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# Applications of soft intersection sets in Γ-near-rings

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Abstract. In this paper, we define “soft int Γ -near-ring”. The structure of soft int Γ-near-ring is based on the inclusion relation and intersection of sets and since this new concept brings the soft set theory, set theory and Γ-near-ring theory together, it is very functional by means of improving the soft set theory with respect to Γ-near-ring structure. From this view, it functions as a bridge among soft set theory, set theory and Γ-near-ring theory. Based on the soft int Γ-near-ring definition, we introduce the concepts of soft int subΓ-near-ring and soft int Γ-ideal. Moreover, investigate these notions with respect to soft image, soft pre-image and $\alpha$-inclusion of soft sets. Finally, we give some applications of soft int Γ-near-ring to Γ-near-ring theory.

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## 1. Introduction

The notion of near ring was first introduced by Dickson and Leonard in 1905 [11]. They showed that there do exist ”fields with one distributive law” (near fields). It was Zassenhaus who was able to determine all finite near rings. Now a days, near fields are mighty tools in characterizing doubly transitive groups, incidence groups and Frobenius groups. We note that the ideals of near rings play a central role in the structure theory, however, they do not in general coincide with the usual ring ideals of a ring. In 1984, Satyanarayana introduced Γ-nearring in his doctoral thesis and obtained some basic results [29]. For further see [7, 6].

To solve complicated problems in economics, engineering, environmental science and social science. Methods in classical mathematics are not always successful because of various types of uncertainties presented in these problems. While probability theory, fuzzy set theory [34], rough set theory [26, 27], and other mathematical tools are well known and often useful approaches to describing uncertainty, each of these theories has its inherent difficulties as pointed out in [24, 25]. In 1999, Molodtsov [24] introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainties. This so-called soft set theory is free from the difficulties affecting existing methods. Presently, works on soft set theory are progressing rapidly. Maji et al. [21] defined several operations on soft sets and made a theoretical study on the theory of soft sets. Since its inception, it has received much attention in the mean of algebraic structures such as groups [3], semirings [12], rings [1], BCK/BCI-algebras [17, 18, 19], normalistic soft groups [30], BL-algebras [35], BCH-algebras [20] and nearrings [31]. Atagün and Sezgin [4] defined the concepts of soft subrings and ideals of a ring, soft subfields of a field and soft submodules of a module and studied their related properties with respect to soft set operations. Also union soft substructures of nearrings and nearring modules are studied in [32]. M. Aslam et al. defined soft LA-semigroups and Soft ideals over LA-semigroups []. Çağman et al. defined two new types of group actions on a soft set, called group $SI$-action and group $SU$-action [8], which are based on the inclusion relation and the intersection of sets and union of sets, respectively.

Ali et al. [2] introduced several operations of soft sets. Sezgin and Atagün [33] studied on soft set operations as well. Soft set relations and functions [5] and soft mappings [23] were proposed and many related concepts were discussed too. Moreover, the theory of soft sets has gone through remarkably rapid strides with a wide-ranging applications especially in soft decision making as in the following studies: [9, 10, 22] and some other fields such as [13, 14, 15, 16].

Çağman and Enginoğlu [10] redefined the operations of soft sets to develop the soft set theory. By using their definitions, in this paper, we define "*soft int* $\Gamma$-*nearring*". The structure of soft int $\Gamma$-nearring is based on the inclusion relation and intersection of sets and since this new concept brings the soft set theory, set theory and $\Gamma$-nearring theory together, it is very functional by means of improving the soft set theory with respect to $\Gamma$-nearring structure. From this view, it functions as a bridge among soft set theory, set theory and $\Gamma$-nearring theory. Based on the soft int $\Gamma$-nearring definition, we introduce the concepts of soft int sub$\Gamma$-nearring and soft int $\Gamma$-ideal. Moreover, we investigate these notions with respect to soft image, soft pre-image and $\alpha$-inclusion of soft sets. Finally, we give some applications of soft int $\Gamma$-nearring to $\Gamma$-nearring theory.

## 2. Preliminaries

We first recall some basic concepts for the sake of completeness from [28]. A nearring $N$ is an algebraic system $(N, +, \cdot)$ consisting of a non-empty set $N$ together with two binary operations addition "+" and multiplication "$\cdot$" such that (1) $(N, +)$ is a group and (2) $(N, \cdot)$ is a semigroup and (3) $(x+y)\cdot z = x\cdot z+y\cdot z$ for all $x, y, z \in N$. We will use word "nearring" to mean "right distributive nearring".

A $\Gamma$-nearring [29] is a triple $(N, +, \Gamma)$, where

(i) $(N, +)$ is a group,

(ii) $\Gamma$ is a non-empty set of binary operators on $N$ such that for each $\alpha \in \Gamma$, $(N, +, \alpha)$ is a nearring,

(iii) $x\alpha(y\alpha z) = (x\alpha y)\beta z$ for all $x, y, z \in N$ and $\alpha, \beta \in \Gamma$.

A subset $S$ of a $\Gamma$-nearring $N$ is called sub$\Gamma$-nearring $N$, if for all $x, y \in S$ and $\alpha \in \Gamma \Rightarrow x + y \in S$ and $x\alpha y \in S$ or equivalently $S + S \subseteq S$ and $S\Gamma S \subseteq S$.

A subset $A$ of a $\Gamma$-nearring $N$ is called a left (resp. right) $\Gamma$-ideal of $N$, if

(i) $(A, +)$ is a normal subgroup of $(N, +)$,

(ii) $x\alpha(y + z)- x\alpha z \in A$ (resp. $x\alpha y \in A$) for all $x \in A$, $\alpha \in \Gamma$ and $y, z \in N$.

We will use word "$\Gamma$-nearring" to mean "right distributive $\Gamma$-nearring".

We now review some soft set concepts. From now on, $X$ refers to an initial universe, $E$ is a set of parameters, $P(X)$ is the power set of $X$ and $A, B, C \subset E$. A soft set $\lambda_A$ over $X$ is a set defined by

$$\lambda_A : E \longrightarrow P(X) \text{ such that } \lambda_A (x) = \varnothing \text{ if } x \notin A.$$

Here $\lambda_A$ is also called an approximate function. A soft set over $X$ can be represented by the set of ordered pairs

$$\lambda_A = \{(x, \lambda_A (x)) : x \in E, \lambda_A (x) \in P(X)\}.$$

It is clear to see that a soft set is a parametrized family of subsets of the set $X$. Note that the set of all soft sets over $X$ will be denoted by $S(X)$.

**Definition 2.1** ([25]). Let $\lambda_A$ and $\lambda_B$ be two soft sets over a common universe $X$. Then, we say that $\lambda_A$ is a soft subset of $\lambda_B$, if

(i) $A \subseteq B$ and

(ii) $\lambda_A (e) \subseteq \lambda_B (e)$ for all $e \in A$.

We write $\lambda_A \mathrel{\widetilde{\subset}} \lambda_B$. A soft set $\lambda_B$ is said to be a soft super set of $\lambda_A$, if $\lambda_A$ is a soft subset of $\lambda_B$. We denote it by $\lambda_B \mathrel{\widetilde{\supset}} \lambda_A$.

**Definition 2.2** ([25]). Let $\lambda_A$,$\lambda_B \in S(X)$. Then, union of $\lambda_A$ and $\lambda_B$, denoted by $\lambda_A \widetilde{\cup} \lambda_B$, is defined as $\lambda_A \widetilde{\cup} \lambda_B = \lambda_{A \widetilde{\cup} B}$, where $\lambda_{A \widetilde{\cup} B} (x) = \lambda_A (x) \cup \lambda_B (x)$ for all $x \in E$.

**Definition 2.3** ([25]). Let $\lambda_A$,$\lambda_B \in S(X)$. Then, intersection of $\lambda_A$ and $\lambda_B$, denoted by $\lambda_A \widetilde{\cap} \lambda_B$, is defined as $\lambda_A \widetilde{\cap} \lambda_B = \lambda_{A \widetilde{\cap} B}$, where $\lambda_{A \widetilde{\cap} B} (x) = \lambda_A (x) \cap \lambda_B (x)$ for all $x \in E$.

**Definition 2.4.** [25]] Let $\lambda_A$,$\lambda_B \in S(X)$. Then, $\wedge$-product of $\lambda_A$ and $\lambda_B$, denoted by $\lambda_A \wedge \lambda_B$, is defined as $\lambda_A \wedge \lambda_B = \lambda_{A \wedge B}$, where $\lambda_{A \wedge B} (x, y) = \lambda_A (x) \cap \lambda_B (y)$ for all $(x, y) \in E \times E$.

**Definition 2.5** ([25]). A soft set $\lambda_A$ over $X$ is said to be a NULL soft set denoted by $\Phi$ if for all $e \in A$, $\lambda_A(e) = \varnothing$(empty set).

**Definition 2.6** ([25]). A soft set $\lambda_A$ over $X$ is said to be an absolute soft set denoted by $\widetilde{\lambda}_A$ if for all $e \in A$, $\lambda_A(e) = X$.

**Definition 2.7** ([8]). Let $\lambda_A$,$\lambda_B \in S(X)$. and $f$ be a function from $A$ to $B$. Then,

(i) The soft image of $\lambda_A$ under $f$ , denoted by $f(\lambda_A)$, is a soft set over $X$ by

$$(f(\lambda_A))(x) = \begin{cases} \cup \{\lambda_A (y) : y \in N,\ f (y) = x\}, & f^{-1} (y) \neq \varnothing \\ \varnothing & \text{otherwise,} \end{cases}$$

for all $y \in B$.

(ii) The soft pre-image (or soft inverse image) of $\lambda_B$ under $f$, denoted by $f^{-1} (\lambda_B)$, is a soft set over $X$ by $\left(f^{-1} (\lambda_B)\right) = \lambda_B (f (a))$ for all $x \in A$.

## 3. Soft Int Γ-near ring

In this section, we first define soft intersection Γ -near ring which is abbreviated as soft int Γ-near ring. We then define soft int subΓ-nearring, soft int-Γ-ideal of a Γ-nearring with some examples and study their fundamental properties with respect to soft set operations.

In what follows, $N$ will denote right distributive Γ-nearring, unless otherwise specified.

**Definition 3.1.** Let $N$ be a Γ-near ring and $\lambda_N$ be a soft set over $X$. Then, $\lambda_N$ is said to be soft int Γ-near ring over $X$ if it satisfies the following conditions hold:
(i) $\lambda_N(x+y) \supseteq \lambda_N(x) \cap \lambda_N(y)$,
(ii) $\lambda_N(-x) = \lambda_N(x)$,
(iii) $\lambda_N(x\alpha y) \supseteq \lambda_N(x) \cap \lambda_N(y)$, for all $x, y \in M$ and $\alpha \in \Gamma$.

**Example 3.2.** Let $N = \{0, a, b, c\}$ and $\Gamma = \{\alpha,\beta\}$ be non-empty sets. The binary operations defined as;

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| α | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | b |
| b | 0 | 0 | 0 | 0 |
| c | 0 | b | 0 | b |

| β | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | 0 |
| b | 0 | 0 | b | 0 |
| c | 0 | 0 | 0 | b |

Clearly $(N, +, \Gamma)$ is a Γ-nearring. Assume that $N$ is the set of parameters and $X = \left\{ \begin{bmatrix} x & x \\ y & y \end{bmatrix} |x, y \in \mathbb{Z}_4 \right\}$, $2 \times 2$ matrices with $\mathbb{Z}_4$ terms, is the universal set. We construct a soft set $\lambda_N$ over $X$ by

$$\begin{aligned}
\lambda_N(0) &= \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \right\}, \\
\lambda_N(a) &= \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \right\}, \\
\lambda_N(b) &= \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}, \right\}, \\
\lambda_N(c) &= \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \right\}.
\end{aligned}$$

Then, one can easily show that the soft set $\lambda_N$ is a soft int Γ-nearring over $X$.

**Example 3.3.** In Example 3.2, assume that $N = \{0, a, b, c\}$ is again the set of parameters and $X = S_3$, symmetric group, is the universal set. we defined a soft set $\lambda_N$ by

$$\begin{aligned}
\lambda_N(0) &= \{(12), (13)\}, \\
\lambda_N(a) &= \{(12), (13), (123)\}, \\
\lambda_N(b) &= \{(12), (23)(123)\}, \\
\lambda_N(c) &= \{(12), (13)(123)\}.
\end{aligned}$$

Then $\lambda_N$ is a not a soft int Γ-nearring, because

$$\lambda_N (a.a) = \lambda_N (0) = \{(12), (13)\} \not\supseteq \{(12), (13), (123)\}.$$

**Example 3.4.** Let $N = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$ be non-empty sets. The binary operations defined as;

| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

| $\alpha$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $b$ | $c$ |
| $b$ | $0$ | $0$ | $0$ | $0$ |
| $c$ | $0$ | $a$ | $b$ | $c$ |

| $\beta$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $0$ | $0$ | $0$ | $b$ |
| $c$ | $a$ | $a$ | $a$ | $c$ |

Then, $(N, +, \Gamma)$ is a Γ-nearring. Assume that $N$ is the set of parameters and

$$X = D_2 = \left\{\langle x,\ y\rangle : x^2 = y^2 = (xy)^2 = e, xy = yx\right\} = \{e, x, y, xy\},$$

dihedral group, is the universal set. We define a soft set $\lambda_N$ over $X$ by

$$\lambda_N (0) = D_2,\ \lambda_N (a) = \{e, x, xy\},\ \lambda_N (b) = \lambda_N (c) = \{e, x\}.$$

By routine calculation, $\lambda_N$ is a soft int Γ-near ring over $X$.

**Remark 3.5.** If $\lambda_N$ is a soft int Γ-near ring over $X$, then $\lambda_N (0) \supseteq \lambda_N (x)$, for all $x \in N$.

**Theorem 3.6.** *Let $N$ be a Γ-near ring and $\lambda_N$ a soft set over $X$. Then, $\lambda_N$ is a soft int Γ-near ring if and only if $\lambda_N (x - y) \supseteq \lambda_N (x) \cap \lambda_N (y)$ and $\lambda_N (x\alpha y) \supseteq \lambda_N (x) \cap \lambda_N (y)$, for all $x, y \in N$ and $\alpha \in \Gamma$.*

*Proof.* Assume that $\lambda_N$ is a soft int Γ-near ring over $X$. Then, by Definition of a soft int Γ-near ring, we have

$$\begin{aligned} \lambda_N (x - y) &\supseteq \lambda_N (x) \cap \lambda_N (-y) = \lambda_N (x) \cap \lambda_N (y) \\ \lambda_N (x - y) &\supseteq \lambda_N (x) \cap \lambda_N (y) \text{ and} \\ \lambda_N (x\alpha y) &\supseteq \lambda_N (x) \cap \lambda_N (y), \text{for all } x, y \in M \text{ and } \alpha \in \Gamma. \end{aligned}$$

Conversely, assume that $\lambda_N (x - y) \supseteq \lambda_N (x) \cap \lambda_N (y)$ and $\lambda_N (x\alpha y) \supseteq \lambda_N (x) \cap \lambda_N (y)$, for all $x, y \in M$ and $\alpha \in \Gamma$. If we choose $x = 0$, then

$$\lambda_N (0 - y) = \lambda_N (-y) \supseteq \lambda_N (0) \cap \lambda_N (y) = \lambda_N (y).$$

Now, $\lambda_N (y) = \lambda_N (-(-y)) \supseteq \lambda_N (-y)$. Thus, $\lambda_N (y) = \lambda_N (-y)$ for all $y \in M$. Also, by assumption, we have

$$\begin{aligned} \lambda_N (x + y) &= \lambda_N (x - (-y)) \supseteq \lambda_N (x) \cap \lambda_N (-y) = \lambda_N (x) \cap \lambda_N (y) \\ \lambda_N (x + y) &\supseteq \lambda_N (x + y). \end{aligned}$$

Thus, $\lambda_N$ is a soft int Γ-near ring over $X$. □

**Remark 3.7.** Let $\lambda_N$ be a soft int Γ-near ring over $X$.

(1) If $\lambda_N (x - y) = 0$ for any $x, y \in N$, then $\lambda_N (x) = \lambda_N (y)$.

(2) If $\lambda_N (x - y) = \lambda_N (0)$ for any $x, y \in N$, then $\lambda_N (x) = \lambda_N (y)$.

It is known that if $(N, +, \Gamma)$ is a $\Gamma$-nearring, then $(N, +)$ is a group but not necessarily abelian. That is, for any $x, y \in N$, $x + y$ needs not be equal to $y + x$. However, we have the following:

**Theorem 3.8.** *Let $\lambda_N$ be a soft int $\Gamma$-nearring over $X$ and $x \in N$. Then*

$$\lambda_N(x) = \lambda_N(0) \Leftrightarrow \lambda_N(x+y) = \lambda_N(y+x)$$

*for all $y \in N$.*

*Proof.* Straightforward. □

**Theorem 3.9.** *Let $N$ be a $\Gamma$-near-field and $\lambda_N$ a soft set over $X$. If $\lambda_N(0) \supseteq \lambda_N(1_N) = \lambda_N(x)$ for all $0 \neq x \in N$, then $\lambda_N$ is a soft int $\Gamma$-nearring over $X$ .*

*Proof.* Suppose that $\lambda_N(0) \supseteq \lambda_N(1_N) = \lambda_N(x)$ for all $0 \neq x \in N$, In order to prove that $\lambda_N$ is soft int $\Gamma$-nearring over $X$, it is enough to prove that

$$\lambda_N(x-y) \supseteq \lambda_N(x) \cap \lambda_N(y) \text{ and } \lambda_N(x\alpha y) \supseteq \lambda_N(x) \cap \lambda_N(y).$$

Let $x, y \in N$ and $\alpha \in \Gamma$. Then, we have the following cases:

Case (i): Suppose that $x \neq 0$ and $y = 0$ or $x = 0$ and $y \neq 0$. Since $N$ is a $\Gamma$-near-field, it follows that $x\alpha y = 0$ and $\lambda_N(x\alpha y) = \lambda_N(0)$. Since $\lambda_N(0) \supseteq \lambda_N(x)$, for all $x \in N$, $\lambda_N(x\alpha y) = \lambda_N(0) \supseteq \lambda_N(x)$ and $\lambda_N(x\alpha y) = \lambda_N(0) \supseteq \lambda_N(y)$. These imply that $\lambda_N(x\alpha y) \supseteq \lambda_N(x) \cap \lambda_N(y)$.

Case (ii): Suppose that $x \neq 0$ and $y \neq 0$. It follows that $x\alpha y \neq 0$. Then,

$$\lambda_N(x\alpha y) = \lambda_N(1_N) = \lambda_N(x) \text{ and } \lambda_N(x\alpha y) = \lambda_N(1_N) = \lambda_N(y).$$

These imply that $\lambda_N(x\alpha y) \supseteq \lambda_N(x) \cap \lambda_N(y)$.

Case (iii): Suppose that $x = 0$ and $y = 0$. Then clearly $\lambda_N(x\alpha y) \supseteq \lambda_N(x) \cap \lambda_N(y)$. Thus, $\lambda_N(x\alpha y) \supseteq \lambda_N(x) \cap \lambda_N(y)$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Now, let $x, y \in N$ . Then,$x - y = 0$ or $x - y \neq 0$.

If $x - y = 0$ , then either $x = y = 0$ or $x \neq 0$, $y \neq 0$ and $x = y$. But, since $\lambda_N(x-y) = \lambda_N(0) \supseteq \lambda_N(x)$, for all $x \in M$, it follows that

$$\lambda_N(x-y) = \lambda_N(0) \supseteq \lambda_N(x) \cap \lambda_N(y).$$

If $x - y \neq 0$, then either $x \neq 0$ $y \neq 0$ and $x \neq y$ or $x \neq 0$ and $y = 0$ or $x = 0$ and $y \neq 0$. Assume that $x \neq 0$ $y \neq 0$ and $x \neq y$. Then

$$\lambda_N(x-y) = \lambda_N(1_N) = \lambda_N(x) \supseteq \lambda_N(x) \cap \lambda_N(y).$$

Now, let $x \neq 0$ and $y = 0$. Then, $\lambda_N(x-y) \supseteq \lambda_N(x) \cap \lambda_N(y)$. Finally, let $x = 0$ and $y \neq 0$. Then, $\lambda_N(x-y) \supseteq \lambda_N(x) \cap \lambda_N(y)$. Thus, $\lambda_N(x-y) \supseteq \lambda_N(x) \cap \lambda_N(y)$, for all $x, y \in N$. So, $\lambda_N$ is a soft int $\Gamma$-nearring over $X$. □

**Theorem 3.10.** *If $\lambda_N$ and $\lambda_M$ are soft int $\Gamma$-nearrings over $X$, then $\lambda_N \wedge \lambda_M$ is also soft int $\Gamma$-nearring over $X$.*

*Proof.* By Definition , let $\lambda_N \wedge\lambda_M = \lambda_{N\wedge M}$, where $\lambda_{N\wedge M}(x,y) = \lambda_N(x) \cap\lambda_M(y)$ for all $(x,y) \in N \times M$. Since $N$ and $M$ are Γ-nearrings, so is $N \times M$. Now, let $(x_1,y_1)$, $(x_2,y_2) \in N \times M$. Then

$$\begin{aligned}
\lambda_{N\wedge M}((x_1,y_2)-(x_2,y_2)) &= \lambda_{N\wedge M}(x_1-x_2,y_1-y_2)\\
&= \lambda_N(x_1-x_2)\cap\lambda_M(y_1-y_2)\\
&\supseteq (\lambda_N(x_1)\cap\lambda_N(x_2))\cap\lambda_M(y_1)\cap\lambda_M(y_2)\\
&= (\lambda_N(x_1)\cap\lambda_M(y_1))\cap(\lambda_N(x_2)\cap\lambda_M(y_2))\\
&= \lambda_{N\wedge M}(x_1,y_1)\cap\lambda_{N\wedge M}(x_2,y_2)\\
\lambda_{N\wedge M}((x_1,y_2)-(x_2,y_2)) &\supseteq \lambda_{N\wedge M}(x_1,y_1)\cap\lambda_{N\wedge M}(x_2,y_2).
\end{aligned}$$

Now, let $(x_1,y_1)$, $(x_2,y_2) \in N \times M$ and $(\alpha_1,\alpha_2) \in \Gamma_1 \times \Gamma_2$. Then

$$\begin{aligned}
\lambda_{N\wedge M}((x_1,y_2)(\alpha_1,\alpha_2)(x_2,y_2)) &= \lambda_{N\wedge M}(x_1\alpha_1x_2,y_1\alpha_2y_2)\\
&= \lambda_N(x_1\alpha_1x_2)\cap\lambda_M(y_1\alpha_2y_2)\\
&\supseteq (\lambda_N(x_1)\cap\lambda_N(x_2))\cap\lambda_M(y_1)\cap\lambda_M(y_2)\\
&= (\lambda_N(x_1)\cap\lambda_M(y_1))\cap(\lambda_N(x_2)\cap\lambda_M(y_2))\\
&= \lambda_{N\wedge M}(x_1,y_1)\cap\lambda_{N\wedge M}(x_2,y_2)\\
\lambda_{N\wedge M}((x_1,y_2)-(x_2,y_2)) &\supseteq \lambda_{N\wedge M}(x_1,y_1)\cap\lambda_{N\wedge M}(x_2,y_2).
\end{aligned}$$

Thus $\lambda_N \wedge\lambda_M$ is a soft int Γ-nearring over $X$. □

**Definition 3.11.** Let $\lambda_N$ and $\mu_M$ be soft int Γ-nearrings over $X_1$ and $X_2$, respectively. Then, the direct product of soft int Γ-nearrings $\lambda_N$ and $\mu_M$ over $X_1$ and $X_2$, respectively, is denoted by $\lambda_N \times \mu_M = h_{N\times M}$ and defined as:

$$h_{N\times M}(x,y) = \lambda_N(x) \times \mu_M(y) \text{ for all } (x,y) \in N \times M.$$

**Theorem 3.12.** *If $\lambda_N$ and $\mu_M$ are soft int Γ-nearrings over $X_1$ and $X_2$, then $h_{N\times M} = \lambda_N \times\lambda_M$ is also soft int Γ-nearring over $X_1 \times X_2$.*

*Proof.* Let $(x_1,y_1)$, $(x_2,y_2) \in N \times M$. Then

$$\begin{aligned}
h_{N\times M}((x_1,y_2)-(x_2,y_2)) &= h_{N\times M}(x_1-x_2,y_1-y_2)\\
&= \lambda_N(x_1-x_2)\times\mu_M(y_1-y_2)\\
&\supseteq (\lambda_N(x_1)\cap\lambda_N(x_2))\times\mu_M(y_1)\cap\mu_M(y_2)\\
&= (\lambda_N(x_1)\times\mu_M(y_1))\cap(\lambda_N(x_2)\times\cap\mu_M(y_2))\\
&= h_{N\times M}(x_1,y_1)\cap h_{N\times M}(x_2,y_2)\\
h_{N\times M}((x_1,y_2)-(x_2,y_2)) &\supseteq h_{N\times M}(x_1,y_1)\cap h_{N\times M}(x_2,y_2).
\end{aligned}$$

Now, let $(x_1,y_1)$, $(x_2,y_2) \in N \times M$ and $(\alpha_1,\alpha_2) \in \Gamma_1 \times \Gamma_2$. Then,

$$\begin{aligned}
h_{N\times M}((x_1,y_2)(\alpha_1,\alpha_2)(x_2,y_2)) &= h_{N\times M}(x_1\alpha_1x_2,y_1\alpha_2y_2)\\
&= \lambda_N(x_1\alpha_1x_2)\cap\mu_M(y_1\alpha_2y_2)\\
&\supseteq (\lambda_N(x_1)\cap\lambda_N(x_2))\times(\mu_M(y_1)\cap\mu_M(y_2))\\
&= (\lambda_N(x_1)\times\mu_M(y_1))\cap(\lambda_N(x_2)\times\mu_M(y_2))\\
&= h_{N\times M}(x_1,y_1)\cap h_{N\times M}(x_2,y_2)\\
h_{N\times M}((x_1,y_2)-(x_2,y_2)) &\supseteq h_{N\times M}(x_1,y_1)\cap h_{N\times M}(x_2,y_2).
\end{aligned}$$

Thus, $\lambda_N \times \lambda_M$ is a soft int $\Gamma$-nearring over $X_1 \times X_2$. □

**Theorem 3.13.** *If $\lambda_M$ and $\mu_M$ are soft int $\Gamma$-nearrings over $X$, then $\lambda_M \widetilde{\cap} \mu_M$ is also soft int $\Gamma$-nearring over $X$.*

*Proof.* Now, let $x, y \in N$. Then,

$$\begin{aligned}
\lambda_N \widetilde{\cap} \lambda_M (x-y) &= \lambda_N (x-y) \cap \mu_M (x-y) \\
&\supseteq (\lambda_N (x) \cap \lambda_N (y)) \cap (\mu_M (x) \cap \mu_M (y)) \\
&= (\lambda_N (x) \cap \mu_M (x)) \cap (\lambda_N (y) \cap \mu_M (y)) \\
&= \lambda_N \widetilde{\cap} \lambda_M (x) \cap \lambda_N \widetilde{\cap} \lambda_M (y) \\
\lambda_N \widetilde{\cap} \lambda_M (x-y) &\supseteq \lambda_N \widetilde{\cap} \lambda_M (x_1, y_1) \cap \lambda_N \widetilde{\cap} \lambda_M (x_2, y_2).
\end{aligned}$$

Now, let $x$, $y \in N$ and $\alpha \in \Gamma$. Then,

$$\begin{aligned}
\lambda_N \widetilde{\cap} \lambda_M (x\alpha y) &= \lambda_N (x\alpha y) \cap \mu_M (x\alpha y) \\
&\supseteq (\lambda_N (x) \cap \lambda_N (y)) \cap (\mu_M (x) \cap \mu_M (y)) \\
&= (\lambda_N (x) \cap \mu_M (x)) \cap (\lambda_N (y) \cap \mu_M (y)) \\
&= \lambda_N \widetilde{\cap} \lambda_M (x) \cap \lambda_N \widetilde{\cap} \lambda_M (y) \\
\lambda_N \widetilde{\cap} \lambda_M (x\alpha y) &\supseteq \lambda_N \widetilde{\cap} \lambda_M (x_1, y_1) \cap \lambda_N \widetilde{\cap} \lambda_M (x_2, y_2).
\end{aligned}$$

Thus, $\lambda_N \widetilde{\cap} \lambda_M$ is a soft int $\Gamma$-nearring over $X$. □

**Definition 3.14.** Let $S$ be a sub $\Gamma$-nearring of $\Gamma$-nearring $N$, $\lambda_N$ a soft int $\Gamma$-nearring over $X$ and let $\lambda_S$ be a soft subset of $\lambda_N$ over $X$. Then, $\lambda_S$ is called soft int sub $\Gamma$-nearring of $\lambda_N$ over $X$, if $\lambda_S$ is itself a soft int $\Gamma$-nearring over $X$ and denoted by $\lambda_S \widetilde{\leq}_i \lambda_N$ .

**Example 3.15.** In Example 3.2, assume that $N = \{0, a, b, c\}$ is again the set of parameters and $X = D_3 = \left\{\langle x,\ y\rangle : x^3 = y^2 = (xy)^2 = e, xy = yx^2\right\} = \left\{e, x, x^2, y, yx, yx^2\right\}$, dihedral group, the universal set. We define a soft set $\lambda_N$ over $X$ by

$$\begin{aligned}
\lambda_N (0) &= D_3, \\
\lambda_N (a) &= \left\{e, x, x^2, y, yx\right\}, \\
\lambda_N (b) &= \left\{e, x, x^2, y\right\}, \\
\lambda_N (c) &= \left\{e, x, x^2, y\right\}.
\end{aligned}$$

Then, $\lambda_N$ is a soft int $\Gamma$-nearring over $X$. Now, let $S = \{0, a\}$ be a sub$\Gamma$-nearring of $N$, the set of parameters and we defined a soft subset $\lambda_S$ of $\lambda_N$ over $X$ by

$$\begin{aligned}
\lambda_N (0) &= \left\{e, x, x^2, y\right\}, \\
\lambda_N (a) &= \left\{e, x, x^2\right\}.
\end{aligned}$$

It is clear that $\lambda_N$ is a soft int sub$\Gamma$-nearring of soft int $\Gamma$-nearring $\lambda_N$ over $X$.

**Theorem 3.16.** *If $\lambda_N$ is a soft int $\Gamma$-nearring over $X$, $\lambda_M \widetilde{\leq}_i \lambda_N$ and $\lambda_K \widetilde{\leq}_i \lambda_N$ over $X$, then $\lambda_M \widetilde{\cap} \lambda_K \widetilde{\leq}_i \lambda_N$ over $X$.*

*Proof.* Straightforward. □

**Definition 3.17.** Let $N$ be a $\Gamma$-nearring and $\lambda_N$ a soft int $\Gamma$-nearring over $X$. Then, $\lambda_N$ is said to be a soft int-$\Gamma$-ideal of $N$ over $X$, if the following conditions hold:
(i) $\lambda_N(x+y-x) \supseteq \lambda_N(x) \cap \lambda_N(y)$,
(ii) $\lambda_N(x\alpha y) \supseteq \lambda_N(x)$,
(iii) $\lambda_N(x\alpha(y+z) - x\alpha y) \supseteq \lambda_N(z)$,
for all $x, y, z \in N$ and $\alpha \in \Gamma$. If $\lambda_N$ is a soft int $\Gamma$-nearring over $X$ and the conditions (i) and (ii) hold, then $\lambda_N$ is called a soft int right $\Gamma$-ideal of $N$ over $X$ and if conditions (i) and (iii) hold, then $\lambda_N$ is called a soft int left $\Gamma$-ideal of $N$ over $X$.

**Example 3.18.** Let $N = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$ be non-empty sets. The binary operations defined as;

| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

| $\alpha$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $a$ | $a$ |
| $b$ | $0$ | $a$ | $b$ | $c$ |
| $c$ | $0$ | $0$ | $c$ | $b$ |

| $\beta$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $0$ | $a$ |
| $b$ | $0$ | $b$ | $0$ | $c$ |
| $c$ | $0$ | $0$ | $0$ | $b$ |

Clearly, $(N, +, \Gamma)$ is a $\Gamma$-nearring. Assume that $N$ is the set of parameters and $X = D_3 = \left\{\langle x,\ y\rangle : x^3 = y^2 = (xy)^2 = e, xy = yx^2\right\} = \{e, x, x^2, y, yx, yx^2\}$, dihedral group, the universal set. We define a soft set $\lambda_N$ over $X$ by

$$\begin{aligned} \lambda_N(0) &= \lambda_N(b) = D_3, \\ \lambda_N(c) &= \lambda_N(a) = \{e, x, \}. \end{aligned}$$

Then, clearly $\lambda_N$ is a soft int left $\Gamma$-ideal and right $\Gamma$-ideal of $N$ over $X$. Therefore, $\lambda_N$ is a soft int $\Gamma$-ideal of $N$ overs $X$.

**Theorem 3.19.** *Let $N$ be a $\Gamma$-near-field and $\lambda_N$ a soft int $\Gamma$-ideal of $N$ overs $X$. Then, $\lambda_N(0) \supseteq \lambda_N(1_N) = \lambda_N(x)$ for all $0 \neq x \in N$.*

*Proof.* Suppose that $\lambda_N$ is a soft int $\Gamma$-ideal of $N$ overs $X$, then $\lambda_N$ is a soft int $\Gamma$-nearring of $N$ over $X$. Since $\lambda_N(0) \supseteq \lambda_N(x)$, in particular $\lambda_N(0) \supseteq \lambda_N(1_N)$. Now, let $0 \neq x \in N$. Then

$$\lambda_N(x) = \lambda_N(1_N \cdot x) \supseteq \lambda_N(1_N) = \lambda_N\left(x \cdot x^{-1}\right) \supseteq \lambda_N(x).$$

Thus $\lambda_N(x) = \lambda_N(1_N)$ for all $0 \neq x \in N$. □

For a nearring $N$, the zero-symmetric part of $N$ denoted by $N_0$ is defined by $N_0 = \{n \in N \mid n0 = 0\}$. It is known that if $N$ is a zero-symmetric nearring and $I \lhd_l N$, then $NI \subseteq N$ . Here, we have an analog for this case:

**Theorem 3.20.** *Let $N = N_0$ and $\lambda_N$ be a soft of $N$ over $X$. Then $\lambda_N(x\alpha(y+z) - x\alpha y) \supseteq \lambda_N(z)$ implies that $\lambda_N(xz) \supseteq \lambda_N(z)$ for all $x, y, z \in N$.*

*Proof.* Let $N$ be a zero-symmetric nearring, let $\lambda_N$ be a soft soft over $X$ and let $\lambda_N(x\alpha(y+z) - x\alpha y) \supseteq \lambda_N(z)$. Let $y = 0$ in $\lambda_N(x\alpha(y+z) - x\alpha y) \supseteq \lambda_N(z)$. Then $\lambda_N(x\alpha(0+z) - x\alpha 0) \supseteq \lambda_N(z)$. Thus $\lambda_N(xz) \supseteq \lambda_N(z)$. □

**Theorem 3.21.** *If $\lambda_N$ and $\mu_M$ are soft int $\Gamma$-ideals over $X$, then $\lambda_N \wedge \mu_M$ is also soft int $\Gamma$-ideal over $X$.*

*Proof.* Let $\lambda_N$ and $\mu_M$ are soft int $\Gamma$-ideals over $X$. Then, by Theorem 3.10, $\lambda_N \wedge \mu_M$ soft int $\Gamma$-nearrings over $X$. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in N \times M$ and $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$. Then

$$\begin{aligned}&(\lambda_N \wedge \mu_M)((x_1, y_1) + (x_2, y_2) - (x_1, y_1))\\ &= (\lambda_N \wedge \mu_M)(x_1 + x_2 - x_1, y_1 + y_2 - y_1)\\ &= \lambda_N(x_1 + x_2 - x_1) \cap \mu_M(y_1 + y_2 - y_1)\\ &\supseteq \lambda_N(x_2) \cap \mu_M(y_2)\\ &= (\lambda_N \wedge \mu_M)(x_2, y_2),\end{aligned}$$

$$\begin{aligned}&(\lambda_N \wedge \mu_M)((x_1, y_1)(x_2, y_2))\\ &= (\lambda_N \wedge \mu_M)(x_1x_2, y_1y_2) = \lambda_N(x_1x_2) \cap \mu_M(y_1y_2)\\ &\supseteq \lambda_N(x_1) \cap \mu_M(y_1)\\ &= (\lambda_N \wedge \mu_M)(x_1, y_1)\end{aligned}$$

and

$$\begin{aligned}&(\lambda_N \wedge \mu_M)((x_1, y_1)(\gamma_1, \gamma_2)((x_2, y_2) + (x_3, y_3)) - (x_1, y_1)(\gamma_1, \gamma_2)(x_2, y_2))\\ =\;& (\lambda_N \wedge \mu_M)(x_1\gamma_1(x_2 + x_3) - x_1\gamma_1x_2, y_1\gamma_2(y_2 + y_3) - y_1\gamma_2y_2)\\ =\;& \lambda_N(x_1\gamma_1(x_2 + x_3) - x_1\gamma_1x_2) \cap \mu_M(y_1\gamma_2(y_2 + y_3) - y_1\gamma_2y_2)\\ \supseteq\;& \lambda_N(x_3) \cap \mu_M(y_3) = (\lambda_N \wedge \mu_M)(x_3, y_3)\end{aligned}$$

Thus $\lambda_N \wedge \mu_M$ is a soft int $\Gamma$-ideal of $N \times M$ over $X$. □

**Theorem 3.22.** *If $\lambda_N$ and $\mu_M$ are soft int $\Gamma$-ideals over $X_1$ and $X_2$, then $h_{N\times M} = \lambda_N \times \lambda_M$ is also soft int $\Gamma$-nearring over $X_1 \times X_2$.*

*Proof.* In Theorem 3.12, it is proved that if $\lambda_N$ and $\mu_M$ are soft int $\Gamma$-nearrings over $X_1$ and $X_2$, then also $h_{N\times M} = \lambda_N \times \lambda_M$. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in N \times M$ and $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$. Then

$$\begin{aligned}&(\lambda_N \times \mu_M)((x_1, y_1) + (x_2, y_2) - (x_1, y_1))\\ &= (\lambda_N \times \mu_M)(x_1 + x_2 - x_1, y_1 + y_2 - y_1)\\ &= \lambda_N(x_1 + x_2 - x_1) \times \mu_M(y_1 + y_2 - y_1)\\ &\supseteq \lambda_N(x_2) \times \mu_M(y_2)\\ &= (\lambda_N \times \mu_M)(x_2, y_2),\end{aligned}$$

$$\begin{aligned}&(\lambda_N \times \mu_M)((x_1, y_1)(x_2, y_2))\\ &= (\lambda_N \times \mu_M)(x_1x_2, y_1y_2) = \lambda_N(x_1x_2) \times \mu_M(y_1y_2)\\ &\supseteq \lambda_N(x_1) \times \mu_M(y_1)\\ &= (\lambda_N \times \mu_M)(x_1, y_1)\end{aligned}$$

and

$$\begin{aligned}&(\lambda_N \times \mu_M)((x_1, y_1)(\gamma_1, \gamma_2)((x_2, y_2) + (x_3, y_3)) - (x_1, y_1)(\gamma_1, \gamma_2)(x_2, y_2))\\ =\;& (\lambda_N \times \mu_M)(x_1\gamma_1(x_2 + x_3) - x_1\gamma_1x_2, y_1\gamma_2(y_2 + y_3) - y_1\gamma_2y_2)\\ =\;& \lambda_N(x_1\gamma_1(x_2 + x_3) - x_1\gamma_1x_2) \times \mu_M(y_1\gamma_2(y_2 + y_3) - y_1\gamma_2y_2)\\ \supseteq\;& \lambda_N(x_3) \times \mu_M(y_3) = (\lambda_N \times \mu_M)(x_3, y_3)\end{aligned}$$

tHUS, $\lambda_N \times \mu_M$ is a soft int $\Gamma$-ideal of $N \times M$ over $X_1 \times X_2$. □

**Theorem 3.23.** *If $\lambda_N$ and $\mu_N$ are soft int $\Gamma$-nearrings over $X$, then $\lambda_N \widetilde{\cap} \mu_N$ is also soft int $\Gamma$-nearring over $X$.*

## 4. Soft union-intersection product and soft characteristic function

**Definition 4.1.** Let $\lambda_N$ and $\mu_N$ be two soft sets over $X$. Then, $\lambda_N \oplus \mu_N$, $\lambda_N *_\Gamma \mu_N$ are soft sets over $X$ define as:

$$(\lambda_N \oplus \mu_N)(x) = \begin{cases} \bigcup\limits_{x=yz} \{\lambda_N(y) \cap \mu_N(z)\}, & \text{if } x = y + z, \text{ for some } y, z \in N, \\ \varnothing & \text{otherwise} \end{cases}$$

and

$$(\lambda_N *_\Gamma \mu_N)(x) = \begin{cases} \bigcup\limits_{x=y\alpha z} \{\lambda_N(y) \cap \mu_N(z)\}, & \text{if } x = y\alpha z, \text{ for } y, z \in N \text{ and } \alpha \in \Gamma, \\ \varnothing & \text{otherwise} \end{cases}$$

and

$$(\lambda_N \otimes \mu_N)(x) = \begin{cases} \bigcup\limits_{x=y\alpha(z+n)-y\alpha z} \{\lambda_N(y) \cap \mu_N(n)\}, & \text{if } x = y\alpha(z+n) - y\alpha z, \\ \varnothing & \text{otherwise,} \end{cases}$$

for all $x, y, z, n \in N$ and $\alpha \in \Gamma$.

**Definition 4.2.** Let $A$ be a subset of $\Gamma$-nearring $N$. We denote by $\mathcal{S}_A$ the soft characteristic function of $A$ and define as:

$$\mathcal{S}_A =: \begin{cases} X & \text{if } x \in A, \\ \varnothing & \text{if } x \notin A. \end{cases}$$

Its clearly that soft characteristic function is a soft set over $X$, that is,

$$\mathcal{S}_A : N \longrightarrow P(X).$$

**Proposition 4.3.** *Let $A$ and $B$ be non-empty subsets of a $\Gamma$-nearring $N$. Then, we have*

(1) *If $A \subseteq B$, then $\mathcal{S}_A \subseteq \mathcal{S}_B$,*
(2) *If $\mathcal{S}_A \widetilde{\cap} \mathcal{S}_B = \mathcal{S}_{A\cap B}$ and $\mathcal{S}_A \widetilde{\cup} \mathcal{S}_B = \mathcal{S}_{A\cup B}$,*
(3) *If $\mathcal{S}_A *_\Gamma \mathcal{S}_B = \mathcal{S}_{A\Gamma B}$ and $\mathcal{S}_A \oplus \mathcal{S}_B = \mathcal{S}_{A\oplus B}$.*

*Proof.* (1) and (2) are straightforward.

(3) Let $x \in N$. Then, we have two cases: either $x = y\alpha z$ or $x \neq y\alpha z$.

Case (i): If $x \in A\Gamma B$, then $x = y\alpha z$ for $y \in A$, $z \in B$ and $\alpha \in \Gamma$. Thus, we have

$$\begin{aligned} (\mathcal{S}_A *_\Gamma \mathcal{S}_B)(x) &= \bigcup_{x=y\alpha z} \{\mathcal{S}_A(y) \cap \mathcal{S}_B(z)\} \supseteq X \cap X = X \\ (\mathcal{S}_A *_\Gamma \mathcal{S}_B)(x) &= X \text{ Because } (\mathcal{S}_A *_\Gamma \mathcal{S}_B)(x) \subseteq X. \end{aligned}$$

and $\mathcal{S}_{A\Gamma B}(x) = X$, because $x \in A\Gamma B$. This implies $\mathcal{S}_A *_\Gamma \mathcal{S}_B = \mathcal{S}_{A\Gamma B}$. If $x \notin A\Gamma B$, then $x \neq y\alpha z$ for $y \notin A$, $z \notin B$. If $x = n\alpha t$ for some $n, t \in N$, then we have

$$\begin{aligned} (\mathcal{S}_A *_\Gamma \mathcal{S}_B)(x) &= \bigcup_{x=n\alpha t} \{\mathcal{S}_A(n) \cap \mathcal{S}_B(t)\} = \varnothing \\ (\mathcal{S}_A *_\Gamma \mathcal{S}_B)(x) &= \varnothing = (\mathcal{S}_A *_\Gamma \mathcal{S}_B)(x) \text{ Because } (\mathcal{S}_A *_\Gamma \mathcal{S}_B)(x) \subseteq X. \end{aligned}$$

Case (ii): If$x \neq y\alpha z$, then clearly $(\mathcal{S}_A *_\Gamma \mathcal{S}_B)(x) = \varnothing = (\mathcal{S}_A *_\Gamma \mathcal{S}_B)(x)$.
This complete the proof. □

## 5. Applications of soft int Γ-nearring and soft int-Γ-ideals

In this section, we give the applications of soft image, soft pre-image and upper-inclusion of sets to nearring theory with respect to soft int nearrings and soft int-ideals of a nearring.

**Theorem 5.1.** *If $\lambda_N$ is a soft int-$\Gamma$-ideal of $\Gamma$-nearring $N$ over $X$, then $N_\lambda = \{x \in N : \lambda_N(x) = \lambda_N(0)\}$ is a $\Gamma$-ideal of $N$ over $X$.*

*Proof.* It is obvious that $0 \in N_\lambda \subseteq N$. We need to prove that
(i) $x - y \in N_\lambda$,
(ii) $n + x - n \in N_\lambda$,
(iii) $x\alpha n \in N_\lambda$
and
(iv) $n\alpha(i + x) - n\alpha i \in N_\lambda$,
for all $x, y \in N_\lambda$, $n, i \in N$ and $\alpha \in \Gamma$.

If $x, y \in N_\lambda$, then $\lambda_N(x) = \lambda_N(y) = \lambda_N(0)$. Thus, by Remark 3.5, it follows that
$f_N(0) \supseteq \lambda_N(x - y)$,
$f_N(0) \supseteq \lambda_N(n + x - n)$,
$f_N(0) \supseteq \lambda_N(x\alpha n)$
and
$f_N(0) \supseteq \lambda_N(n\alpha(i + x) - n\alpha i)$,
for all $x, y \in N_\lambda$, $n, i \in N$ and $\alpha \in \Gamma$.
Since $\lambda_N$ is a soft int-Γ-ideal of Γ-nearring $N$ over $X$,
$\lambda_N(x - y) \supseteq \lambda_N(x) \cap \lambda_N(y) = \lambda_N(0)$,
$\lambda_N(n + x - n) \supseteq \lambda_N(x) = \lambda_N(0)$,
$\lambda_N(x\alpha n) \supseteq \lambda_N(x) = \lambda_N(0)$ and
$\lambda_N(n\alpha(i + x) - n\alpha i) \supseteq \lambda_N(x) = \lambda_N(0)$.
These imply that
$\lambda_N(x - y) = \lambda_N(0)$,
$\lambda_N(n + x - n) = \lambda_N(0)$,
$\lambda_N(x\alpha n) = \lambda_N(0)$
and
$\lambda_N(n\alpha(i + x) - n\alpha i) = \lambda_N(0)$,
for all $x, y \in N_\lambda$, $n, i \in N$ and $\alpha \in \Gamma$. Thus, $N_\lambda$ is a Γ-ideal of $N$ over $X$. □

**Theorem 5.2.** *Let $\lambda_N$ be a soft set over $X$ and $\beta$ be a subset of $X$ such that $\varnothing \neq \beta \subseteq \lambda_N(0)$. If $\lambda_N$ is a soft int-$\Gamma$-ideal of $\Gamma$-nearring $N$ over $X$, then $\lambda_N^{\supseteq\beta} = \{x \in N : \lambda_N(x) \supseteq \beta\}$ is a $\Gamma$-ideal of $N$ over $X$.*

*Proof.* Since $\lambda_N(0) \supseteq \beta$, $0 \in \lambda_N^{\supseteq\beta}$ and $\varnothing \neq \lambda_N^{\supseteq\beta} \subseteq N$. Take $x, y \in \lambda_N^{\supseteq\beta}$, $n, i \in N$ and $\alpha \in \Gamma$, which implies that $\lambda_N(x) \supseteq \beta$ and $\lambda_N(y) \supseteq \beta$. Now, we need to prove that
(i) $x - y \in \lambda_N^{\supseteq\beta}$,
(ii) $n + x - n \in \lambda_N^{\supseteq\beta}$,
(iii) $x\alpha n \in \lambda_N^{\supseteq\beta}$
and

(iv) $n\alpha(i+x) - n\alpha i \in \lambda_N^{\supseteq\beta}$,
for all $x, y \in \lambda_N^{\supseteq\beta}$, $n, i \in N$ and $\alpha \in \Gamma$.
Since $\lambda_N$ is a soft int-Γ-ideal of Γ-nearring $N$ over $X$, it follows that

$\lambda_N(x-y) \supseteq \lambda_N(x) \cap \lambda_N(y)\,\beta \cap \beta = \beta,$
$\lambda_N(n+x-n) \supseteq \lambda_N(x) \supseteq \beta,$
$\lambda_N(x\alpha n) \supseteq \lambda_N(x) \supseteq \beta$

and

$\lambda_N(n\alpha(i+x) - n\alpha i) \supseteq \lambda_N(x) \supseteq \beta.$

Thus, this completes the proof. □

**Theorem 5.3.** *Let $\lambda_N$ and $\lambda_M$ be soft sets over $X$ and $f$ be a Γ-nearring isomorphism from Γ-nearring $N$ to Γ-nearring $M$.*

*(1) If $\lambda_N$ is a soft int-Γ-ideal of Γ-nearring $N$ over $X$, then $f(\lambda_N)$ is a soft int-Γ-ideal of Γ-nearring $M$ over $X$*

*(2) If $\lambda_M$ is a soft int-Γ-ideal of Γ-nearring $M$ over $X$, then $f^{-1}(\lambda_M)$ is a soft int-Γ-ideal of Γ-nearring $N$ over $X$.*

*Proof.* (1) Let $x_1, x_2 \in M$. Since $f$ is surjective, there exist $y_1, y_2 \in N$ such that

$$f(y_1) = x_1,\ f(y_2) = x_2 \text{ and } f(y_3) = x_3.$$

Then

$$\begin{aligned}
& f(\lambda_N)(x_1 - x_2) \\
= \ & \cup\{\lambda_N(y) : y \in N,\ f(y) = x_1 - x_2\} \\
= \ & \cup\left\{\lambda_N(y) : y \in N,\ y = f^{-1}(x_1 - x_2)\right\} \\
= \ & \cup\left\{\lambda_N(y) : y \in N,\ y = f^{-1}(f(y_1 - y_2)) = y_1 - y_2\right\} \\
= \ & \cup\{\lambda_N(y_1 - y_2) : y_i \in N,\ f(y_i) = x_i,\ i = 1, 2\} \\
\supseteq \ & \cup\{\lambda_N(y_1) \cap \lambda_N(y_2) : y_i \in N,\ f(y_i) = x_i,\ i = 1, 2\} \\
= \ & \cup\{\lambda_N(y_1) : y_1 \in N,\ f(y_1) = x_1\} \cap \{\lambda_N(y_2) : y_2 \in N,\ f(y_2) = x_2\} \\
= \ & f(\lambda_N)(x_1) \cap f(\lambda_N)(x_2).
\end{aligned}$$

Thus $f(\lambda_N)(x_1 - x_2) \supseteq f(\lambda_N)(x_1) \cap f(\lambda_N)(x_2)$.

Similarly, we can prove that $f(\lambda_N)(x_1\alpha_1 x_2) \supseteq f(\lambda_N)(x_1) \cap f(\lambda_N)(x_2)$.

Now, we prove that

$$\begin{aligned}
& f(\lambda_N)(x_1 + x_2 - x_1) \\
= \ & \cup\{\lambda_N(y) : y \in N,\ f(y) = x_1 + x_2 - x_1\} \\
= \ & \cup\left\{\lambda_N(y) : y \in N,\ y = f^{-1}(x_1 + x_2 - x_1)\right\} \\
= \ & \cup\left\{\lambda_N(y) : y \in N,\ y = f^{-1}(f(y_1 + y_2 - y_1)) = y_1 + y_2 - y_1\right\} \\
= \ & \cup\{\lambda_N(y_1 - y_2) : y_i \in N,\ f(y_i) = x_i,\ i = 1, 2\} \\
\supseteq \ & \cup\{\lambda_N(y_2) : y_2 \in N,\ f(y_2) = x_2\} \\
= \ & f(\lambda_N)(x_2) \\
= \ & f(\lambda_N)(x_2).
\end{aligned}$$

So $f(\lambda_N)(x_1 + x_2 - x_1) \supseteq f(\lambda_N)(x_2)$.

Now, let $x_1, x_2 \in M$, $y_1, y_2 \in N$, $\alpha \in \Gamma$ and $\alpha_1 \in \Gamma_1$. Then

$$\begin{aligned}
& f(\lambda_N)(x_1\alpha_1 x_2) \\
= & \cup\{\lambda_N(y) : y \in N,\ f(y) = x_1\alpha_1 x_2\} \\
= & \cup\left\{\lambda_N(y) : y \in N,\ y = f^{-1}(x_1\alpha_1 x_2)\right\} \\
= & \cup\left\{\lambda_N(y) : y \in N,\ y = f^{-1}(f(y_1\alpha y_2)) = y_1\alpha y_2\right\} \\
= & \cup\{\lambda_N(y_1\alpha y_2) : y_i \in N,\ f(y_i) = x_i,\ i = 1,2\} \\
\supseteq & \cup\{\lambda_N(y_2) : y_2 \in N,\ f(y_2) = x_2,\} \\
= & f(\lambda_N)(x_1) \cap f(\lambda_N)(x_2).
\end{aligned}$$

Now, let $x_1, x_2, x_3 \in M$, $y_1, y_2, y_3 \in N$, $\alpha \in \Gamma$ and $\alpha_1 \in \Gamma_1$. Then

$$\begin{aligned}
& f(\lambda_N)(x_1\alpha_1(x_2 + x_3) - x_1\alpha_1 x_2) \\
= & \cup\{\lambda_N(y) : y \in N,\ f(y) = x_1\alpha_1(x_2 + x_3) - x_1\alpha_1 x_2\} \\
= & \cup\left\{\lambda_N(y) : y \in N,\ y = f^{-1}(x_1\alpha_1(x_2 + x_3) - x_1\alpha_1 x_2)\right\} \\
= & \cup\left\{\begin{array}{c} \lambda_N(y) : y \in N,\ y = f^{-1}(f(y_1\alpha(y_2 + y_3) - y_1\alpha y_2)) \\ = y_1\alpha(y_2 + y_3) - y_1\alpha y_2 \end{array}\right\} \\
= & \cup\{\lambda_N(y_1\alpha(y_2 + y_3) - y_1\alpha y_2) : y_i \in N,\ f(y_i) = x_i,\ i = 1,2,3\} \\
\supseteq & \cup\{\lambda_N(y_3) : y_3 \in N,\ f(y_3) = x_3\} \\
= & f(\lambda_N)(x_3).
\end{aligned}$$

Hence $f(\lambda_N)$ is a a soft int-$\Gamma$-ideal of $\Gamma$-nearring $M$ over $X$.

(2) Let $x_1, x_2, x_3 \in M$ and $\alpha_1 \in \Gamma$ and $\beta_1 \in \Gamma_1$. Then

$$\begin{aligned}
\left(f^{-1}(\lambda_M)\right)(x_1\alpha_1 x_2) &= \lambda_M(f(x_1\alpha_1 x_2)) = \lambda_M(f(x_1)\beta_1 f(x_2)) \\
&\supseteq \lambda_M(f(x_1)) \cap \lambda_M(f(x_2)) \\
&= \left(f^{-1}(\lambda_M)\right)(x_1) \cap \left(f^{-1}(\lambda_M)\right)(x_2)
\end{aligned}$$

Similarly, $\left(f^{-1}(\lambda_M)\right)(x_1 - x_2) \supseteq \left(f^{-1}(\lambda_M)\right)(x_1) \cap \left(f^{-1}(\lambda_M)\right)(x_2)$. Also,

$$\begin{aligned}
\left(f^{-1}(\lambda_M)\right)(x_1 + x_2 - x_1) &= \lambda_M(f(x_1 + x_2 - x_1)) \\
&= \lambda_M(f(x_1) + f(x_2) - f(x_1)) \\
&\supseteq \lambda_M(f(x_2)) = \left(f^{-1}(\lambda_M)\right)(x_2) \\
\left(f^{-1}(\lambda_M)\right)(x_1 + x_2 - x_1) &\supseteq \left(f^{-1}(\lambda_M)\right)(x_2).
\end{aligned}$$

Now, let $x_1, x_2 \in M$ and $\alpha_1 \in \Gamma$ and $\beta_1 \in \Gamma_1$. Then

$$\begin{aligned}
\left(f^{-1}(\lambda_M)\right)(x_1\alpha_1 x_2) &= \lambda_M(f(x_1\alpha_1 x_2)) = \lambda_M(f(x_1)\beta_1 f(x_2)) \\
&\supseteq \lambda_M(f(x_1)) = \left(f^{-1}(\lambda_M)\right)(x_1) \\
\left(f^{-1}(\lambda_M)\right)(x_1\alpha_1 x_2) &\supseteq \left(f^{-1}(\lambda_M)\right)(x_1).
\end{aligned}$$

Finally, let $x_1, x_2, x_3 \in M$ and $\alpha_1 \in \Gamma$ and $\beta_1 \in \Gamma_1$. Then

$$\begin{aligned} & \left(f^{-1}\left(\lambda_M\right)\right)\left(x_1\alpha_1\left(x_2+x_3\right)-x_1\alpha_1x_2\right) \\ = \;& \left(\lambda_M\right)\left(f\left(x_1\alpha_1\left(x_2+x_3\right)-x_1\alpha_1x_2\right)\right) \\ = \;& \left(\lambda_M\right)\left(f\left(x_1\right)\beta_1 f\left(x_2\right)+f\left(x_3\right)-f\left(x_1\right)\beta_1 f\left(x_2\right)\right) \\ \supseteq \;& \lambda_M\left(f\left(x_3\right)\right)=\left(f^{-1}\left(\lambda_M\right)\right)\left(x_3\right). \end{aligned}$$

Thus $f^{-1}\left(\lambda_M\right)$ is a soft int-Γ-ideal of Γ-nearring $N$ over $X$. □

## 6. Conclusion

Fuzzy set theory, rough set theory and soft set theory are all mathematical tools for dealing with uncertainties. This paper is devoted to discussion of combination of soft set theory, set theory and Γ-nearring. By using soft sets and intersection operation of sets, we have defined a new concept, called soft int Γ-nearring. This new notion brings the soft set theory, set theory and Γ-nearring theory together and therefore is very functional for obtaining results by means of Γ-nearring structure. Based on the definition, we have introduced the concepts of soft int subΓ-nearrings and soft int-Γ-ideals of a Γ-nearring with illustrative examples. We have then investigated these notions with respect to soft image, soft pre-image and $\alpha$-inclusion of soft sets. Finally, we give some applications of soft int Γ-nearrings to Γ-nearring theory.

In further research, we focus on the following topics,

(1) We will define soft int-bi-Γ-ideals and soft int-quasi-Γ-ideals of a Γ-nearring and its applications.

(2) We will characterize Γ-nearrings by the properties of soft int Γ-ideals

(3)We will define fuzzy soft int Γ-nearring, fuzzy soft int-Γ-ideals of a Γ-nearring,

(4) We extend this study, one can further study the other algebraic structures such as different algebras by means of soft intersections.

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