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Analytical solutions of fuzzy partial differential equations

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Abstract. In this paper, we study a general form of the fuzzy partial differential equations using successive of the Adomian decomposition method to concept the strongly generalized partial derivative for fuzzy-valued functions from $\mathbb{R}^2$ into E. Existence and convergence theorems of fuzzy solution obtain by presented method. At last, we exhibit two illustrative examples.

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1. Introduction

Fuzzy systems are powerful tools to study a variety of problems ranging in mathematic models, for example, population models [20], the golden mean [15], transport of water and pesticide in an unsaturated layered soil profile [35]. Initially, the derivative of fuzzy-valued functions is defined at the various modes. Namely, Blasi differential, Fréchet differential [16] and H-derivative [30] and so on. The H-derivative is a popular and practical concept, and an its important application is exhibited in differential equations which in fuzzy setting are a natural way to model uncertainty systems [1, 2, 4, 13, 19, 22, 26, 28, 29].

Under H-derivative, mainly the existence and uniqueness theorems of solution for a fuzzy differential equation have been obtained [24, 27, 31, 33, 34, 38]. Solving the fuzzy differential equations with H-derivative lead to solutions where have an increasing support. This shortcoming is solved by interpreting a fuzzy differential equation as a system of differential inclusions [17, 23]. The main shortcoming of using differential inclusions is that we do not have a derivative of a fuzzy-valued function. In another approach in [12, 11] instead of derivative concept, the extension principle is used to extend crisp differential equations to the fuzzy case.

For solving the above mentioned shortcomings, the strongly generalized differentiability concept has been introduced in [8] and studied in [7, 9, 10] by Bede and others. Indeed, the strongly generalized derivative is defined for a larger class of fuzzy-valued functions than the Hukuhara derivative. In [9], Bede and Gal studied several characterizations under this interpretation for a fuzzy-valued function from $\mathbb{R}$ into E, and as an application obtained the existence and uniqueness theorems of the solutions for a fuzzy differential equation.

The directional derivative has been presented and studied in [36] for a fuzzy-valued function from $\mathbb{R}^n$ into E. By the strongly generalized differentiability concept in [36], can introduce the strongly generalized partial derivative for a fuzzy-valued function from $\mathbb{R}^n$ into E. In this work, we obtain existence and uniqueness theorems of solution for a fuzzy partial differential equation by successive iterations of the Adomian decomposition method [3, 21] using concept derivative in [9, 36].

After a preliminary section, we study the Adomian decomposition method. In Section 3, by generalized partial derivative and successive iterations of the Adomian decomposition method, we introduce a fuzzy partial differential equation and give existence theorems. In Section 4, we provide two examples and in the last section we represent some conclusions.

2. Preliminaries

Let $E = \{u | u : \mathbb{R} \to [0, 1]$ has the following properties $(1) - (4)\}$:

(1) $\forall u \in E$, u is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$.

(2) $\forall u \in E$, u is a convex fuzzy set (i.e. $u(rx + (1-r)y) \geq \min(u(x), u(y))$, $\forall r \in [0, 1]$, $x, y \in \mathbb{R}$),

(3) $\forall u \in E$, u is upper semi-continuous on $\mathbb{R}$,

(4) $[u]^0 = cl\{x \in \mathbb{R} | u(x) > 0\}$ is a compact set.

Then E is called fuzzy number space [18] and $\forall u \in E$, u is called a fuzzy number. Obviously, $\mathbb{R} \subset E$.

For each $r \in [0, 1]$ and $u \in E$, $[u]^r = [u(r)_*, u(r)^*]$ denotes a bounded closed interval and is defined by $[u]^r = \{x \in \mathbb{R} | u(x) \geq r\}$.

For $u, v \in E$ and $\lambda \in \mathbb{R}$ we can define sum and scalar multiplication on E, respectively, by

$$[u \oplus v]^r = [u]^r + [v]^r,$$

$$[\lambda \odot u]^r = \lambda [u]^r, \;\; \forall r \in [0, 1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ [18] and

$$(u+v)(r)_* = u(r)_* + v(r)_*, \qquad (u+v)(r)^* = u(r)^* + v(r)^*,$$

$$(\lambda u)(r)_* = \begin{cases} \lambda u(r)_* & \lambda \geq 0, \\ \lambda u(r)^* & \lambda < 0, \end{cases}$$

and

$$(\lambda u)(r)^* = \begin{cases} \lambda u(r)^* & \lambda \geq 0, \\ \lambda u(r)_* & \lambda < 0, \end{cases} \qquad for\, any\, r \in [0, 1].$$

Definition 2.1 ([30])**.** Let $u, v \in E$. If there exists $w \in E$ such that $u = v \oplus w$, then w is called H-difference of u and v and it is denoted by $u - v = w$.

It is obvious that if the H-difference $u - v$ exists, then $(u-v)(r)_* = u(r)_* - v(r)_*$ and $(u-v)(r)^* = u(r)^* - v(r)^*$.

Theorem 2.2 ([6]). (1) *If we show* $\tilde{0} = \chi_{\{0\}}$*, the characteristic function of zero, then* $\tilde{0} \in E$ *is neutral element with respect to* $\oplus$*, i.e.* $u \oplus \tilde{0} = \tilde{0} \oplus u = u$*, for all* $u \in E$*.*

(2) *With respect to* $\tilde{0}$*, none of* $u \in E \setminus \mathbb{R}$*, has inverse in* E *(with respect to* $\oplus$*).*

(3) *For each* $a, b \in \mathbb{R}$ *with* $a, b \geq 0$ *or* $a, b \leq 0$ *and each* $u \in E$*, we have* $(a+b) \odot u = a \odot u \oplus b \odot u$*. For general* $a, b \in \mathbb{R}$*, the above property does not hold.*

(4) *For each* $\lambda \in \mathbb{R}$ *and each* $u, v \in E$*, we have* $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$*.*

(5) *For each* $\lambda, \mu \in \mathbb{R}$ *and each* $u \in E$*, we have* $\lambda \odot (\mu \odot u) = (\lambda\mu) \odot u$*.*

Theorem 2.3 ([16]). *If* $u \in E$*, then* $u(r)^*$ *and* $u(r)_*$ *are functions on* $[0,1]$ *satisfying the following conditions (1)-(4):*

(1) $u(r)_*$ *is a nondecreasing function on* $[0,1]$*,*

(2) $u(r)^*$ *is a nonincreasing function on* $[0,1]$*,*

(3) $u(r)_*$ *and* $u(r)^*$ *are bounded and left continuous on* $(0,1]$*, and right continuous at* $r = 0$*,*

(4) $u(r)_* \leq u(r)^*$*, for each* $r \in [0,1]$*.*

Conversely, if functions $u(r)_*$ *and* $u(r)^*$ *on* $[0,1]$ *satisfy conditions (1)-(4), then there exists a unique* $u \in E$ *such that* $[u]^r = [u(r)_*, u(r)^*]$ *for any* $r \in [0,1]$*.*

The Hausdorff distance is defined as $D : E \times E \to [0, +\infty)$ by $D(u,v) = \sup_{r\in[0,1]} \max\{|u(r)_* - v(r)_*|, |u(r)^* - v(r)^*|\}$. The following properties are well-known [16]:

(1) $D(u \oplus w, v \oplus w) = D(u,v), \ \forall u, v, w \in E$,

(2) $D(k \odot u, k \odot v) = |k| D(u,v), \ \forall k \in \mathbb{R}, \forall u, v \in E$,

(3) $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \ \forall u, v, w, e \in E$

and (E, D) is a complete metric space.

Definition 2.4 ([19]). Let $f : \mathbb{R} \to E$ be a fuzzy-valued function. If for arbitrary fixed $t_0 \in \mathbb{R}$ and $\epsilon > 0$, there exist $\delta > 0$ such that $|t - t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \epsilon$. f is said to be continuous.

Suppose that $F : M(\subset \mathbb{R}^n) \to E$, F is called a fuzzy-valued function. For each $r \in [0,1]$, r-cuts of F is defined by $[F(x)]^r = [F(x,r)_*, F(x,r)^*]$.

Definition 2.5 ([36]). Let $F : M(\subset \mathbb{R}^n) \to E$ be a fuzzy-valued function and $x_0 \in M$. If for $y \in \mathbb{R}^n$, there exists $\delta > 0$ such that $x_0 + hy, x_0 - hy \in M$ and the H-differences $F(x_0 + hy) - F(x_0)$ and $F(x_0) - F(x_0 - hy)$ exist for any real number $h \in (0, \delta)$, and there exists $D_y F(x_0) \in E$ such that

$$\begin{aligned} &\lim_{h\to 0^+} \tfrac{F(x_0+hy) - F(x_0)}{h} \\ &= \lim_{h\to 0^+} \tfrac{F(x_0) - F(x_0-hy)}{h} \\ &= D_y F(x_0). \end{aligned}$$

Then we say F to be H-differentiable in the direction y at x_0 and call $D_y F(x_0)$ the H-derivative of F at x_0 in the direction y. (Here h in denominator means $\frac{1}{h}\odot$.)

Definition 2.6 ([36]). Let $F : M(\subset \mathbb{R}^n) \to (E, D)$ be a fuzzy-valued function, $x \in M$. If F is the strongly generalized directional differentiable in the direction e_i at x, then we say that F is generalization of partially differentiable at x with respect to the i-th component, call $F_{x_i}(x)$ i.e. strongly generalized partial derivative of F at x with respect to the i-th component.

Finally, we define the strongly generalized partial derivative for fuzzy-valued function F from $M(\subset \mathbb{R}^2)$ into E.

Definition 2.7 ([9, 36]). Let $F : M(\subset \mathbb{R}^2) \to E$. We say that F is the strongly generalized partial differentiable at (t_0, x_0) with respect to t, if there exists an element $F_t(t_0, x_0) \in E$ such that

(1) for all $h \in (0, \delta)$ sufficiently small, there exist $(t_0 + h, x_0), (t_0 - h, x_0) \in M$, $F(t_0 + h, x_0) - F(t_0, x_0)$, $F(t_0, x_0) - F(t_0 - h, x_0)$ and the limits (in the metric D)

$$\begin{aligned} &\lim_{h\to 0^+} \tfrac{F(t_0+h,x_0)-F(t_0,x_0)}{h} \\ &= \lim_{h\to 0^+} \tfrac{F(t_0,x_0)-F(t_0-h,x_0)}{h} \\ &= F_t(t_0, x_0), \end{aligned}$$

or

(2) for all $h \in (0, \delta)$ sufficiently small, there exist $(t_0 + h, x_0), (t_0 - h, x_0) \in M$, $F(t_0, x_0) - F(t_0 + h, x_0)$, $F(t_0 - h, x_0) - F(t_0, x_0)$ and the limits

$$\begin{aligned} &\lim_{h\to 0^+} \tfrac{F(t_0,x_0)-F(t_0+h,x_0)}{(-h)} \\ &= \lim_{h\to 0^+} \tfrac{F(t_0-h,x_0)-F(t_0,x_0)}{(-h)} \\ &= F_t(t_0, x_0), \end{aligned}$$

or

(3) for all $h \in (0, \delta)$ sufficiently small, there exist $(t_0 + h, x_0), (t_0 - h, x_0) \in M$, $F(t_0 + h, x_0) - F(t_0, x_0)$, $F(t_0 - h, x_0) - F(t_0, x_0)$ and the limits

$$\begin{aligned} &\lim_{h\to 0^+} \tfrac{F(t_0+h,x_0)-F(t_0,x_0)}{h} \\ &= \lim_{h\to 0^+} \tfrac{F(t_0-h,x_0)-F(t_0,x_0)}{(-h)} \\ &= F_t(t_0, x_0), \end{aligned}$$

or

(4) for all $h \in (0, \delta)$ sufficiently small, there exist $(t_0 + h, x_0), (t_0 - h, x_0) \in M$, $F(t_0, x_0) - F(t_0 + h, x_0)$, $F(t_0, x_0) - F(t_0 - h, x_0)$ and the limits

$$\begin{aligned} &\lim_{h\to 0^+} \tfrac{F(t_0,x_0)-F(t_0+h,x_0)}{(-h)} \\ &= \lim_{h\to 0^+} \tfrac{F(t_0,x_0)-F(t_0-h,x_0)}{h} \\ &= F_t(t_0, x_0). \end{aligned}$$

(h and $(-h)$ in denominators mean $\frac{1}{h}\odot$ and $-\frac{1}{h}\odot$, respectively.)

Remark 2.8. The Definition 2.7 can be written with respect to the variable x.

Theorem 2.9 ([14]). *Let $F : M \to E$ be a function and denote* $[F(t, x)]^r = [F(t, x, r)_*, F(t, x, r)^*]$ *for each $r \in [0, 1]$. Then*

(1) *If F is (1)-differentiable with respect to t, $F(t,x,r)_*$ and $F(t,x,r)^*$ are differentiable functions which are the lower and upper functions of fuzzy-valued function F in the parametric form and* $[F_t(t,x)]^r = [F_t(t,x,r)_*, F_t(t,x,r)^*]$.

(2) *If F is (2)-differentiable with respect to t, $F(t,x,r)_*$ and $F(t,x,r)^*$ are differentiable functions and* $[F_t(t,x)]^r = [F_t(t,x,r)^*, F_t(t,x,r)_*]$.

Remark 2.10. Theorem 2.9 can be written with respect to the variable x.

Theorem 2.11 ([37])**.** *Let $F(t,x)$ be a fuzzy-valued function on M represented by $(F(t,x,r)_*, F(t,x,r)^*)$. For any fixed $r \in [0,1]$, assume $F(t,x,r)_*$ and $F(t,x,r)^*$ are Riemann-integrable with respect to t on $[t_0,t_1]$ for every $t_1 \geq t_0$, and assume there are two positive functions $N(r)_*$ and $N(r)^*$ such that $\int_{t_0}^{t_1} |F(t,x,r)_*|dt \leq N(r)_*$ and $\int_{t_0}^{t_1} |F(t,x,r)^*|dt \leq N(r)^*$ for every $t_1 \geq t_0$. Then $F(t,x)$ is improper fuzzy Riemann- integrable with respect to t on $[t_0,+\infty]$ and the improper fuzzy Riemann-integral is a fuzzy number. Furthermore, we have*

$$\int_{t_0}^{\infty} F(t,x)dt = (\int_{t_0}^{\infty} F(t,x,r)_* dt, \int_{t_0}^{\infty} F(t,x,r)^* dt).$$

Remark 2.12. Theorem 2.11 can be written with respect to the variable x.

Definition 2.13 ([5])**.** Let $F : M \to E$ and $(t_0,x_0) \in M$. We define the second-order differential of F as follows: we say that F is strongly generalized differentiable of the second-order at (t_0,x_0), if there exists an element $F_{tt}(t_0,x_0) \in E$, such that

(1) for all $h \in (0,\delta)$ sufficiently small, there exist $(t_0+h,x_0),(t_0-h,x_0) \in M$, $F_t(t_0+h,x_0) - F_t(t_0,x_0)$, $F_t(t_0,x_0) - F_t(t_0-h,x_0)$ and the limits (in the metric D)

$$\begin{aligned} &\lim_{h\to 0^+} \tfrac{F_t(t_0+h,x_0)-F_t(t_0,x_0)}{h} \\ &= \lim_{h\to 0^+} \tfrac{F_t(t_0,x_0)-F_t(t_0-h,x_0)}{h} \\ &= F_{tt}(t_0,x_0), \end{aligned}$$

or

(2) for all $h \in (0,\delta)$ sufficiently small, there exist $(t_0+h,x_0),(t_0-h,x_0) \in M$, $F_t(t_0,x_0) - F_t(t_0+h,x_0)$, $F_t(t_0-h,x_0) - F_t(t_0,x_0)$ and the limits

$$\begin{aligned} &\lim_{h\to 0^+} \tfrac{F_t(t_0,x_0)-F_t(t_0+h,x_0)}{(-h)} \\ &= \lim_{h\to 0^+} \tfrac{F_t(t_0-h,x_0)-F_t(t_0,x_0)}{(-h)} \\ &= F_{tt}(t_0,x_0), \end{aligned}$$

or

(3) for all $h \in (0,\delta)$ sufficiently small, there exist $(t_0+h,x_0),(t_0-h,x_0) \in M$, $F_t(t_0+h,x_0) - F_t(t_0,x_0)$, $F_t(t_0-h,x_0) - F_t(t_0,x_0)$ and the limits

$$\begin{aligned} &\lim_{h\to 0^+} \tfrac{F_t(t_0+h,x_0)-F_t(t_0,x_0)}{h} \\ &= \lim_{h\to 0^+} \tfrac{F_t(t_0-h,x_0)-F_t(t_0,x_0)}{(-h)} \\ &= F_{tt}(t_0,x_0), \end{aligned}$$

or

(4) for all $h \in (0, \delta)$ sufficiently small, there exist $(t_0 + h, x_0), (t_0 - h, x_0) \in M$, $F_t(t_0, x_0) - F_t(t_0 + h, x_0)$, $F_t(t_0, x_0) - F_t(t_0 - h, x_0)$ and the limits

$$\begin{aligned} &\lim_{h\to 0^+} \tfrac{F_t(t_0,x_0) - F_t(t_0+h,x_0)}{(-h)} \\ &= \lim_{h\to 0^+} \tfrac{F_t(t_0,x_0) - F_t(t_0-h,x_0)}{h} \\ &= F_{tt}(t_0, x_0). \end{aligned}$$

(h and $(-h)$ in denominators mean $\frac{1}{h}\odot$ and $-\frac{1}{h}\odot$, respectively.)

Remark 2.14. The Definition 2.13 can be written with respect to the variable x.

Theorem 2.15 ([32])**.** *Let $F(t, x)$ and $F_t(t, x)$ are two differentiable fuzzy-valued functions. Moreover, we denote r-cut representation fuzzy-valued function $F(t, x)$ with $[F(t, x)]^r = [F(t, x, r)_*, F(t, x, r)^*]$. Then*

(1) Let $F(t, x)$ and $F_t(t, x)$ be (1)-differentiable with respect to t, or, let $F(t, x)$ and $F_t(t, x)$ be (2)-differentiable with respect to t. Then, $F(t, x, r)_*$ and $F(t, x, r)^*$ have first-order and second-order derivatives and $[F_{tt}(t, x)]^r = [F_{tt}(t, x, r)_*, F_{tt}(t, x, r)^*]$.

(2) Let $F(t, x)$ be (1)-differentiable with respect to t and $F_t(t, x)$ be (2)-differentiable with respect to t, or, let $F(t, x)$ be (2)-differentiable with respect to t and $F_t(t, x)$ be (1)-differentiable with respect to t. Then, $F(t, x, r)_*$ and $F(t, x, r)^*$ have first-order and second-order derivatives and $[F_{tt}(t, x)]^r = [F_{tt}(t, x, r)^*, F_{tt}(t, x, r)_*]$.

Remark 2.16. Theorem 2.15 can be written with respect to the variable x.

In results, in the same way can be defined higher-order derivatives.

3. Adomian decomposition method

Consider the crisp partial differential equation as

$$L_t u(t, x) = \rho(t, x, L_x)u(t, x), \tag{3.1}$$

subject to

$$u(t_0, x) = q(x), \tag{3.2}$$

where $L_t = \frac{\partial}{\partial t}$ and $L_x = \frac{\partial}{\partial x}$ and $(t, x) \in M = [t_0, +\infty)\times\mathbb{R}$ with $t_0 \geq 0$. The operator $\rho(t, x, L_x)$ will be a polynomial, with continuous variable coefficient respect to t and x on M. The L_x, $L_x(L_x) = L_{xx}$ and so on, denote the order of partial derivative with respect to x. Also, $q : \mathbb{R} \to \mathbb{R}$ is a known continuous function.

Let us formally define the inverse integral operator

$$L_t^{-1} = \int_{t_0}^{t} ds.$$

Applying the inverse operator L_t^{-1} to the Eq. (3.1), and using the initial condition (3.2) yields

$$u(t, x) = q(x) + \int_{t_0}^{t} \rho(s, x, L_x)u(s, x)ds. \tag{3.3}$$

The linear terms $u(t,x)$ can be decomposed by an infinite series of components

$$u(t,x) = \sum_{n=0}^{+\infty} u_n(t,x). \tag{3.4}$$

Because of not exist nonlinear terms in the Eq. (3.1), therefore isn't expressed the infinite series of the so-called Adomian polynomials. Now by replacing (3.4) onto (3.3) we will have

$$\sum_{n=0}^{+\infty} u_n(t,x) = q(x) + \int_{t_0}^{t} \rho(s,x,L_x) \sum_{n=0}^{+\infty} u_n(s,x)ds. \tag{3.5}$$

Following Adomian analysis, Adomian decomposition method uses the recursive relations

$$\begin{aligned} &u_0(t,x) = q(x), \\ &u_n(t,x) = \textstyle\int_{t_0}^{t} \rho(s,x,L_x)u_{n-1}(s,x)ds, \quad n \geq 1. \end{aligned} \tag{3.6}$$

We assume $\varphi_n(t,x) = \sum_{i=0}^{n} u_i(t,x)$, obviously we have

$$u(t,x) = \lim_{n\to+\infty} \varphi_n(t,x),$$

therefore we rewrite successive iterations (3.6) as follows

$$\begin{aligned} &\varphi_0(t,x) = q(x), \\ &\varphi_{n+1}(t,x) = q(x) + \textstyle\sum_{i=1}^{n+1} \int_{t_0}^{t} \rho(s,x,L_x)u_{i-1}(s,x)ds, \\ &n = 0, 1, 2, ... \end{aligned} \tag{3.7}$$

4. Fuzzy partial differential equation

In this section we discuss about existence and uniqueness of solution of a fuzzy partial differential equation by the successive iterations Adomian decomposition method. In this section $\sum$ means the sum of fuzzy numbers for each (t,x) in domain.

Consider the fuzzy partial differential equation

$$u_t(t,x) = \rho(t,x,L_x)u(t,x), \tag{4.1}$$

subject to

$$u(t_0,x) = f(x), \tag{4.2}$$

where $L_x = \frac{\partial}{\partial x}$ and $(t,x) \in M = [t_0,+\infty) \times \mathbb{R}$ with $t_0 \geq 0$. The operator $\rho(t,x,L_x)$ will be a polynomial, with continuous variable coefficient respect to t and x on M. The L_x, $L_x(L_x) = L_{xx}$ and so on, denote the order of partial derivative with respect to x. Also $f : \mathbb{R} \to E$ and $u : M \to E$ are continuous fuzzy-valued functions where f is strongly generalized differentiable in the sense of Definition 5 in [9] and u is strongly generalized partial differentiable with respect to t and x.

Theorem 4.1. *Let us suppose the following conditions hold:*
(a) *$f : \mathbb{R} \to E$ be a continuous and bounded function.*
(b) *There exist $\gamma > 0$, $\beta > 1$ and $e^{-\beta t_0}\gamma \leq 1$ such that*

$$D(\rho(t,x,L_x)u(t,x), \rho(t,x,L_x)v(t,x)) \leq \gamma e^{-\beta t} D(u,v), \tag{4.3}$$

and $\rho(t,x,L_x)u(t,x)$ and $\rho(t,x,L_x)v(t,x)$ are continuous.
Then the fuzzy partial differential equation (4.1) with the fuzzy initial condition (4.2) has two solutions (one (1)-differentiable and the other one (2)-differentiable) $u,\overline{u}: M\rightarrow E$ with respect to t and the successive iterations

$$\begin{aligned}&\varphi_0(t,x)=f(x),\\ &\varphi_{n+1}(t,x)=f(x)\oplus\textstyle\sum_{i=1}^{n+1}\int_{t_0}^{t}\rho(s,x,L_x)u_{i-1}(s,x)ds,\\ &(n=0,1,2,...),\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}&\overline{\varphi}_0(t,x)=f(x),\\ &\overline{\varphi}_{n+1}(t,x)=f(x)-(-1)\odot\\ &\quad\textstyle\sum_{i=1}^{n+1}\int_{t_0}^{t}\rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds,\ (n=0,1,...),\end{aligned}\tag{4.5}$$

uniformly convergent to these two solutions, respectively.

Proof. The case (1)-differentiable is obtained as the case (2)-differentiable and is omitted. To prove the case (2)-differentiable, by the invariance to translation of distance D and the hypotheses for uniform convergence of the sequence $\{\overline{\varphi}_n(t,x)\}$ we have

$$\begin{aligned}&D(\overline{\varphi}_{n+1}(t,x),\overline{\varphi}_n(t,x))\\ &=D(f(x)-\overline{\varphi}_{n+1}(t,x),f(x)-\overline{\varphi}_n(t,x))\\ &=D((-1)\odot\textstyle\sum_{i=1}^{n+1}\int_{t_0}^{t}\rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds,\\ &\quad(-1)\odot\textstyle\sum_{i=1}^{n}\int_{t_0}^{t}\rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds)\\ &=D((-1)\odot\textstyle\sum_{i=1}^{n}\int_{t_0}^{t}\rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds+\\ &\quad(-1)\odot\int_{t_0}^{t}\rho(s,x,L_x)\overline{u}_{n}(s,x)ds,(-1)\\ &\quad\odot\textstyle\sum_{i=1}^{n}\int_{t_0}^{t}\rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds)\\ &=D((-1)\odot\int_{t_0}^{t}\rho(s,x,L_x)\overline{u}_{n}(s,x)ds,\tilde{0})\\ &\le\int_{t_0}^{t}D(\rho(s,x,L_x)\overline{u}_{n}(s,x),\tilde{0})ds\\ &\le\gamma\sup_{(t,x)\in M}D(\overline{u}_n(t,x),\tilde{0})\int_{t_0}^{t}e^{-\beta s}ds\\ &\le\tfrac{\gamma e^{-\beta t_0}}{\beta}\sup_{(t,x)\in M}D(\overline{u}_n(t,x),\tilde{0})\\ &\le\tfrac{1}{\beta}\sup_{(t,x)\in M}D(\overline{u}_n(t,x),\tilde{0}),\end{aligned}$$

in result, we have

$$D(\overline{\varphi}_{n+1}(t,x),\overline{\varphi}_n(t,x))\le\frac{1}{\beta}\sup_{(t,x)\in M}D(\overline{u}_n(t,x),\tilde{0}).\tag{4.6}$$

On the other hand, from (3.6) we can obtain for $n \geq 1$,

$$\begin{aligned}
&D(\overline{u}_n(t,x),\tilde{0})\\
&= D((-1)\odot \textstyle\int_{t_0}^{t} \varphi(s,x,L_x)\overline{u}_{n-1}(s,x)ds,\tilde{0})\\
&\leq \textstyle\int_{t_0}^{t} D(\varphi(s,x,L_x)\overline{u}_{n-1}(s,x),\tilde{0})ds\\
&\leq \tfrac{\gamma e^{-\beta t_0}}{\beta}\sup_{(t,x)\in M} D(\overline{u}_{n-1}(t,x),\tilde{0})\\
&\leq \tfrac{1}{\beta}\sup_{(t,x)\in M} D(\overline{u}_{n-1}(t,x),\tilde{0})\\
&\qquad\vdots\\
&\leq \tfrac{1}{\beta^n}\sup_{(t,x)\in M} D(\overline{u}_0(t,x),\tilde{0}) = \tfrac{1}{\beta^n}\sup_{x\in\mathbb{R}} D(f(x),\tilde{0}).
\end{aligned}$$

Thus, we have

$$\sup_{(t,x)\in M} D(\overline{u}_n(t,x),\tilde{0}) \leq \frac{Q}{\beta^n}, \tag{4.7}$$

where $Q = \sup_{x\in\mathbb{R}} D(f(x),\tilde{0})$. In result, from (4.6) and (4.7), we get

$$\sup_{(t,x)\in M} D(\overline{\varphi}_{n+1}(t,x),\overline{\varphi}_n(t,x)) \leq \frac{Q}{\beta^{n+1}},$$

which denotes the series $\frac{Q}{\beta}\sum_{n=0}^{+\infty}\frac{1}{\beta^n}$ is convergent. So the series

$$\sum_{n=0}^{+\infty} D(\overline{\varphi}_{n+1}(t,x),\overline{\varphi}_n(t,x)),$$

is uniformly convergent on M. If we show $\overline{u}(t,x) = \lim_{n\to+\infty}\overline{\varphi}_n(t,x)$, then $\overline{u}(t,x)$ satisfies (4.1).

To prove the uniqueness of solution by $\overline{\varphi}_n(t,x)$, assume $\overline{u}(t,x)$ and $\overline{v}(t,x)$ be two solutions of (4.1) on M. Then

$$\begin{aligned}
0 &\leq D(\overline{u}(t,x),\overline{v}(t,x))\\
&= D(\overline{u}(t,x)+\overline{\varphi}_n(t,x),\overline{v}(t,x)+\overline{\varphi}_n(t,x))\\
&\leq D(\overline{u}(t,x),\overline{\varphi}_n(t,x)) + D(\overline{v}(t,x),\overline{\varphi}_n(t,x)).
\end{aligned}$$

Since $\overline{\varphi}_n(t,x)$ is convergent to solution of (4.1),

$$D(\overline{u}(t,x),\overline{\varphi}_n(t,x)) \to 0,$$

$$D(\overline{v}(t,x),\overline{\varphi}_n(t,x)) \to 0,$$

when $n \to +\infty$. Thus $D(\overline{u}(t,x),\overline{v}(t,x)) = 0$, i.e., $\overline{u}(t,x) = \overline{v}(t,x)$.

Let $t_0 \leq t < t+h < +\infty$, we observe that

$$\begin{aligned}
&\overline{\varphi}_{n+1}(t,x) - \overline{\varphi}_{n+1}(t+h,x)\\
&= (-1)\odot \textstyle\sum_{i=1}^{n+1}\int_{t}^{t+h} \rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds.
\end{aligned} \tag{4.8}$$

Indeed, we have by direct computation

$$\overline{\varphi}_{n+1}(t+h,x) \oplus (-1) \odot \sum_{i=1}^{n+1} \int_t^{t+h} \rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds$$
$$= f(x) - (-1) \odot \sum_{i=1}^{n+1} \int_{t_0}^{t+h} \rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds \oplus$$
$$(-1) \odot \sum_{i=1}^{n+1} \int_t^{t+h} \rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds$$
$$= f(x) - (-1) \odot \sum_{i=1}^{n+1} \int_{t_0}^{t+h} \rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds \oplus$$
$$(-1) \odot \sum_{i=1}^{n+1} \int_{t_0}^{t+h} \rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds$$
$$- (-1) \odot \sum_{i=1}^{n+1} \int_{t_0}^{t} \rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds$$
$$= \overline{\varphi}_{n+1}(t,x).$$

With multiplying $\frac{1}{(-h)}$ and passing to limit with $h \to 0^+$ we have by Definition 2.7,

$$\lim_{h\to 0^+} \tfrac{\overline{\varphi}_{n+1}(t,x) - \overline{\varphi}_{n+1}(t+h,x)}{(-h)}$$
$$= \lim_{h\to 0^+} \tfrac{1}{h} \odot \textstyle\sum_{i=1}^{n+1} \int_t^{t+h} \rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds$$
$$= \lim_{h\to 0^+} \tfrac{1}{h} \odot \textstyle\sum_{i=0}^{n} \int_t^{t+h} \rho(s,x,L_x)\overline{u}_{i}(s,x)ds.$$

By $\overline{\varphi}_n(t,x) = \sum_{i=0}^{n} \overline{u}_i(t,x)$ we observe that

$$D(\tfrac{1}{h} \odot \textstyle\sum_{i=0}^{n} \int_t^{t+h} \rho(s,x,L_x)\overline{u}_i(s,x)ds, \rho(t,x,L_x)\overline{\varphi}_n(t,x))$$
$$= D(\tfrac{1}{h} \odot \textstyle\int_t^{t+h} \rho(s,x,L_x)\overline{\varphi}_n(s,x)ds, \rho(t,x,L_x)\overline{\varphi}_n(t,x))$$
$$= D(\tfrac{1}{h} \odot \textstyle\int_t^{t+h} \rho(s,x,L_x)\overline{\varphi}_n(s,x)ds,$$
$$\tfrac{1}{h} \odot \textstyle\int_t^{t+h} \rho(t,x,L_x)\overline{\varphi}_n(t,x)ds)$$
$$\leq \tfrac{1}{h} \odot \textstyle\int_t^{t+h} D(\rho(s,x,L_x)\overline{\varphi}_n(s,x), \rho(t,x,L_x)\overline{\varphi}_n(t,x))ds$$
$$\leq \sup_{|s-t|\leq h} D(\rho(s,x,L_x)\overline{\varphi}_n(s,x), \rho(t,x,L_x)\overline{\varphi}_n(t,x)),$$

and thus for $h \to 0^+$ the last term $\searrow 0^+$ where means that

$$\lim_{h\to 0^+} \frac{\overline{\varphi}_{n+1}(t,x) - \overline{\varphi}_{n+1}(t+h,x)}{(-h)} = \rho(t,x,L_x)\overline{\varphi}_n(t,x).$$

Analogous (4.8) we can obtain

$$\overline{\varphi}_{n+1}(t-h,x) - \overline{\varphi}_{n+1}(t,x)$$
$$= (-1) \odot \textstyle\sum_{i=1}^{n+1} \int_{t-h}^{t} \rho(s,x,L_x)\overline{u}_{i-1}(s,x)ds,$$

where by similar reasonings leads to

$$\lim_{h\to 0^+}\frac{\overline{\varphi}_{n+1}(t-h,x)-\overline{\varphi}_{n+1}(t,x)}{(-h)}=\rho(t,x,L_x)\overline{\varphi}_n(t,x).$$

Finally, it follows that $\overline{\varphi}_{n+1}(t,x)$ is (2)-differentiable with respect to t and

$$(\overline{\varphi}_{n+1}(t,x))_t=\rho(t,x,L_x)\overline{\varphi}_n(t,x),\ \forall(t,x)\in M,$$

where this completes the proof of theorem. □

According to Theorem 4.1, we restrict our attention to functions which are (1) or (2) differentiable on their domain except on a finite number of points.

Remark 4.2. In the Theorem 4.1 was introduced two solutions of the fuzzy differential eequation (4.1) with respect to t. In the right hand of Eq. (4.1) exists derivative with respect to x, that by attention to order derivative with respect to x are added several states other to solutions. In particularly,

I: if the Eq. (4.1) be as the following

$$u_t=\rho(t,x)u,$$

then it has two solutions by Theorem 2.9,

II: if the Eq. (4.1) be as the following

$$u_t=\rho(t,x)u_{xx},$$

then it has four solutions by Theorem 2.9 and Theorem 2.15,

III: and so on states.

Notice that, all solutions or some of them may be valid [25].

Lemma 4.3. *If the conditions of Theorem 4.1 hold and*

$$\overline{u}_0(t,x)=f(x),$$
$$\overline{u}_n(t,x)=(-1)\odot\int_{t_0}^t\rho(s,x,L_x)\overline{u}_{n-1}(s,x)ds,\ (n\geq 1),$$

then

(1) $\overline{u}_n(t,x)$ *is bounded on* M,
(2) $\overline{u}_n(t,x)$ *is continuous on* M.

Proof. (1) By the hypothesis, $\overline{u}_0(t,x)=f(x)$ is bounded. Assume $\overline{u}_{n-1}(t,x)$ is bounded. By Theorem 4.1(b), we observe that

$$\begin{aligned}&D(\overline{u}_n(t,x),\tilde{0})\\&=D((-1)\odot\int_{t_0}^t\rho(s,x,L_x)\overline{u}_{n-1}(s,x)ds,\tilde{0})\\&\leq\int_{t_0}^t D(\rho(s,x,L_x)\overline{u}_{n-1}(s,x),\tilde{0})ds\\&\leq\tfrac{1}{\beta}\sup_{(t,x)\in M}D(\overline{u}_{n-1}(t,x),\tilde{0}),\end{aligned}$$

and by induction $\overline{u}_n(t,x)$ is bounded on M.

(2) Suppose $t_0 < t \leq \hat{t} < +\infty$ and $-\infty < x \leq \hat{x} < +\infty$. Then, we have

$$\begin{aligned}
&D(\overline{u}_n(t,x), \overline{u}_n(\hat{t},\hat{x})) \\
&= D((-1) \odot \int_{t_0}^{t} \rho(s,x,L_x)\overline{u}_{n-1}(s,x)ds, \\
&\quad (-1) \odot \int_{t_0}^{\hat{t}} \rho(s,\hat{x},L_{\hat{x}})\overline{u}_{n-1}(s,\hat{x})ds) \\
&= D(\int_{t_0}^{t} \rho(s,x,L_x)\overline{u}_{n-1}(s,x)ds, \int_{t_0}^{t} \rho(s,\hat{x},L_{\hat{x}})\overline{u}_{n-1}(s,\hat{x})ds \\
&\quad + \int_{t}^{\hat{t}} \rho(s,\hat{x},L_{\hat{x}})\overline{u}_{n-1}(s,\hat{x})ds) \\
&\leq D(\int_{t_0}^{t} \rho(s,x,L_x)\overline{u}_{n-1}(s,x)ds, \int_{t_0}^{t} \rho(s,\hat{x},L_{\hat{x}})\overline{u}_{n-1}(s,\hat{x})ds) \\
&\quad + D(\int_{t}^{\hat{t}} \rho(s,\hat{x},L_{\hat{x}})\overline{u}_{n-1}(s,\hat{x})ds, \tilde{0}) \\
&\leq \int_{t_0}^{t} D(\rho(s,x,L_x)\overline{u}_{n-1}(s,x), \rho(s,\hat{x},L_{\hat{x}})\overline{u}_{n-1}(s,\hat{x}))ds \\
&\quad + \int_{t}^{\hat{t}} D(\rho(s,\hat{x},L_{\hat{x}})\overline{u}_{n-1}(s,\hat{x}), \tilde{0})ds \\
&\leq (t-t_0)\sup_{x,\hat{x}\in\mathbb{R}, t\in[t_0,+\infty)} \\
&\qquad D(\rho(t,x,L_x)\overline{u}_{n-1}(t,x), \rho(t,\hat{x},L_{\hat{x}})\overline{u}_{n-1}(t,\hat{x})) \\
&\quad + \gamma \sup_{\hat{x}\in\mathbb{R}, t\in[t_0,+\infty)} D(\overline{u}_{n-1}(t,\hat{x}), \tilde{0}) \int_{t}^{\hat{t}} e^{-\beta s} ds.
\end{aligned}$$

Thus, we obtain

$$D(\overline{u}_n(t,x), \overline{u}(\hat{t},\hat{x})) \to 0 \quad as \quad (t,x) \to (\hat{t},\hat{x}).$$

So $\overline{u}_n(t,x)$ is continuous on M. □

Lemma 4.4. *If the conditions of Theorem 4.1 hold and*

$$\begin{aligned}
&u_0(t,x) = f(x), \\
&u_n(t,x) = \int_{t_0}^{t} \rho(s,x,L_x)u_{n-1}(s,x)ds, \ (n \geq 1),
\end{aligned}$$

then

(1) $u_n(t,x)$ *is bounded on* M,
(2) $u_n(t,x)$ *is continuous on* M.

Proof. It is an immediately consequence of Lemma 4.3. □

Theorem 4.5. *If the conditions of Theorem 4.1 hold, then* $\overline{\varphi}_{n+1}(t,x)$ *and* $\varphi_{n+1}(t,x)$ *are bounded and continuous on* M.

Proof. It is an immediately consequence of Lemma 4.3 and Lemma 4.4, respectively. □

Theorem 4.6. *If the conditions of Theorem 4.1 hold, then* $u(t,x), \overline{u}(t,x) \in E$ *for each* $(t,x) \in M$.

Proof. In the same way Theorem 3.1 in [33], we can prove $u(t,x), \overline{u}(t,x) \in E$ for each $(t,x) \in M$. □

Theorem 4.7. *If the conditions of Theorem 4.1 hold, then the largest interval of existence of any fuzzy solution $u(t,x)$ of (4.1) is M and the limit*

$$\lim_{x\to\pm\infty}\lim_{t\to+\infty} u(t,x) = \xi \in E,$$

exists. In the same way, we can also present for $\overline{u}(t,x)$.

Proof. By Theorem 4.5, $u(t,x)$ is continuous and bounded on M. Then the limit

$$[\xi_*(r),\xi^*(r)] = [\lim_{x\to\pm\infty}\lim_{t\to+\infty} u(t,x)_*(r),$$
$$\lim_{x\to\pm\infty}\lim_{t\to+\infty} u(t,x)^*(r)],$$

exists. On the other hand by Theorem 4.6, we can obtain that the intervals

$$[\xi_*(r),\xi^*(r)],\ 0 < r \le 1,$$

define a fuzzy number $\xi \in E$. $\overline{u}(t,x)$ can similarly be proved. □

Theorem 4.8. *Assume the following conditions hold:*

(a) *$f : \mathbb{R} \to E$ be a continuous and bounded function.*

(b) *There exist $\gamma > 0$, $0 < \beta \le 1$ and $e^{-\beta t_0}\gamma \le \frac{\beta^2}{2}$ such that*

$$D(\rho(t,x,L_x)u(t,x), \rho(t,x,L_x)v(t,x)) \le \gamma e^{-\beta t} D(u,v), \tag{4.9}$$

and $\rho(t,x,L_x)u(t,x)$ and $\rho(t,x,L_x)v(t,x)$ are continuous. Then the fuzzy partial differential equation (4.1) with the fuzzy initial condition (4.2) has two solutions (one (1)-differentiable and the other one (2)-differentiable) $u,\overline{u} : M \to E$ with respect to t and the successive iterations (4.4) and (4.5) uniformly convergent to these two solutions, respectively.

Proof. Proving of theorem is similar to proving of Theorem 4.1. □

Remark 4.9. If we consider the conditions of Theorem 4.8 instead of Theorem 4.1, we obtain that the results of Lemma 4.3, Lemma 4.4, Theorem 4.5, Theorem 4.6 and Theorem 4.7 are hold.

5. Example

In this section we denote application by two examples of fuzzy patrial differential equations using fuzzy-valued functions.

Example 5.1. Let us consider the fuzzy partial differential equation

$$\begin{aligned} u_t(t,x) &= (-1)\odot u(t,x), \\ u(t_0,x) &= f(x),\ \ x\in\mathbb{R},\ t\ge t_0,\ t_0\ge 0, \end{aligned} \tag{5.1}$$

where $u : [t_0,+\infty)\times\mathbb{R} \to E$, the $f : \mathbb{R} \to E$ is the strongly generalized differentiable [9] and bounded.

Case I: assume u be (1)-differentiable with respect to t. Therefore, the solution of Eq. (5.1) is obtained by Adomian decomposition method as the following

$$\begin{aligned} u(t,x) &= [u(t,x,r)_*, u(t,x,r)^*] \\ &= [f(x,r)_* cosht - f(x,r)^* sinht, f(x,r)^* cosht - f(x,r)_* sinht]. \end{aligned}$$

Case II: assume u be (2)-differentiable with respect to t. In this case, if we denote $u(t,x) = e^{-(t-t_0)} \odot f(x)$, then u is the strongly generalized partial differentiable with respect to all t, $u_t(t,x) = (-1)e^{-(t-t_0)} \odot f(x)$ (see [9], and [36]) and $u(t,x)$ satisfies the fuzzy partial differential equation (5.1).

On the other hand, because $e^{-(t-t_0)} > 0$ and $(e^{-(t-t_0)})' < 0$, $\forall t \in (t_0, +\infty)$, we cannot say nothing about the existence of $u_t(t,x)$ in the H-differentiability sense, since there is not exist the H-difference $u(t+h,x) - u(t,x)$, as $u(t,x) \in E \backslash \mathbb{R}$.

This denotes the advantage of the strongly generalized partial differentiability with respect to the usual differentiability.

Example 5.2. Let us consider the fuzzy partial differential equation

$$\begin{aligned} u_t(t,x) &= \alpha \odot u_x(t,x), \\ u(t_0,x) &= \tilde{c} \odot f(x), \ \ x \in \mathbb{R}^+, \, t \geq t_0, \, t_0 \geq 0, \end{aligned} \tag{5.2}$$

where $\tilde{c} \in E$, $\alpha \in (0, +\infty)$, the $f : \mathbb{R} \to \mathbb{R}^+$ is continuous and bounded on $\mathbb{R}^+$.

Case I: assume u be (1)-differentiable with respect to t and x, or (2)-differentiable with respect to t and x. So, the solution of Eq. (5.2) is obtained by Adomian decomposition method as the following

$$u(t,x) = \tilde{c} \odot \sum_{n=0}^{\infty} \frac{\alpha^n t^n}{n!} f^{(n)}(x).$$

Case II: assume u be (1)-differentiable with respect to t and (2)-differentiable with respect to x, or (2)-differentiable with respect to t and (1)-differentiable with respect to x. In this case, we have a system of partial differential equations and solution of fuzzy partial differential equation as follows

$$\begin{aligned} u(t,x) &= [u(t,x,r)_*, u(t,x,r)^*] \\ &= [c(r)_* \textstyle\sum_{n=0}^{\infty} \frac{\alpha^{2n} t^{2n}}{(2n)!} f^{(2n)}(x) + c(r)^* \sum_{n=0}^{\infty} \frac{\alpha^{2n+1} t^{2n+1}}{(2n+1)!} f^{(2n+1)}(x), \\ &\quad c(r)^* \textstyle\sum_{n=0}^{\infty} \frac{\alpha^{2n} t^{2n}}{(2n)!} f^{(2n)}(x) + c(r)_* \sum_{n=0}^{\infty} \frac{\alpha^{2n+1} t^{2n+1}}{(2n+1)!} f^{(2n+1)}(x)]. \end{aligned}$$

6. Conclusions

In this article, we used the concept of strongly generalized directional derivative for fuzzy-valued functions. By this concept, a fuzzy partial differential equation may has several solutions (two solutions locally with respect to t). This is the advantage of existence of the solutions and we can choose the solution that has a better reflex on the behavior of the modelled real-world system.

As an application we represented the fuzzy partial differential equation to the strongly generalized partial derivative and obtained the existence and convergence theorems of solutions for this equations.

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