d):

$$\lim_{h\to 0}\frac{(x_0+h)\ominus f(x_0)}{h}=\lim_{h\to 0}\frac{f(x_0-h)\ominus f(x_0)}{-h}=f'(x_0),$$

or

(iv) $\forall h > 0$ sufficiently small, $\exists\ f(x_0)\ominus f(x_0+h)$, $f(x_0)\ominus f(x_0-h)$, then the following limits holds(in metric d):

$$\lim_{h\to 0}\frac{f(x_0)\ominus f(x_0+h)}{-h}=\lim_{h\to 0}\frac{f(x_0-h)\ominus f(x_0)}{h}=f'(x_0).$$

The denominators h and $-h$ denote multiplication by $\frac{1}{h}$ $\frac{-1}{h}$ respectively.

Theorem 2.8 ([16])**.** *Let $f:\mathbb{R}\to E$ be a function denoted by $f(t)=(\underline{f}(t,r),\overline{f}(t,r))$ for each $r\in[0,1]$. Then*

(1) *If f is* (i)*-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and $f'(t)=(\underline{f}'(t,r),\overline{f}'(t,r))$.*

(2) *If f is* (ii)*-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and $f'(t)=(\overline{f}'(t,r),\underline{f}'(t,r))$.*

Lemma 2.9 ([17])**.** *Let $x_0\in\mathbb{R}$, then the FDE $y'=f(x,y)$, $y(x_0)=y_0\in\mathbb{R}$ and $f:\mathbb{R}\times E\to E$ is supposed to be a continuous and equivalent to one of the following integral equations.*

$$y(x)=y_0+\int_{x_0}^{x}f(t,y(t))dt\quad\forall\quad x\in[x_0,x_1],$$

or

$$y(0)=y^1(x)+(-1)\odot\int_{x_0}^{x}f(t,y(t))dt\quad\forall\quad x\in[x_0,x_1],$$

on some interval $(x_0,x_1)\subset\mathbb{R}$ depending on the strongly generalized differentiability. Integral equivalency shows that if one solution satisfies the given equation, then the other will also satisfy.

Remark 2.10 ([12])**.** In the case of strongly generalized differentiability to the FDE's $y'=f(x,y)$ we use two different integral equations. But in the case of differentiability as the definition of H-derivative, we use only one integral. The second integral equation as in Lemma 2.9 will be in the form of $y^1(t)=y_0^1\ominus(-1)\int_{x_0}^{x}f(t,y(t))dt$. The following theorem related to the existence of solution of FIVP under the generalized differentiability.

Theorem 2.11. *Let us suppose that the following conditions are satisfied:*

(1) *Let $\mathbb{R}_0=[x_0,x_0+s]\times B(y_0,q),s,q>0,y\in E$, where $B(y_0,q)=\{y\in E: B(y,y_0)\leq q\}$ which denotes a closed ball in E and let $f:\mathbb{R}_0\to E$ be continuous functions such that $D(0,f(x,y))\leq M$, $\forall(x,y)\in\mathbb{R}_0$ and $0\in E$.*

(2) *Let $g:[x_0,x_0+s]\times[0,q]\to\mathbb{R}$ such that $g(x,0)\equiv 0$ and $0\leq g(x,u)\leq M$, $\forall x\in[x_0,x_0+s],0\leq u\leq q$, such that $g(x,u)$ is increasing in u, and g is such that the FIVP $u'(x)=g(x,u(x)),u(x)\equiv 0$ on $[x_0,x_0+s]$.*

(3) *We have $D[f(x,y),f(x,z)\leq g(x,D(y,z))]$,$\forall$ (x,y), (x, z)$\in\mathbb{R}_0$ and $D(y,z)\leq q$.*

(4) $\exists d > 0$ *such that for* $x \in [x_0, x_0 + d]$, *the sequence* $y_n^1 : [x_0, x_0 + d] \to E$ *given by* $y_0^1(x) = y_0$, $y_{n+1}^1(x) = y_0 \ominus (-1) \int_{x_0}^{x} f(t, y_n^1)dt$ *defined for any* $n \in N$.
Then the FIVP $y' = f(x, y)$, $y(x_0) = y_0$ *has two solutions that is* (i)*-differentiable and* (ii)*-differentiable for* y.

$y^1 = [x_0, x_0 + r] \to B(y_0, q)$, *where* $r = \min\{s, \frac{q}{M}, \frac{q}{M_1}, d\}$ *and the successive iterations* $y_0(x) = y_0$, $y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$ *and* $y_{n+1}^1 = y_0$, $y_{n+1}^1(x) = y_0 \ominus (-1) \int_{x_0}^{x} f(t, y_n^1(t))dt$ *converge to these two solutions, respectively.*

Now according to Theorem 2.11, we restrict our attention to function which are (i) or (ii)-differentiable on their domain except on a finite number of points as discussed in [21].

3. Fuzzy Laplace transform (FLT)

Suppose that f is a fuzzy-valued function and p is a real parameter, then according to [6, 17] FLT of the function f is defined as follows:

Definition 3.1. The FLT of fuzzy-valued function is [6]

$$\widehat{F}(p) = L[f(t)] = \int_0^\infty e^{-pt} f(t)dt, \tag{3.1}$$

$$\widehat{F}(p) = L[f(t)] = \lim_{\tau \to \infty} \int_0^\tau e^{-pt} f(t)dt, \tag{3.2}$$

$$\widehat{F}(p) = \left[\lim_{\tau \to \infty} \int_0^\tau e^{-pt} \underline{f}(t)dt, \lim_{\tau \to \infty} \int_0^\tau e^{-pt} \overline{f}(t)dt\right], \tag{3.3}$$

whenever the limits exist.

Definition 3.2. Classical fuzzy Laplace transform: Now consider the fuzzy-valued function in which the lower and upper FLT of the function are represented by

$$\widehat{F}(p; r) = L[f(t; r)] = [l(\underline{f}(t; r)), l(\overline{f}(t; r))], \tag{3.4}$$

where

$$l[\underline{f}(t; r)] = \int_0^\infty e^{-pt} \underline{f}(t; r)dt = \lim_{\tau \to \infty} \int_0^\tau e^{-pt} \underline{f}(t; r)dt, \tag{3.5}$$

$$l[\overline{f}(t; r)] = \int_0^\infty e^{-pt} \overline{f}(t; r)dt = \lim_{\tau \to \infty} \int_0^\tau e^{-pt} \overline{f}(t; r)dt. \tag{3.6}$$

Definition 3.3 ([8])**.** Let $f : (a, b) \to E$ and $x_0 \in (a, b)$. Then the nth order derivative of the function is as follows:

Let $f : (a, b) \to E$, where $x_0 \in (a, b)$. Then we say that f is strongly generalized differentiable of the nth order at x_0, if $\exists$ an element $f^k(x_0) \in E$ such that $\forall\ k = 1, 2 \cdots, n$, one of the following holds:

(i) $\forall h > 0$ sufficiently small,

$$\exists f^{k-1}(x_0 + h) \ominus f^{k-1}(x_0) \text{ and } f^{k-1}(x_0) \ominus f^{k-1}(x_0 - h),$$

then the following limits hold (in the metric d):

$$\lim_{h} \to 0 \frac{f^{k-1}(x_0+h) \ominus f^{k-1}(x_0)}{h} = \lim_{h\to 0} \frac{f^{k-1}(x_0) \ominus f^{k-1}(x_0-h)}{h} = f^k(x_0),$$

(ii) $\forall h > 0$ sufficiently small,

$$\exists \quad f^{k-1}(x_0) \ominus f^{k-1}(x_0+h) \text{ and } f^{k-1}(x_0-h) \ominus f^{k-1}(x_0),$$

then the following limits holds (in the metric d):

$$\lim_{h\to 0} \frac{f^{k-1}(x_0) \ominus f^{k-1}(x_0+h)}{-h} = \lim_{h\to 0} \frac{f^{k-1}(x_0-h) \ominus f^{k-1}(x_0)}{-h} = f^k(x_0),$$

(iii) $\forall h > 0$ sufficiently small,

$$\exists \quad f^{k-1}(x_0+h) \ominus f^{k-1}(x_0) \text{ and } f^{k-1}(x_0-h) \ominus f^{k-1}(x_0),$$

then following limits holds (in metric d):

$$\lim_{h\to 0} \frac{f^{k-1}(x_0+h) \ominus f^{k-1}(x_0)}{h} = \lim_{h\to 0} \frac{f^{k-1}(x_0-h) \ominus f^{k-1}(x_0)}{-h} = f^k(x_0),$$

(iv) $\forall h > 0$ sufficiently small,

$$\exists \quad f^{k-1}(x_0) \ominus f^{k-1}(x_0+h) \text{ and } f^{k-1}(x_0) \ominus f^{k-1}(x_0-h),$$

then the following limits holds(in metric d):

$$\lim_{h\to 0} \frac{f^{k-1}(x_0) \ominus f^{k-1}(x_0+h)}{-h} = \lim_{h\to 0} \frac{f^{k-1}(x_0-h) \ominus f^{k-1}(x_0)}{h} = f^k(x_0).$$

Theorem 3.4 ([7]). *Let $F(t), F'(t), F''(t), \cdots, F^{(n)}(t)$ are nth order differentiable fuzzy-valued functions and we denote r-level set of a fuzzy-valued function $F(t)$ with $[F(t)]^r = [f_r(t), g_r(t)]$, then $[F^n(t)] = [f_r^n(t), g_r^n(t)]$'*

Proof. Here $F(t)$ and $F'(t)$ are differentiable, then we can write as $[F''(t)]^r = [f_r''(t), g_r''(t)]$. Since $F''(t)$ is differentiable, then by Definition 2.6 of [7], a fuzzy-valued function $F : U \to F_0(\mathbb{R}^n)$ is called Hukuhara differentiable at $t_0 \in U$ if $\exists\, DF(t_0) = F'(t_0) \in F_0 \times \mathbb{R}^n$ such that the limits $\lim_{h\to 0} \frac{F(t_0+h) \ominus F(t_0)}{h}$ and $\lim_{h\to 0} \frac{F(t_0) \ominus F(t_0-h}{h}$ exist and is equal to $DF(t_0)$.

Similarly, for $D^2F(t_0)$, we have

$$\begin{aligned} [F'(t_0+h) \ominus F'(t_0)]^r &= [f_r'(t_0+h), g_r'(t_0+h)] \ominus [f_r'(t_0), g_r'(t_0)] \\ &= [f_r'(t_0+h) \ominus f_r'(t_0), g_r'(t_0+h) \ominus g_r'(t_0)] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} [F'(t_0) \ominus F'(t_0-h)]^r &= [f_r'(t_0), g_r'(t_0)] \ominus [f_r'(t_0-h), g_r'(t_0-h)] \\ &= [f_r'(t_0) \ominus f_r'(t_0-h), g_r'(t_0) \ominus g_r'(t_0-h)]. \end{aligned} \tag{3.8}$$

Similarly, for third order, fourth order and continuing up to nth order $i.e$ $D^nF(t_0)$ we have

$$\begin{aligned} &[F^{(n-1)}(t_0+h) \ominus F^{(n-1)}(t_0)]^r \\ &= [f_r^{(n-1)}(t_0+h), g_r^{(n-1)}(t_0+h)] \ominus [f_r^{(n-1)}(t_0), g_r^{(n-1)}(t_0)] \\ &= [f_r^{(n-1)}(t_0+h) \ominus f_r^{(n-1)}(t_0), g_r^{(n-1)}(t_0+h) \ominus g_r^{(n-1)}(t_0)], \end{aligned} \tag{3.9}$$

$$\begin{aligned} & [F^{(n-1)}(t_0) \ominus F^{(n-1)}(t_0 - h)]^r \\ (3.10) \qquad & = [f_r^{(n-1)}(t_0), g_r^{(n-1)}(t_0)] \ominus [f_r^{(n-1)}(t_0 - h), g_r^{(n-1)}(t_0 - h)] \\ & = [f_r^{(n-1)}(t_0) \ominus f_r^{(n-1)}(t_0 - h), g_r^{(n-1)}(t_0) \ominus g_r^{(n-1)}(t_0 - h)]. \end{aligned}$$

Now multiplying $\frac{1}{h}$ to the second order, third order and so on up to nth order and then applying limit as $h \to 0$ on both sides we get the general form. According to [18], if n is a positive integer so in the case of (i) and (ii)-differentiability we can write the nth derivative of the functions $F, F', \cdots, F^{(n-1)}$ in the form of $D^n_{k_1 \cdots k-n} F(t_0)$, where $k_i = 1, 2$ for $i = 1, \cdots, n$. Now if we want to compute the nth derivative of F at t_0 Moreover $D^{(n-1)}_{11} F(t_0)$ is (i)-differentiable and $D^{(n-1)}_{22} F(t_0)$ is (ii)-differentiable. Also $D^{(n-1)}_{12} F(t_0)$ is (i)and (ii)-differentiable and $D^{(n-1)}_{21} F(t_0)$ is (ii) and (i)-differentiable and hence proof is completed. □

3.1. **Convergence.** The FLT can be applied to a large number of fuzzy-valued functions [6], and in some of the examples FLT does not converge as explained below and reported in [6].

Example 3.5. Let the fuzzy-valued function $f(t) = Ce^{t^2}$, where $C \in E$, then we get

$$\lim_{\tau \to \infty} \int_0^\tau Ce^{-pt} e^{t^2} dt \to \infty$$

for any choice of variable p so the integral grows with out bounds as $\tau \to \infty$.

In the fuzzy Laplace theory, we have to use absolute value of fuzzy-valued functions. Here we will define two types of absolute value of fuzzy-valued functions as discussed in [6] and is given in the following definition.

Definition 3.6. Let us consider a fuzzy-valued function whose parametric form is given as:

$$f(t;r) = [\underline{f}(t;r), \overline{f}(t;r)].$$

Now if f is (i)-absolute value function, then $\forall\ r \in [m_1, m_2] \subset [0, 1]$

$$[f(t;r)] = [|\underline{f}(t;r)|, |\overline{f}(t;r)|].$$

If f is (2)-absolute value function, then $\forall\ r \in [m_1, m_2] \subseteq [0, 1]$

$$[f(t;r)] = [|\overline{f}(t;r)|, |\underline{f}(t;r)|],$$

provided that the r-cut or r-level set is satisfied by the fuzzy-valued function $|f(t;r)|$

Theorem 3.7. *According to [6], if a fuzzy-valued function f defined as $[f(t;r)] = [|\underline{f}(t;r)|, |\overline{f}(t;r)|]$, where $\underline{f}(t;r)$ and $\overline{f}(t;r)$ are lower and upper end points fuzzy-valued functions for $r \in [0, 1]$ respectively, then*

(1) *If $\underline{f}(t;r) \geq 0\ \forall r$ then f is* (i)*-absolute value fuzzy function.*
(2) *If $\overline{f}(t;r) \leq 0\ \forall r$ then f is* (ii)*-absolute value fuzzy function.*

Example 3.8. Let us consider $f(t;r) = a(r)e^t$, which is discussed in [6] where $a(r) = [1 + r; 2 - r]$, then $f(t)$ is (1)-absolute and $\forall\ r \in [0, 1]$, we have

$$|f(t;r)| = [|(1 + r)e^t|, |(2 - r)e^{-t}|] = [(1 + r)e^t, (2 - r)e^t].$$

Definition 3.9. The integral (3.1) is absolute convergent, if

$$\lim_{\tau\to\infty}\int_0^{\tau}|e^{-pt}f(t)|dt \tag{3.11}$$

exists, that is

$$\lim_{\tau\to\infty}\int_0^{\tau}e^{-pt}|\underline{f}(t;r)|dt,\ \lim_{\tau\to\infty}\int_0^{\tau}e^{-pt}|\overline{f}(t;r)|dt \tag{3.12}$$

exist.

If $L[f(t)]$ does not converge absolutely and if $f(t)$ is (i)-absolute, then

$$|\int_{\tau}^{\acute{\tau}}e^{-pt}f(t)dt\ |=[|\int_{\tau}^{\acute{\tau}}e^{-pt}\underline{f}(t;r)dt\ |,|\int_{\tau}^{\acute{\tau}}e^{-pt}\overline{f}(t;r)dt|], \tag{3.13}$$

$$|\int_{\tau}^{\acute{\tau}}e^{-pt}f(t)dt|\preceq[\int_{\tau}^{\acute{\tau}}e^{-pt}|\underline{f}(t;r)|dt,\int_{\tau}^{\acute{\tau}}e^{-pt}|\overline{f}(t;r)|dt], \tag{3.14}$$

$$|\int_{\tau}^{\acute{\tau}}e^{-pt}f(t)dt|=\int_{\tau}^{\acute{\tau}}e^{-pt}|f(t)|dt\to\widetilde{0}, \tag{3.15}$$

as $\tau\to\infty$, $\forall\ \acute{\tau}>\tau$. This implies that $L[f(t)]$ also converges.

Similar case holds when f is (ii)-absolute. The symbol $\leq$ is an ordering relation defined as follows:
For any two arbitrary fuzzy numbers u and v, $u\leq v\Leftrightarrow\underline{u}(r)\leq\underline{v}(r)$ and $\overline{u}(r)\leq\overline{v}(r)$, for all $r\in[0,1]$.

Theorem 3.10 ([6])**.** *Suppose that f,f' are continuous fuzzy-valued functions on [0,∞) and of exponential order and that f'' is piecewise continuous fuzzy-valued function [0,∞). Then*

$$L(f''(t))=p^2L(f(t))\ominus pf(0)\ominus f'(0), \tag{3.16}$$

if f and f' are (i)*-differentiable.*

$$L(f''(t))=\ominus(f'(0))\ominus(-p^2)L(f(t))\ominus pf(0), \tag{3.17}$$

if f is (i)*-differentiable and f' is* (ii)*-differentiable.*

$$L(f''(t))=\ominus(pf(0))\ominus(-p^2)L(f(t))\ominus f'(0), \tag{3.18}$$

if f is (ii)*-differentiable and f' is* (i)*-differentiable.*

$$L(f''(t))=p^2L(f(t))\ominus pf(0)-f'(0), \tag{3.19}$$

if f and f' are (ii)*-differentiable.*

Theorem 3.11. *According to [7], suppose that $f(t),f'(t),f''(t)$ are the continuous fuzzy-valued function on [0,∞) and of exponential order while $f'''(t)$ is piecewise continuous fuzzy-valued function on [0,∞) with $f(t)=(\underline{f}(t,r),\overline{f}(t,r))$. Then notations of the nth derivative of the function is given by*

$$L[f'''(t,r)]=p^3L[f(t)]\ominus p^2f(0)\ominus pf'(t)\ominus f''(0). \tag{3.20}$$

.

Proof. Here the notations $\underline{f}', \underline{f}^{''}, \underline{f}^{'''}$ means the lower end points functions derivatives and $\overline{f}', \overline{f}^{''}, \overline{f}^{'''}$ are the upper end points functions derivatives. By using Theorem 3.2, we have

$$L[f'''(t)] = L[\underline{f}^{'''}(t,r), \overline{f}^{'''}(t,r)] = [l\underline{f}^{'''}(t,r), l\overline{f}^{'''}(t,r)]. \tag{3.21}$$

Now for any arbitrary fixed number $r \in [0,1]$, using the definition of classical transform, we get

$$l[\underline{f}'''(t,r)] = p^3 l[\underline{f}(t,r)] - p^2\underline{f}(0,r) - p\underline{f}'(0,r) - \underline{f}^{''}(0,r), \tag{3.22}$$

$$l[\overline{f}^{n}(t,r)] = p^3 l[\overline{f}(t,r)] - p^2\overline{f}(0,r) - p\overline{f}'(0,r) - \overline{f}^{''}(0,r). \tag{3.23}$$

□

Now in order to solve the nth order FIVP, we need the FLT of nth derivative of the fuzzy-valued functions under the generalized H-differentiability. So we will prove the following theorem for the fuzzy Laplace transform for nth order FIVP as follows.

Theorem 3.12 ([22]). *Suppose that $f, f', \cdots, f^{(n-1)}$ are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and that $f^{(n)}$ is piecewise continuous fuzzy-valued function on $[0,\infty)$. Then*

$$\begin{aligned} L(f^{(n)}(t)) = p^n L(f(t)) \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0) \ominus p^{n-3}f^{''}(0) \\ \ominus \cdots \ominus f^{n-1}(0)^{,} \end{aligned} \tag{3.24}$$

if $f, f' \cdots f^{(n-1)}$ are (i)*-differentiable.*

$$\begin{aligned} L(f^{(n)}(t)) = \ominus(f^{(n-1)}(0)) \ominus (-p^n)L(f(t)) \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0) \\ \ominus \cdots \ominus p^{n-(n-1)}f^{(n-2)}(0), \end{aligned} \tag{3.25}$$

if $f, f' \cdots f^{(n-2)}$ are (i)*-differentiable and $f^{(n-1)}$ is* (ii)*-differentiable.*

$$\begin{aligned} L(f^{(n)}(t)) = \ominus(p^{n-(n-1)}f^{(n-2)}) \ominus f^{(n-1)}(0) \ominus (-p^n)L(f(t)) \ominus p^{n-1}f(0) \\ \ominus p^{n-2}f'(0) \ominus \cdots \ominus p^{n-(n-2)}f^{(n-3)}(0), \end{aligned} \tag{3.26}$$

if $f, f' \cdots f^{(n-3)}$ are (i)*-differentiable and $f^{(n-1)}, f^{(n-2)}$ are* (ii)*-differentiable. Similarly*

$$\begin{aligned} L(f^{(n)}(t)) = \ominus(p^{n-1}f(0)) \ominus (-p^n)L(f(t)) \ominus p^{n-2}f'(0) \\ \ominus \cdots \ominus f^{(n-1)}(0), \end{aligned} \tag{3.27}$$

if $f', \cdots, f^{(n-1)}$ are (i)*-differentiable and f is* (ii)*-differentiable.*
Continuing the process until we obtain 2^n system of differential equations, hence according to [17] the last equation is

$$L(f^{(n)}(t)) = p^n L(f(t)) \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0) \ominus p^{n-3}f''(0) \cdots - f^{n-1}(0), \tag{3.28}$$

if $f, f' \cdots f^{(n-1)}$ are (ii)*-differentiable.*

Proof. This theorem is already proved in [22]. In [22], they have used principle of induction to prove the nth derivative theorem. We follow the approach of [18] to generalize the Allahviranloo and Ahmadi derivative theorem.

$$p^nL[f(t)] \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0) \ominus \cdots \ominus f^{(n-1)}(0) = (p^nl[\underline{f}(t,r)] - p^{n-1}\underline{f}(0,r) -p^{n-2}\underline{f}'(0,r)\cdots - \underline{f}^{(n-1)}(0,r), p^nl[\overline{f}(t,r)] - p^{n-1}\overline{f}(0,r) - p^{n-2}\overline{f}'(0,r) \cdots - \overline{f}^{(n-1)}(0,r)).$$

Since

$$l(\underline{f}^{(n)}(t,r)) = l(\underline{f^{(n)}}(t,r)) = p^nl[\underline{f}(t,r)]-p^{n-1}\underline{f}(0,r)-p^{n-2}\underline{f}'(0,r)\cdots-\underline{f}^{(n-1)}(0,r),$$

$$l(\overline{f^{(n)}}(t,r)) = l(\overline{f}^{(n)}(t,r)) = p^nl[\overline{f}(t,r)]-p^{n-1}\overline{f}(0,r)-p^{n-2}\overline{f}'(0,r)\cdots-\overline{f}^{(n-1)}(0,r),$$

where $\underline{f}^{(n-1)}(0,r) = \underline{f}^{(n-1)}(0,r))$ and $\overline{f}^{(n-1)}(0,r) = \overline{f}^{(n-1)}(0,r)$.

$$p^nL[f(t)] \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0)\cdots \ominus f^{(n-1)}(0) = (l(\underline{f}^{(n)})(t,r), l(\overline{f}^{(n)})(t,r)),$$

$$p^nL[f(t)] \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0)\cdots \ominus f^{(n-1)}(0) = L(\underline{f}^{(n)}(t,r), (\overline{f}^{(n)}(t,r))),$$

$$p^nL[f(t)] \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0)\cdots \ominus f^{(n-1)}(0) = L(f^{(n)}(t)).$$

Hence the proof is completed.

Now we are going to prove the final equation (3.28), while the equations in the middle are almost analogous to the proof of (3.24) and (3.28).

$$p^nL[f(t)] \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0)\cdots - f^{(n-1)}(0) = (p^nl[\underline{f}(t,r)] - p^{n-1}\underline{f}(0,r)- p^{n-2}\underline{f}'(0,r)\cdots - \overline{f}^{(n-1)}(0,r), p^nl[\overline{f}(t,r)] - p^{n-1}\overline{f}(0,r) - p^{n-2}\overline{f}'(0,r)\cdots -\underline{f}^{(n-1)}(0,r)).$$

Since

$$l(\underline{f}^{(n)}(t,r)) = l(\underline{f^{(n)}}(t,r)) = p^nl[\underline{f}(t,r)]-p^{n-1}\underline{f}(0,r)-p^{n-2}\underline{f}'(0,r)\cdots-\underline{f}^{(n-1)}(0,r),$$

$$l(\overline{f^{(n)}}(t,r)) = l(\overline{f}^{(n)})(t,r)) = p^nl[\overline{f}(t,r)]-p^{n-1}\overline{f}(0,r)-p^{n-2}\overline{f}'(0,r)\cdots-\overline{f}^{(n-1)}(0,r).$$

Also we know that
$\overline{f}^{(n-1)}(0,r) = \underline{f}^{(n-1)}(0,r)$ and $\underline{f}^{(n-1)}(0,r) = \overline{f}^{(n-1)}$. Therefore

$$p^nL[f(t)] \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0)\cdots - f^{(n-1)}(0) = (l(\underline{f}^{(n)})(t,r), l(\overline{f}^{(n)})(t,r)),$$

$$p^nL[f(t)] \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0)\cdots \ominus f^{(n-1)}(0) = L(\underline{f}^{(n)}(t,r), (\overline{f}^{(n)}(t,r))),$$

$$p^nL[f(t)] \ominus p^{n-1}f(0) \ominus p^{n-2}f'(0)\cdots \ominus f^{(n-1)}(0) = L(f^{(n)}(t)),$$

which is the required result. □

4. Constructing solutions of FIVP via FLT

In this section we are going to consider the following nth order FIVP under generalized H-differentiability proposed in [17].

$$y^{(n)}(t) = f(t, y(t), y'(t), \cdots, y^{(n-1)}(t)), \tag{4.1}$$

subject to the nth order initial conditions

$$y(0) = (\underline{y}(0;r), \overline{y}(0;r)),$$
$$y'(0) = (\underline{y}'(0;r), \overline{y}'(0;r)),$$
$$y''(0) = (\underline{y}''(0;r), \overline{y}''(0;r)).$$

continuing for nth initial conditions

$$y^{(n-1)}(0) = (\underline{y}^{(n-1)}(0;r), \overline{y}^{(n-1)}(0;r)).$$

Now we use FLT

$$L[y^{(n)}(t)] = L[f(t, y(t), y'(t), \cdots, y^{(n-1)}(t))]. \tag{4.2}$$

Using theorem 3.12 and equation (4.2)

$$p^n L[y(t)] \ominus p^{n-1} y(0) \ominus p^{n-2} y'(0) \ominus \cdots \ominus y^{(n-1)}(0) = L[f(t, y(t), y'(t), \cdots, y^{n-1}(t))].$$

In the classical form

$$\begin{aligned} p^n l[\underline{y}(t;r)] - p^{n-1}\underline{y}(0;r) - p^{n-2}\underline{y}'(0;r) - \cdots - \underline{y}^{(n-1)}(0;r) \\ = l[\underline{f}(t, y(0;r), y'(0;r), \cdots, y^{(n-1)}(0;r))], \end{aligned} \tag{4.3}$$

$$\begin{aligned} p^n l[\overline{y}(t;r)] - p^{n-1}\overline{y}(0;r) - p^{n-2}\overline{y}'(0;r) - \cdots - \overline{y}^{(n-1)}(0;r) \\ = l[\overline{f}(t, y(0;r), y'(0;r), \cdots, y^{(n-1)}(0;r))]. \end{aligned} \tag{4.4}$$

In order to solve (4.3) and (4.4) we assume that $A(p;r)$ and $B(p;r)$ are the solutions of (4.3) and (4.4) respectively. Then, we have

$$l[\underline{y}(t;r)] = A(p;r), \tag{4.5}$$

$$l[\overline{y}(t;r)] = B(p;r). \tag{4.6}$$

Using inverse Laplace transform (ILT), we have

$$[\underline{y}(t;r)] = l^{-1}[A(p;r)], \tag{4.7}$$

$$[\overline{y}(t;r)] = l^{-1}[B(p;r)]. \tag{4.8}$$

In case of g_h-differentiability, if we apply FLT on fourth order FIVP, then we have a system of sixteen differential equations. It can be listed in the form of a differential operator on the function $f(t_0)$. The list is below

Table 1. Differentiable operators of sixteen possible solution for order four

S. No	1 and 2 differentiability	S. No	1 and 2 differentiability
1	$(D_1D_1D_1D_1)f(t_0)$	9	$(D_1D_2D_1D_2)f(t_0)$
2	$(D_2D_1D_1D_1)f(t_0)$	10	$(D_2D_1D_2D_1)f(t_0)$
3	$(D_2D_2D_1D_1)f(t_0)$	11	$(D_2D_2D_1D_2)f(t_0)$
4	$(D_2D_2D_2D_1)f(t_0)$	12	$(D_1D_2D_1D_1)f(t_0)$
5	$(D_2D_2D_2D_2)f(t_0)$	13	$(D_1D_1D_2D_1)f(t_0)$
6	$(D_1D_2D_2D_2)f(t_0)$	14	$(D_2D_1D_2D_2)f(t_0)$
7	$(D_1D_1D_2D_2)f(t_0)$	15	$(D_2D_1D_1D_2)f(t_0)$
8	$(D_1D_1D_1D_2)f(t_0)$	16	$(D_1D_2D_2D_1)f(t_0)$

5. Examples

Example 5.1. Consider the following fourth order FIVP

$$y^{(iv)}(t) + y^{''}(t) = y^{'''}(t), \qquad \text{t} \in [-5, 5] \tag{5.1}$$

subject to the fuzzy initial conditions:

$y(0) = (r-1, 1-r)$,
$y'(0) = (r+1, 3-r)$,
$y''(0) = (2+r, 4-r)$,
$y'''(0) = (3+r, 5-r)$.

To solve this example, we will discuss two cases.

Case (i): If $y, y', \cdots, y^{'''}$ are (1)-differentiable. Applying FLT on both sides of (5.1), we get

$$L[y^{(iv)}(t)] + L[y''(t)] = L[y'''(t)],$$

$$p^4L[y(t)] \ominus p^3y(0) \ominus p^2y'(0) \ominus py''(0) \ominus y'''(0) + p^2L[y(t)] \ominus py(0) \ominus y'(0)$$
$$= p^3L[y(t)] \ominus p^2y(0) \ominus py'(0) \ominus y''(0).$$

Now the classical FLT form of the above equation is

$$p^4l[\underline{y}(t,r)] - p^3\underline{y}(0,r) - p^2\underline{y}'(0,r) - p\underline{y}^{''}(0,r) - \underline{y}^{'''}(0,r) + p^2l[\underline{y}(t,r)]$$
$$-p\underline{y}(0,r) - \underline{y}'(0,r) = p^3l[\underline{y}(t,r)] - p^2\underline{y}(0,r) - p\underline{y}'(0,r) - \underline{y}''(0,r).$$

$$p^4l[\overline{y}(t,r)] - p^3\overline{y}(0,r) - p^2\overline{y}'(0,r) - p\overline{y}^{''}(0,r) - \overline{y}^{'''}(0,r) + p^2l[\overline{y}(t,r)]$$
$$-p\overline{y}(0,r) - \overline{y}'(0,r) = p^3l[\overline{y}(t,r)] - p^2\overline{y}(0,r) - \overline{y}'(0,r) - \overline{y}''(0,r).$$

Applying the initial conditions, we have

$$p^4l[\underline{y}(t,r)] - p^3(r-1)p^2(r+1) - p(2+r) - (3+r) + p^2l[\underline{y}(t,r)]p(r-1) - (r+1)$$
$$= p^3l[\underline{y}(t,r)] - p^2(r-1) - p(r+1) - (2+r),$$

$$p^4l[\overline{y}(t,r)] - p^3(1-r)p^2(3-r) - p(4-r) - (5-r) + p^2l[\overline{y}(t,r)]p(1-r) - (3-r))$$
$$= p^3l[\overline{y}(t,r)] - p^2(1-r) - p(3-r) - (4-r).$$

Taking the inverse Laplace transform of the above equation, then solve for $\underline{y}(t,r)$ and $\overline{y}(t,r)$, we get

$$\underline{y}(t,r) = (r-1) + (r+1)t + (2+r)\Big[-t + \frac{2e^{t/2}\mathrm{Sin}\left[\frac{\sqrt{3}t}{2}\right]}{\sqrt{3}}\Big] \\ +[1 + t - \frac{1}{3}e^{t/2}\left(3\mathrm{Cos}\left[\frac{\sqrt{3}t}{2}\right] + \sqrt{3}\mathrm{Sin}\left[\frac{\sqrt{3}t}{2}\right]\right)\Big],$$

$$\overline{y}(t,r) = (1-r) + (3-r)t + (4-r)\Big[-t + \frac{2e^{t/2}\mathrm{Sin}\left[\frac{\sqrt{3}t}{2}\right]}{\sqrt{3}}\Big] \\ +\left[1 + t - \frac{1}{3}e^{t/2}\left(3\mathrm{Cos}\left[\frac{\sqrt{3}t}{2}\right] + \sqrt{3}\mathrm{Sin}\left[\frac{\sqrt{3}t}{2}\right]\right)\right].$$

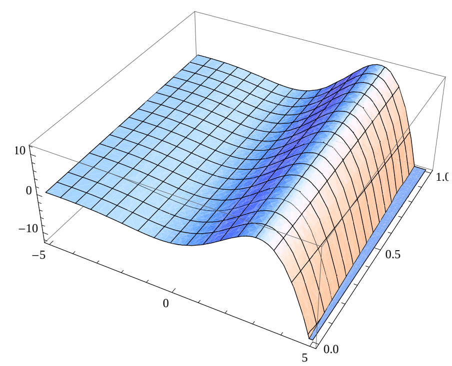

FIGURE 1. $\underline{y}(t,r)$ for example 5.1 (case 1)

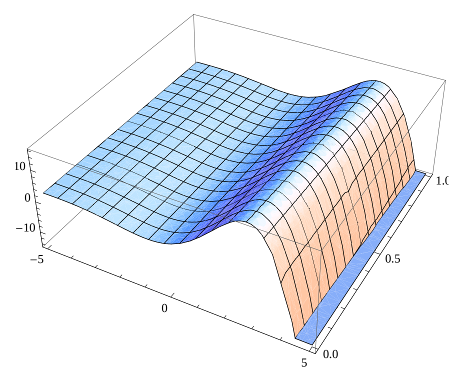

FIGURE 2. $\overline{y}(t,r)$ for example 5.1 (case 1)

Case (ii): If y, y' are (1)-differentiable and $y^{''}, y^{'''}$ are (2)-differentiable. Applying FLT on both sides of (5.1), we get

$$L[y^{(iv)}(t)] + L[y''(t)] = L[y'''(t)]$$

$$\ominus(y'''(0)) \ominus py''(0) \ominus (-p^4)L[y(t)] \ominus p^3y(0) \ominus p^2y'(0) + p^2L[y(t)] \ominus py(0) \ominus y'(0)$$
$$= \ominus(y''(0)) \ominus (-p^3)L[y(t)] \ominus p^2y(0) \ominus py'(0).$$

Now the classical FLT form of the above equation is

$$p^4l[\underline{y}(t,r)] - p^3\underline{y}(0,r) - p^2\underline{y}'(0,r) - p\overline{y}^{''}(0,r) - \overline{y}^{'''}(0,r) + p^2l[\underline{y}(t,r)]$$
$$-p\underline{y}(0,r) - \underline{y}'(0,r) = p^3l[\underline{y}(t,r)] - p^2\underline{y}(0,r) - p\underline{y}'(0,r) - \overline{y}^{''}(0,r).$$

$$p^4l[\overline{y}(t,r)] - p^3\overline{y}(0,r) - p^2\overline{y}'(0,r) - p\underline{y}^{''}(0,r) - \underline{y}^{'''}(0,r) + p^2l[\overline{y}(t,r)]$$
$$-p\overline{y}(0,r) - \overline{y}'(0,r) = p^3l[\overline{y}(t,r)] - p^2\overline{y}(0,r) - \overline{y}'(0,r) - \underline{y}^{''}(0,r).$$

Applying the initial conditions, we have

$$p^4l[\underline{y}(t,r)] - p^3(r-1) - p^2(r+1) - p(4-r) - (5-r) + p^2l[\underline{y}(t,r)] - p(r-1) - (r+1)$$
$$= p^3l[\underline{y}(t,r)] - p^2(r-1) - p(r+1) - (4-r),$$

$$p^4l[\overline{y}(t,r)] - p^3(1-r) - p^2(3-r) - p(2+r) - (3+r) + p^2l[\overline{y}(t,r)] - p(1-r) - (3-r)$$
$$= p^3l[\overline{y}(t,r)] - p^2(1-r) - p(3-r) - (2+r).$$

Taking the inverse Laplace transform of the above equation, then solve for $\underline{y}(t,r)$ and $\overline{y}(t,r)$, we get

$$\underline{y}(t,r) = (r-1) + (r+1)t + (4-r)\left[-t + \frac{2e^{t/2}\text{Sin}\left[\frac{\sqrt{3}t}{2}\right]}{\sqrt{3}}\right]$$
$$+(5-r)\left[1 + t - \frac{1}{3}e^{t/2}\left(3\text{Cos}\left[\frac{\sqrt{3}t}{2}\right] + \sqrt{3}\text{Sin}\left[\frac{\sqrt{3}t}{2}\right]\right)\right],$$

$$\overline{y}(t,r) = (1-r) + (3-r)t + (2+r)\left[-t + \frac{2e^{t/2}\text{Sin}\left[\frac{\sqrt{3}t}{2}\right]}{\sqrt{3}}\right]$$
$$+(3+r)\left[1 + t - \frac{1}{3}e^{t/2}\left(3\text{Cos}\left[\frac{\sqrt{3}t}{2}\right] + \sqrt{3}\text{Sin}\left[\frac{\sqrt{3}t}{2}\right]\right)\right].$$

Example 5.2. Consider the following fourth order FIVP

$$y^{(iv)}(t) = y(t), \text{t} \in [-5, 5], \tag{5.2}$$

subject to the following fuzzy initial conditions:

$y(0) = (r-1, 1-r)$,
$y'(0) = (r-1, 1-r)$,
$y''(0) = (r-1, 1-r)$,
$y'''(0) = (r-1, 1-r)$.

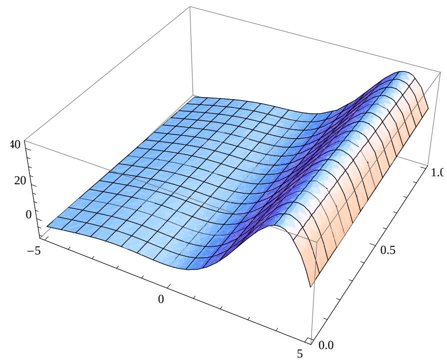

FIGURE 3. $\underline{y}(t,r)$ for example 5.1 (case 2)

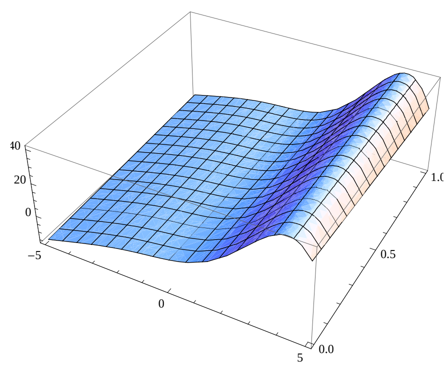

FIGURE 4. $\overline{y}(t,r)$ for example 5.1 (case 2)

Case (i): If $y, y', \cdots, y^{'''}$ are (1)-differentiable. Applying FLT on both sides of (5.2), we get

$$L[y^{(iv)}(t)] = L[y(t)],$$
$$p^4 L[y(t)] \ominus p^3 y(0) \ominus p^2 y'(0) \ominus p y''(0) \ominus y'''(0) = L[y(t)].$$

Now the classical FLT form of the above equation is

$$p^4 l[\underline{y}(t,r)] - p^3 \underline{y}(0,r) - p^2 \underline{y}'(0,r) - p\underline{y}^{''}(0,r) - \underline{y}^{'''}(0,r) = l[\underline{y}(t,r)],$$
$$p^4 l[\overline{y}(t,r)] - p^3 \overline{y}(0,r) - p^2 \overline{y}'(0,r) - p\overline{y}^{''}(0,r) - \overline{y}^{'''}(0,r) = l[\overline{y}(t,r)].$$

Applying the initial conditions, we have

$$p^4 l[\underline{y}(t,r)] - p^3(r-1) - p^2(r-1) - p(r-1) - (r-1) = l[\underline{y}(t,r)],$$
$$p^4 l[\overline{y}(t,r)] - p^3(1-r)p^2(1-r) - p(1-r) - (1-r) = l[\overline{y}(t,r)].$$

Taking the inverse Laplace transform of the above equation, then solve for $\underline{y}(t,r)$ and $\overline{y}(t,r)$, we get

$$\underline{y}(t,r) = (r-1)l^{-1}\left[\frac{p^3+p^2+p+1}{p^4-1}\right],$$

$$\underline{y}(t,r) = (1-r)e^t.$$

And

$$\overline{y}(t,r) = (1-r)l^{-1}\left[\frac{p^3+p^2+p+1}{p^4-1}\right],$$
$$\overline{y}(t,r) = (1-r)e^t.$$

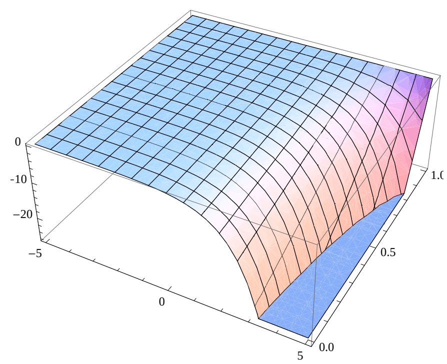

FIGURE 5. $\underline{y}(t,r)$ for example 5.2 (case 1)

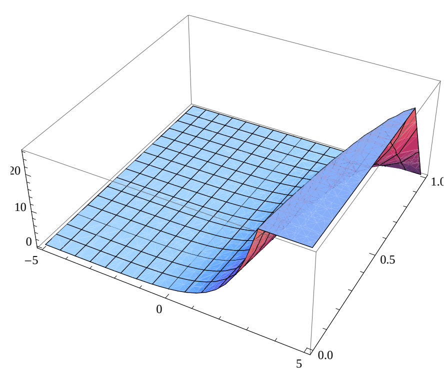

FIGURE 6. $\overline{y}(t,r)$ for example 5.2 (case 1)

Case (ii): If y is (1)-differentiable and y', y'', y''' are (2)-differentiable. Applying FLT on both sides of (5.2), we get

$$L[y^{(iv)}(t)] = L[y(t)], \qquad \text{t} \in [-5,5]$$
$$\ominus(p^3y(0)) \ominus (-p^4)L[y(t)] \ominus p^2y'(0) \ominus py''(0) \ominus y'''(0) = L[y(t)].$$

Now the classical FLT form of the above equation is

$$p^4l[\overline{y}(t,r)] - p^3\overline{y}(0,r) - p^2\underline{y}'(0,r) - p\underline{y}''(0,r) - \underline{y}'''(0,r) = l[\overline{y}(t,r)],$$

$$p^4 l[\underline{y}(t,r)] - p^3\underline{y}(0,r) - p^2\overline{y}'(0,r) - p\overline{y}''(0,r) - \overline{y}'''(0,r) = l[\underline{y}(t,r)].$$

Applying the initial conditions, we have

$$p^4 l[\overline{y}(t,r)] - p^3(1-r) - p^2(r-1) - p(r-1) - (r-1) = l[\overline{y}(t,r)],$$

$$p^4 l[\underline{y}(t,r)] - p^3(r-1) - p^2(1-r) - p(1-r) - (1-r) = l[\underline{y}(t,r)].$$

Taking the inverse Laplace transform of the above equation, then solve for $\underline{y}(t,r)$ and $\overline{y}(t,r)$, we get

$$\underline{y}(t,r) = (r-1)l^{-1}\left[\frac{p^3}{p^4-1}\right] + (1-r)l^{-1}\left[\frac{p^2+p+1}{p^4-1}\right],$$

$$\underline{y}(t,r) = (r-1)\left[\frac{e^{-t}}{4} + \frac{e^t}{4} + \frac{\mathrm{Cos}[t]}{2}\right] + (1-r)\left[-\frac{e^{-t}}{4} + \frac{3e^t}{4} - \frac{\mathrm{Cos}[t]}{2}\right].$$

And

$$\overline{y}(t,r) = (1-r)\left[\frac{e^{-t}}{4} + \frac{e^t}{4} + \frac{\mathrm{Cos}[t]}{2}\right] + (r-1)\left[-\frac{e^{-t}}{4} + \frac{3e^t}{4} - \frac{\mathrm{Cos}[t]}{2}\right].$$

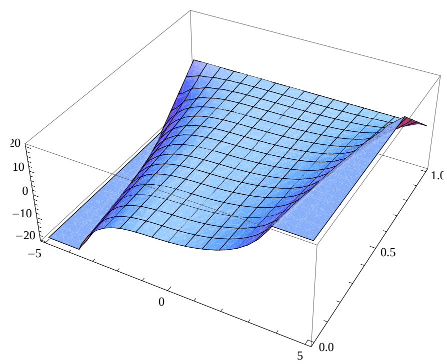

FIGURE 7. $\underline{y}(t,r)$ for example 5.2 (case 2)

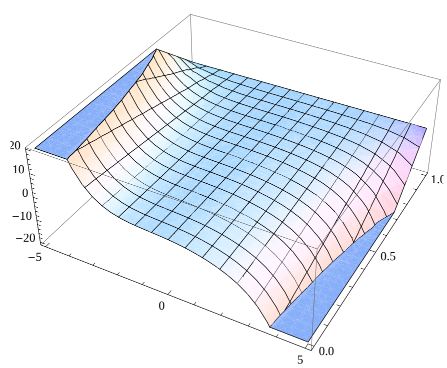

FIGURE 8. $\overline{y}(t,r)$ for example 5.2 (case 2)

6. Conclusion

We proved the FLT for the nth derivative of a fuzzy-valued function with a different approach than [22] and provided method for the solution of an nth order FIVP using the generalized differentiability concept. We also generalized the r-level set of the fuzzy valued functions in the form of (i) and (ii)-differentiable for nth derivative. We have solved a number of different problems using this new approach. However some more research is needed to apply this method for the solution of system of FDEs which is in progress.

References

[1] S. S. L. Chang and L. Zadeh, On fuzzy mapping and control, IEEE Trans on Syst Man and Cybern 2 (1972) 30–34.

[2] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301–317.

[3] S. Seikala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319–330.

[4] T. Allahviranloo, N. Ahmadi and E. Ahmadi, Numerical solution of fuzzy differential equations by predictor-corrector method, Inform. Sci. 177 (2007) 1633–1647.

[5] T. Allahviranloo, N. A. Kiani and M. Barkhordari, Toward the existence and uniqueness of solutions of second-order fuzzy differential equations, Inform. Sci. 179 (2009) 1207–1215.

[6] S. Salahshour and T. Allahviranloo, Applications of fuzzy Laplace Transforms, Soft Computing. 17 (1) (2013) 145–158.

[7] F. Hawrra and K. H. Amal, On fuzzy Laplace transform for fuzzy differential equations of the third order, Journal of Kerbala University 11 (3) (2013) 251–256.

[8] T. Allahviranloo, N. A. Kiani and N. Motamedi, Solving fuzzy differential equatios by differential transformation method, Inform. Sci. 179 (2009) 956–966.

[9] J. Xu, Z. Liao and J. J. Nieto, A class of linear differential dynamical systems with fuzzy matrices, J. Math. Anal. Appl. 368 (2010) 54–68.

[10] H. J. Zimmermann, Fuzzy Set Theory and Its Applications, Revised Edition, Springer 1992.

[11] B. Bede and S. G. Gal, Remark on the new solutions of fuzzy differential equations, Chaos, Solitons & Fractals 2006.

[12] M. L. Puri and D. A. Ralescu, Fuzzy random variables, J. Math. Anal. Appl. 114 (1986) 409–422.

[13] S. Song and C. Wu, Existence and uniqueness of solutions to Cauchy problem of fuzzy differential equations, Fuzzy Sets and Systems 110 (1986) 55–67.

[14] Hsien-Chung Wu, The improper fuzzy Riemann integral and its numerical integration, Inform. Sci. 111 (1998) 109–13, .

[15] W. Fei, Existence and uniqueness of solution for fuzzy random differential equations with non-Lipschitz coefficients, Inform. Sci. 177 (2007) 4329–4337.

[16] Y. Chalco-Cano and H. RomÁn-Flores, On new solutions of fuzzy differential equations, Chaos, Solitons & Fractals 38 (1) (2008) 112–119.

[17] T. Allahviranloo and M. Barkhordari Ahmadi, Fuzzy Laplace Transforms, Soft Computing. 14 (3) (2010) 235–243.

[18] A. Khastan, F. Bahrami and K. Ivaz, New results on multiple solutions for Nth-order fuzzy differential equations under generalized differentiability, Hindawi Publishing Corporation: Boundary-Value Problems 2009 (2009) Article ID 395714, 13 pages.

[19] S. Abbasbandy and T. Allahviranloo, Numerical solutions of fuzzy differential equations by Taylor method, Journal of Computational Methods in Applied Mathematics 2 (2002) 113–124.

[20] M. Ma, M. Friedman and A. Kandel, Numerical Solutions of fuzzy differential equations, Fuzzy Sets and Systems 105 (1999) 133–138.

[21] B. Bede and S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 581–599.

[22] M. Barkhordari Ahmadi, N. A. Kiani and N. Mikaeilvand, Laplace transform formula on fuzzy nth-order derivative and its application in fuzzy ordinary differential equations, Soft Computing. , 1-9, 2014. doi:10.1007/s00500-014-1224-x

Latif Ahmad (ahmad49960@yahoo.com)
1. Shaheed Benazir Bhutto University,
Sheringal, Upper Dir, Khyber Pakhtunkhwa, Pakistan
2. Department of Mathematics, University of Peshawar,
25120, Khyber Pakhtunkhwa, Pakistan

Muhammad Farooq (mfarooq@upesh.edu.pk)
Department of Mathematics, University of Peshawar,
25120, Khyber Pakhtunkhwa, Pakistan

Saleem Abdullah (saleemabdullah81@yahoo.com)
Department of Mathematics, Abdul Wali Khan University,
Mardan, Khyber Pakhtunkhwa, Pakistan