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Some fixed point theorems on D*-fuzzy metric spaces

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Abstract. In this paper, concepts of convergent sequence, cauchy sequence in D*-fuzzy metric space are introduced and some common fixed point theorems for some generalized contraction mapping are established in such papers.

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1. Introduction

Many authors generalized the idea of metric space in different approaches. One of such generalization is D- metric space introduced by Dhage [6] in 1992. Sedghi et al.[11] modified the definition of D-metric space and introduced an idea of D*-metric space and established some fixed point theorems in such space. Veerapandi and Pillai [13] established some fixed point theorems of contractive mappings on D*-complete metric spaces. In this context, it is worth mentioning the work of Deng [5], Erceg [7] and Singh et al. [12].

On the other hand, different authors generalized the idea of fuzzy metric space in different directions. Sedghi et al.[10] introduced the concept of M- fuzzy metric space which is a generalization of fuzzy metric space due to George and Veeramoni [8]. Recently Bag [1] modified the definition of M- fuzzy metric space introduced by Sedghi et al.[10] and call it D*- fuzzy metric space. It has been possible to achieve two decomposition theorems of D*- fuzzy metric into a family of D*- metrices. Sh. Rezapour et al. [9], N. R. Das et al. [4] and Bag [2, 3] studied many results in generalized fuzzy metric spaces.

In this paper, we consider D*- fuzzy metric space introduced by Bag [1] and introduce the concepts of convergent sequence, Cauchy sequence etc. and some common fixed point theorems for some generalized contraction mappings have been established.

The organization of the paper is as follows:
In Section 2, some preliminary results are given. Some basic principles are studied in Section 3. In Section 4,some fixed point theorems are established.

2. Preliminaries

Definition 2.1 ([1])**.** A 3-tuple $(X, D^*, *)$ is called an D^*-fuzzy metric space, if X is an arbitrary (non-empty) set and D^* is a fuzzy set on $X^3 \times [0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s \in [0, \infty)$;

(FD^*1) $D^*(x, y, z, 0) = 0$,

(FD^*2) $\forall t > 0,\ D^*(x, y, z, t) = 1$ iff $x = y = z$,

(FD^*3) $D^*(x, y, z, t) = D^*(p\{x, y, z\}, t)$ (symmetry), where p is a permutation function,

(FD^*4) $D^*(x, y, a, t) * D^*(a, z, z, s) \leq D^*(x, y, z, t + s)$,

(FD^*5) $\lim\limits_{t\to\infty} D^*(x, y, z, t) = 1$.

3. Some basic properties of D^*-fuzzy metric spaces.

In this section an example of D^*-fuzzy metric space is given and some properties are studied.

Example 3.1. Let (X, D) be a D*- metric space. Define
$D^* : X \times X \times X \times [0, \infty)$ by

$$D^*(x, y, z, t) \ = \begin{cases} 1 & \text{if } t > D(x, y, z) \\ \frac{1}{2} & \text{if } 0 < t \leq D(x, y, z) \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then (X, D*, $*$) is a D*- fuzzy metric space.
Where X is a non-empty set and $\forall x, y, z \in X$, $(X\ ,\ D^*)$ is a D^*- metric space $\forall t \in [0, \infty)$ and D^* is a function defined on X above,
Then we shall prove that $(X\ ,\ D^*)$ is a fuzzy D^*- metric space.

Proof. (i) $D^*(x, y, z, 0) = 0\ \forall x, y, z \in X$.

(ii) $D^*(x, y, z, t) = D^*(p(x, y, z), t)\ \forall t \in [0, \infty), \forall x, y, z \in X$ [Since $D(p\{x, y, z\}) = D(x, y, z)$].

(iii) $D^*(x, y, z, t) = 1\ \forall t > 0 \Leftrightarrow t > D(x, y, z)\ \forall t > 0$.

(iv) For all $x, y, z, a \in X,\ s, t \in [0, \infty) \Leftrightarrow D(x, y, z) = 0 \Leftrightarrow x = y = z$.

We consider the following cases.

Case (i): Suppose $D^*(x, y, z, t) = 1$ and $D^*(a, z, z, s) = 1$. Then $t > D^*(x, y, a)$ and $s > D^*(a, z, z)$. Now $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z) < t + s$. Thus $t + s > D^*(x, y, z)$. So $D^*(x, y, z, t + s) = 1 = 1 * 1 = D^*(x, y, a, t) * D^*(a, z, z, s)$.

Case (ii): Suppose $D^*(x, y, a, t) = \frac{1}{2}$ and $D^*(a, z, z, s) = 1$. Then $0 < t \leq D^*(x, y, a)$ and $s > D^*(a, z, z) \geq 0$. Thus $s + t > 0$. So $D^*(x, y, z, t + s) \geq \frac{1}{2} = \frac{1}{2} * 1 = D^*(x, y, a, s) * D^*(a, z, z, t)$.

Case (iii): Suppose $D^*(x, y, a, t) = 0$ and $D^*(a, z, z, s) = 1$. Then $D^*(x, y, z, t + s) \geq 0 = 0 * 1 = D^*(x, y, a, t) * D^*(a, z, z, s)$.

Case (iv): Similarly, we can prove that

$$D^*(x, y, z, t + s) \geq D^*(x, y, a, t) * D^*(a, z, z, s),$$

whenever $D^*(x,y,a,t)=1$, $D^*(a,z,z,s)=1$ or $\frac{1}{2}$ or 0.

Case (v): Suppose $D^*(x,y,a,t)=\frac{1}{2}$ and $D^*(a,z,z,s)=\frac{1}{2}$. Then

$$0<t=D^*(x,y,a), 0<s\leq D^*(a,z,z).$$

Thus $t+s>0$. Now

$$D^*(x,y,z,t+s)\geq \tfrac{1}{2}=\tfrac{1}{2}*1\geq \tfrac{1}{2}*\tfrac{1}{2}=D^*(x,y,a,t)*D^*(a,z,z,s).$$

So $\forall t,s,\in[0,\infty]$ and $\forall x,y,z,a\ X$,

$$D^*(x,y,z,t+s)\geq D^*(x,y,a,t)*D^*(a,z,z,s).$$

Hence (X, D*, $*$) is a D*- fuzzy metric space. □

Definition 3.2. A sequence $\{x_n\}$ in a D^*-fuzzy metric space X is said to be convergent to a point $x\epsilon X$ if $\lim\limits_{n\to\infty} D^*\ (x_n,x_n,x,t)=\lim\limits_{n\to\infty} D^*\ (x,x,x_n,t)=1\ \forall t>0$.

Definition 3.3. A sequence $\{x_n\}$ in a D^*-fuzzy metric space X is said to be a Cauchy sequence if $\lim\limits_{n\to\infty} D^*\ (x_n,x_n,x_{n+p},t)=1\ \forall t>0$ and p=1,2,3...

Definition 3.4. Let X be a D^*-fuzzy metric space. A nonempty set A of X is said to be complete if every Cauchy sequence $\{x_n\}$ in A converges to some point in A.

Example 3.5. Let X be a nonempty set and D' be a D^* metric on X and $(X\ ,\ D')$ is complete.

Choose $a*b=ab\ \forall a,b\in[0,1]$. For each $t\in[0,\infty)$, we define

$$D^*(x,y,z,t)=\tfrac{t}{t+D'(x,y,z)}\ \forall x,y,z\in X.$$

Then $(X\ ,\ D^*)$ is a complete D^*-fuzzy metric space.

Proposition 3.6. *Let (X, D*, $*$) be a D*- fuzzy metric space where $*$ is a continuous t- norm. Then limit of a convergent sequence is unique.*

Proof. Let $\{x_n\}$ be a sequence in a D^*-fuzzy metric space X and suppose $x_n\to x$ and $x_n\to y$ for some $x,y\in X$. We shall show that $x=y$.
We have

$$D^*(x,x,y,t+s)\geq D^*(x,x,x_n,t)*D^*(x_n,y,y,s)\ \forall t,s\in(0,\infty),\ \text{n=1,2,3,....}$$

Let $n\to\infty$. Then

$$D^*(x,x,y,t+s)\geq \lim_{n\to\infty} D^*(x,x,x_n,t)*\lim_{n\to\infty} D^*(x_n,y,y,s)=1*1=1.$$

Thus $D^*(x,x,y,t+s)=1$, $\forall t,s>0$. So $x=y$. □

Proposition 3.7. *Every convergent sequence is Cauchy.*

Proof. Let $\{x_n\}$ be a sequence in X and $x_n\to x$ for some $x\in X$. Now

$$D^*(x,x_n,x_{n+p},t+s)\geq D^*(x_n,x_n,x,t)*D^*(x,x,x_{n+p},s)\ \forall t,s\in(0,\infty) \text{ and } p\in N.$$

Let $n \to \infty$. Then

$$\lim_{n\to\infty} D^*(x_n, x_n, x_{n+p}, t+s) \geq \lim_{n\to\infty} D^*(x_n, x_n, x, t) * \lim_{n\to\infty} D^*(x, x_{n+p}, x_{n+p}, s)$$
$$= 1 * 1 = 1.$$

Thus $\lim_{n\to\infty} D^*(x_n, x_n, x_{n+p}, t+s) = 1$ for p=1,2,3,.... and $\forall t, s \in (0, \infty)$. So $\{x_n\}$ is a Cauchy sequence in X. □

4. Some fixed point theorems.

In this section some fixed point theorems in D^*-fuzzy metric spaces are established.

Theorem 4.1. *Let (X, D^*) be a complete D^*- fuzzy metric space and $T_1, T_2, T_3 : X \to X$ be three mappings satisfying that*

$$D^*(T_1x, T_2y, T_3z, t) \geq D^*(x, y, z, \tfrac{t}{a})\ \forall t > 0,\ \forall x, y, z \in X \text{ and } 0 < a < 1.$$

Then T_1, T_2, T_3 have a unique fixed point in X.

Proof. Let $x_0 \in X$ be a fixed arbitrary point . Define sequence $\{x_n\}$ in X

$T_1x_n = x_{n+1}$,
$T_2x_{n+1} = x_{n+2}$,
$T_3x_{n+2} = x_{n+3}$,
- - - - - - - - - - - .

Then

$$D^*(x_n, x_n, x_{n+1}, t) = D^*(T_1x_{n-1}, T_2x_{n-1}, T_3x_n, t)\ \forall t > 0$$
$$\geq D^*(x_{n-1}, x_{n-1}, x_n, \tfrac{t}{a}),\ a \in (0, 1)$$
$$- -$$
$$\geq D^*(x_0, x_0, x_1, \tfrac{t}{a^n}).$$

Thus $\lim_{n\to\infty} D^*(x_0, x_0, x_1, \dfrac{t}{a^n}) = 1\ \forall t > 0$. So $\lim_{n\to\infty} D^*(x_n, x_n, x_{n+1}, t) = 1\ \forall t > 0$.

Similarly, we can prove that

$$\lim_{n\to\infty} D^*(x_n, x_n, x_{n+p}, t) = 1 \text{ for p=1,2,3,..... and } \forall t > 0.$$

Hence $\{x_n\}$ is a Cauchy sequence.

Since, X is complete, $\lim x_n = x$, for some $x \in X$.

Now we prove that $T_1x = x$. Clearly,

$$D^*(T_1x, T_2x, T_3x, t) \geq D^*(x, x, x, \tfrac{t}{a}),\ \forall t > 0,\ 0 < a < 1.$$

Then $D^*(T_1x, T_2x, T_3x, t) = 1\ \forall t > 0$. Thus $T_1x = T_2x = T_3x$. Again

$$D^*(T_1x, T_1x, x, t) = D^*(T_1x, T_2x, x, t)\ \forall t > 0$$
$$\geq D^*(T_1x, T_2x, x_{n+3}, \tfrac{t}{2}) * D^*(x_{n+3}, x, x, \tfrac{t}{2})\ \forall t > 0$$
$$= D^*(T_1x, T_2x, T_3x_{n+2}, \tfrac{t}{2}) * D^*(x_{n+3}, x, x, \tfrac{t}{2})\ \forall t > 0$$
$$\geq D^*(x, x, x_{n+2}, \tfrac{t}{2}) * D^*(x_{n+3}, x, x, \tfrac{t}{2}).$$

$n \to \infty$. Then $D^*(T_1x, T_1x, x, t) \geq 1 * 1 = 1$. Thus $D^*(T_1x, T_1x, x, t) = 1\ \forall t > 0$. So $T_1x = T_1x = x$. Hence x is a fixed point of T_1, T_2, T_3.

Uniqueness: Assume that $\exists y (\neq x)$ such that $T_1y = T_2y = T_3y = y$. Then

$$D^*(x, y, y, t) = D^*(T_1x, T_2y, T_3y, t)$$
$$\geq D^*(x, y, y, \tfrac{t}{a})$$

$= D^*(T_1x, T_2y, T_3y, \frac{t}{a})$
$\geq D^*(x, y, y, \frac{t}{a^2})$

- -

$\geq D^*(x, y, y, \frac{t}{a^n})$, for some $n \in N$.
Let $n \to \infty$. Then $D^*(x, y, y, \frac{t}{a^n}) = 1$ $\forall t > 0$. Thus $0 < a < 1$. So $D^*(x, y, y, t) = 1$ $\forall t > 0$. Hence $x = y$ and thus T_1, T_2, T_3 have a unique and common fixed point in X.

The above theorem is justified by the following example. □

Example 4.2. Take $X = R$ and define

$$D'(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ max(x, y, z) & \text{otherwise.} \end{cases}$$

Then D' is a D*-metric space.

Proof. Define $D^* : X \times X \times X \times [0, \infty)$ by $D^*(x, y, z, t) = \frac{t}{t + D'(x,y,z)}$.
Then D*-is a D*-fuzzy metric space.
Now define mappings $T_1, T_2, T_3 : X \to X$ by $T_1(x) = T_2(x) = T_3(x) = \frac{1}{3}x$.
Then for $\frac{1}{3} < a < 1$, $D^*(T_1x, T_2y, T_3z, t) \geq D^*(x, y, z, \frac{t}{a})$, $\forall t > 0$.

On one hand, $D^*(T_1x, T_2y, T_3z, t) = \frac{t}{t + \frac{1}{3}D'(x,y,z)}$. Thus

$$D^*(x, y, z, \frac{t}{a}) = \frac{\frac{t}{a}}{\frac{t}{a} + D'(x, y, z)}.$$

Now

$$\begin{aligned} D^*(T_1x, T_2y, T_3z, t) - D^*(x, y, z, \tfrac{t}{a}) &= \tfrac{t}{t + \frac{1}{3}D'(x,y,z)} - \tfrac{t}{t + aD'(x,y,z)} \\ &= \tfrac{3t}{3t + D'(x,y,z)} - \tfrac{t}{t + aD'(x,y,z)} \\ &= \tfrac{3atD'(x,y,z) - tD'(x,y,z)}{(3t + D'(x,y,z))(t + aD'(x,y,z))} \\ &= \tfrac{(3a-1)tD'(x,y,z)}{(3t + D'(x,y,z))(t + aD'(x,y,z))} \geq 0. \\ &\quad (\text{Since} \tfrac{1}{3} < a < 1) \end{aligned}$$

Then $D^*(T_1x, T_2y, T_3z, t) \geq D^*(x, y, z, \frac{t}{a})$ $\forall t > 0$ and $\frac{1}{3} < a < 1$. Thus, by above theorem, T_1, T_2, T_3 have a common unique fixed point which is 0. So

$$T_1x = T_2x = T_3x = \frac{1}{3}x.$$

Hence $x = \frac{1}{3}x, i.e., 2x = 0$. Therefore $x = 0 \in X$. □

Theorem 4.3. *Let* $(X\ ,\ D^*, *)$ *be a complete* D^**- fuzzy metric space and* $T : X \to X$ *be a mapping such that*

$$D^*(Tx, T^2x, T^3x, t) \geq D^*(x, Tx, T^2x, \tfrac{t}{a})$$

for all $x \in X$ *and* $0 \leq a < 1$, $\forall t > 0$. *Then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ be a fixed arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for n=0,1,2,....
Then for $n \geq 0$, we have

$$\begin{aligned} D^*(x_n, x_n, x_{n+1}, t) &= D^*(Tx_{n-1}, Tx_{n-1}, Tx_n, t) \\ &\geq D^*(x_{n-1}, x_{n-1}, x_n, \tfrac{t}{a}) \end{aligned}$$

- -

$$\geq D^*(x_0, x_0, x_1, \tfrac{t}{a^n}) \to 1 \text{ as } n \to \infty, \text{ since } 0 \leq a < 1.$$

Thus $\lim\limits_{n\to\infty} D^*(x_n, x_n, x_{n+p}, t) = 1 \ \forall t > 0$ and p=1,2,3,..... So $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, $x_n \to x$ for some $x \in X$. Then

$$D^*(x_{n+1}, x_{n+1}, Tx, t) = D^*(Tx_n, Tx_n, Tx, t) \geq D^*(x_n, x_n, x, \tfrac{t}{a}).$$

Thus $\lim\limits_{n\to\infty} D^*(x_{n+1}, x_{n+1}, Tx, t) \geq \lim\limits_{n\to\infty} D^*(x_n, x_n, x, \dfrac{t}{a}) = 1 \ \forall t > 0$.
So $\lim\limits_{n\to\infty} D^*(x_{n+1}, x_{n+1}, Tx, t) = 1 \ \forall t > 0$. Hence $\{x_{n+1}\} \to Tx$. Since limit of a sequence is unique, Tx=x.

Uniqueness: Suppose $\exists y \in X$, $x \neq y$ such that $Ty = y$. Then

$$\begin{aligned} D^*(x, y, y, t) &= D^*(T^3x, T^2y, Ty, t) \ \forall t > 0 \\ &\geq D^*(T^2x, Ty, y, \tfrac{t}{a}) \ \forall t > 0,\ 0 < a < 1 \\ &= D^*(T^3x, T^2y, Ty, \tfrac{t}{a}) \\ &\geq D^*(T^2x, Ty, y, \tfrac{t}{a^2}) \\ &\geq D^*(T^2x, Ty, y, \tfrac{t}{a^2}) \geq D^*(T^2x, Ty, y, \tfrac{t}{a^n}). \text{ (after n steps)} \end{aligned}$$

Let $n \to \infty$. Then $\lim\limits_{n\to\infty} D^*(T^2x, Ty, y, \dfrac{t}{a^n}) = 1$ $(0 \leq a < 1)$ $\forall t > 0$. Thus $D^*(x, y, y, t) = 1 \ \forall t > 0$. So $x = y$. Hence T has a unique fixed point in X. □

5. Conclusions

In an earlier paper concept of D*-fuzzy metric space has been introduced. In this paper some basic results are studied and establish some fixed point theorems. We think that the work of this paper will be helpful to the researchers to study further results on D*-fuzzy metric spaces.

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