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Generalized interval valued anti fuzzy ideals in near-rings

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Abstract. In this paper, we introduce the new notion of generalized interval valued anti fuzzy subnear-rings and ideals of near-rings. We have also discussed some theoretical properties of these algebraic structures and obtained some characterizations.

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1. Introduction

Near-rings are generalization of rings. The fundamental concept of fuzzy set was introduced by Zadeh[13]. After a decade the same author[14] initiated the study of interval valued(i-v) fuzzy subsets where the values of membership functions are the intervals instead of numbers. The study of fuzzy subgroups was initiated by Rosenfeld[7]. Biswas[2] introduced the idea of anti fuzzy subgroups. Abou-Zaid[1] discussed the concept of fuzzy subnear-rings and ideals. Saeid et al.[9] introduced the notion of besideness($<$) and non-quasi-coincidence(γ) with fuzzy subsets and anti fuzzy points. Davvaz[3, 4] discussed the idea of i-v fuzzy sets applied to fuzzy ideals of near-rings and generalized fuzzy H_v-submodules and investigated some of their properties. Kim et al.[5] introduced the notion of anti fuzzy ideals of near-rings. Kyung Ho Kim et al.[6] initiated the concept of anti fuzzy R-subgroups of near-rings. Shabir et al.[8] introduced different types of anti fuzzy ideals in ternary semigroups. Recently, Tariq Anwar et al.[10] have introduced the notion of generalized anti fuzzy ideals of near-rings. Thillaigovindan et al.[12] have initiated the study of i-v anti fuzzy ideals of near-rings and gave some characterizations. In this paper, we introduce the concept of i-v $(<, <)$-fuzzy ideals (subnear-rings) and $(<, < \vee\gamma)$-fuzzy ideals (subnear-rings). We have also obtained some characterizations of these ideals.

2. Preliminaries

Throughout this paper R stands for left near-ring unless otherwise specified. In this section we recall some basic definitions and results.

A near-ring is an algebraic system $(R, +, \cdot)$ consisting of a non empty set R together with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group, not necessarily abelian and $(R, \cdot)$ is a semigroup connected by the distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$ valid for all $x, y, z \in R$. We will use the word 'near-ring' to mean 'left near-ring '. We denote xy instead of $x \cdot y$.

An ideal I of a near-ring R is a subset of R such that
(i) $(I, +)$ is a normal subgroup of $(R, +)$,
(ii) $RI \subseteq I$,
(iii) $(x + a)y - xy \in I$, for any $a \in I$ and $x, y \in R$.
Note that I is called a left ideal of R if I satisfies (i) and (ii) and a right ideal of R if I satisfies (i) and (iii).

Definition 2.1 ([4, 11])**.** By an interval number $\overline{a}$, we mean an interval $[a^-, a^+]$ such that $0 \leq a^- \leq a^+ \leq 1$ where a^- and a^+ are the lower and upper limits of $\overline{a}$ respectively. The set of all closed subintervals of $[0, 1]$ is denoted by $D[0, 1]$. We also identify the interval $[a, a]$ by the number $\overline{a} \in D[0, 1]$. For any interval numbers $\overline{a}_j = [a_j^-, a_j^+], \overline{b}_j = [b_j^-, b_j^+] \in D[0, 1], j \in J$, we define

$$\max\{\overline{a}_j, \overline{b}_j\} = [\max\{a_j^-, b_j^-\}, \max\{a_j^+, b_j^+\}],$$
$$\min\{\overline{a}_j, \overline{b}_j\} = [\min\{a_j^-, b_j^-\}, \min\{a_j^+, b_j^+\}],$$
$$\inf \overline{a}_j = \left[\bigcap_{j\in I} a_j^-, \bigcap_{j\in I} a_j^+\right], \sup \overline{a}_j = \left[\bigcup_{j\in I} a_j^-, \bigcup_{j\in I} a_j^+\right]$$

and put
(i) $\overline{a} \leq \overline{b} \Longleftrightarrow a^- \leq b^-$ and $a^+ \leq b^+$,
(ii) $\overline{a} = \overline{b} \Longleftrightarrow a^- = b^-$ and $a^+ = b^+$,
(iii) $\overline{a} < \overline{b} \Longleftrightarrow \overline{a} \leq \overline{b}$ and $\overline{a} \neq \overline{b}$,
(iv) $k\overline{a} = [ka^-, ka^+]$, whenever $0 \leq k \leq 1$.

Definition 2.2. For any two interval numbers, $\overline{a} = [a^-, a^+]$ and $\overline{b} = [b^-, b^+]$ addition, subtraction, multiplication and division are defined as

$$\overline{a} + \overline{b} = \begin{cases} [a^- + b^-, a^+ + b^+] & \text{if } [a^- + b^-, a^+ + b^+] \leq \overline{1} \\ [\max\{a^-, b^-\}, \max\{a^+, b^+\}] & \text{if } [a^- + b^-, a^+ + b^+] > \overline{1}, \end{cases}$$

$$\overline{a} - \overline{b} = \begin{cases} [a^- - b^+, a^+ - b^-] & \text{if } [a^- - b^+, a^+ - b^-] \geq [0, 0] \\ [\min\{a^-, b^-\}, \min\{a^+, b^+\}] & \text{if } [a^- - b^+, a^+ - b^-] < [0, 0], \end{cases}$$

$$\overline{a} \cdot \overline{b} = [\min\{a^- \cdot b^-, a^+ \cdot b^+\}, \max\{a^- \cdot b^-, a^+ \cdot b^+\}],$$

$$\overline{a}/\overline{b} = \begin{cases} [\min(\frac{a^-}{b^-}, \frac{a^+}{b^+}), \max(\frac{a^-}{b^-}, \frac{a^+}{b^+})] & \text{if } \overline{a} \leq \overline{b} \neq [0, 0] \\ \left[\frac{1}{\max(\frac{a^-}{b^-}, \frac{a^+}{b^+})}, \frac{1}{\min(\frac{a^-}{b^-}, \frac{a^+}{b^+})}\right] & \text{if } \overline{a} > \overline{b} \\ \text{Not defined} & \text{if } \overline{a} = \overline{b} = [0, 0]. \end{cases}$$

Definition 2.3 ([11])**.** Let X be a non-empty set. A mapping $\overline{\mu} : X \to D[0,1]$ is called an i-v fuzzy subset of X. For all $x \in X$, $\overline{\mu}(x) = [\mu^-(x), \mu^+(x)]$, where μ^- and μ^+ are fuzzy subsets of X such that $\mu^-(x) \leq \mu^+(x)$. Thus $\overline{\mu}(x)$ is an interval(a closed subinterval of $[0,1]$) and not a number from the interval $[0,1]$ as in the case of a fuzzy set.

Let $\overline{\mu}, \overline{\nu}$ be i-v fuzzy subsets of X. Then
(i) $\overline{\mu} \leq \overline{\nu} \Leftrightarrow \overline{\mu}(x) \leq \overline{\nu}(x)$.
(ii) $\overline{\mu} = \overline{\nu} \Leftrightarrow \overline{\mu}(x) = \overline{\nu}(x)$.
(iii) $(\overline{\mu} \cup \overline{\nu})(x) = \max\{\overline{\mu}(x), \overline{\nu}(x)\}$.
(iv) $(\overline{\mu} \cap \overline{\nu})(x) = \min\{\overline{\mu}(x), \overline{\nu}(x)\}$.
(v) $\overline{\mu}^c(x) = \overline{1} - \overline{\mu}(x) = [1 - \mu^+(x), 1 - \mu^-(x)]$.

Definition 2.4 ([11])**.** Let $\overline{\mu}$ be an i-v fuzzy subset of X and $[t_1, t_2] \in D[0,1]$. Then
$\overline{U}(\overline{\mu} : [t_1, t_2]) = \{x \in X |\ \overline{\mu}(x) \geq [t_1, t_2]\}$ is called the upper level set of $\overline{\mu}$
and
$\overline{L}(\overline{\mu} : [t_1, t_2]) = \{x \in X |\ \overline{\mu}(x) \leq [t_1, t_2]\}$ is called the lower level set of $\overline{\mu}$.

Definition 2.5 ([12])**.** An i-v fuzzy subset $\overline{\mu}$ of a near-ring R is called an i-v anti fuzzy subnear-ring of R, if
(i) $\overline{\mu}(x - y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y)\}$,
(ii) $\overline{\mu}(xy) \leq \max\{\overline{\mu}(x), \overline{\mu}(y)\}$ for all $x, y \in R$.

Definition 2.6 ([12])**.** An i-v fuzzy subset $\overline{\mu}$ of a near-ring R is called an i-v anti fuzzy ideal of R, if $\overline{\mu}$ is an i-v anti fuzzy subnear-ring of R and
(iii) $\overline{\mu}(y + x - y) \leq \overline{\mu}(x)$,
(iv) $\overline{\mu}(xy) \leq \overline{\mu}(y)$,
(v) $\overline{\mu}((x + z)y - xy) \leq \overline{\mu}(z)$ for all $x, y, z \in R$.

Note that $\overline{\mu}$ is an i-v anti fuzzy left ideal of R, if it satisfies (i),(ii),(iii) and (iv), and $\overline{\mu}$ is an i-v anti fuzzy right ideal of R, if it satisfies (i),(ii),(iii) and (v).

3. $(<, < \vee\gamma)$ Fuzzy ideals

In this section, we introduce the concept of i-v $(<, < \vee\gamma)$ fuzzy ideals of near-ring and study some of their theoretical properties.

Definition 3.1. An i-v fuzzy subset $\overline{\mu}$ of R of the form

$$\overline{\mu}(y) = \begin{cases} \overline{s} \in D[0,1] \neq \overline{1} & \text{if } \ y = x \\ \overline{1} & \text{if } \ y \neq x \end{cases}$$

is called an i-v anti fuzzy point with support x and value $\overline{s}$ and is denoted by $x_{\overline{s}}$.

An i-v fuzzy subset $\overline{\mu}$ of X is said to be non unit if there exists $x \in X$ such that $\overline{\mu}(x) < \overline{1}$.

Definition 3.2. An i-v anti fuzzy point $x_{\overline{s}}$ is said to be beside to (resp. be non-quasi coincident with) a fuzzy subset $\overline{\mu}$, denoted by $x_{\overline{s}} < \overline{\mu}$ (resp. $x_{\overline{s}}\gamma\overline{\mu}$), if $\overline{\mu}(x) \leq \overline{s}$ (resp. $\overline{\mu}(x) + \overline{s} < \overline{1}$). We say that $<$ (resp.γ) is a beside to (resp. non-quasi coincident with) relation between i-v anti fuzzy points and i-v fuzzy subsets. If $x_{\overline{s}} < \overline{\mu}$ or $x_{\overline{s}}\gamma\overline{\mu}$, we say that $x_{\overline{s}} < \vee\gamma\overline{\mu}$ and $x_{\overline{s}}\overline{<}\overline{\mu}$ (resp. $x_{\overline{s}}\overline{\gamma}\,\overline{\mu}, x_{\overline{s}}\overline{< \vee\gamma}\overline{\mu}$) means $x_{\overline{s}} < \overline{\mu}$ (resp. $x_{\overline{s}}\gamma\overline{\mu}, x_{\overline{s}} < \vee\gamma\overline{\mu}$) does not hold.

Definition 3.3. An i-v fuzzy subset $\overline{\mu}$ of R is called an i-v $(<,<)$-fuzzy subnear-ring of R, if for all $x, y \in R$ and $\overline{s}, \overline{t} \in D[0,1] \neq \overline{1}$,
(i) $x_{\overline{s}} < \overline{\mu}, y_{\overline{t}} < \overline{\mu} \Rightarrow (x-y)_{\max\{\overline{s},\overline{t}\}} < \overline{\mu}$,
(ii) $x_{\overline{s}} < \overline{\mu}, y_{\overline{t}} < \overline{\mu} \Rightarrow (xy)_{\max\{\overline{s},\overline{t}\}} < \overline{\mu}$.

Definition 3.4. An i-v fuzzy subset $\overline{\mu}$ of R is called an i-v $(<,<)$-fuzzy left (right) ideal of R, if for all $x, y \in R$ and $\overline{s}, \overline{t} \in D[0,1] \neq \overline{1}$,
(i) $x_{\overline{s}} < \overline{\mu}, y_{\overline{t}} < \overline{\mu} \Rightarrow (x-y)_{\max\{\overline{s},\overline{t}\}} < \overline{\mu}$,
(ii) $x_{\overline{s}} < \overline{\mu}, y \in R \Rightarrow (y+x-y)_{\overline{s}} < \overline{\mu}$,
(iii) $y_{\overline{s}} < \overline{\mu}, x \in R \Rightarrow (xy)_{\overline{s}} < \overline{\mu}$, (resp. $a_{\overline{s}} < \overline{\mu}, x, y \in R \Rightarrow ((x+a)y - xy)_{\overline{s}} < \overline{\mu}$).
An i-v fuzzy subset which is $(<,<)$ fuzzy left and right ideal of R is called i-v $(<,<)$ fuzzy ideal of R.

Example 3.5. Let $R = \{0, a, b, c\}$ be a set with two binary operations $'+'$ and $'\cdot'$ defined as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	a	a

Then $(R, +, \cdot)$ is a left near-ring. Let $\overline{\mu} : R \to D[0,1]$ be an i-v fuzzy subset of R defined by $\overline{\mu}(0) = [0.1, 0.2], \overline{\mu}(a) = [0.4, 0.5]$ and $\overline{\mu}(b) = [0.7, 0.8] = \overline{\mu}(c)$. Then $\overline{\mu}$ is an i-v $(<,<)$-fuzzy ideal of R.

The following theorem gives the connection between i-v $(<,<)$-fuzzy ideal and i-v anti fuzzy ideal.

Theorem 3.6. *An i-v fuzzy subset $\overline{\mu}$ of R is an i-v $(<,<)$-fuzzy ideal of R if and only if it is an i-v anti fuzzy ideal of R.*

Proof. Assume that $\overline{\mu}$ is an i-v anti fuzzy ideal of R. Let $x, y \in R$ and $\overline{t}, \overline{r} \in D[0,1]$ with $\overline{t}, \overline{r} \neq \overline{1}$ be such that $x_{\overline{t}}, y_{\overline{r}} < \overline{\mu}$. Then $\overline{\mu}(x) \leq \overline{t}$ and $\overline{\mu}(y) \leq \overline{r}$. Since $\overline{\mu}$ is an i-v anti fuzzy ideal of R, we have $\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y)\} \leq \max\{\overline{t}, \overline{r}\}$. It follows that $(x-y)_{\max\{\overline{t},\overline{r}\}} < \overline{\mu}$. Now let $x, y \in R$ and $\overline{t} \in D[0,1]$ with $\overline{t} \neq \overline{1}$. Then $x_{\overline{t}} < \overline{\mu}$ and thus $\overline{\mu}(x) \leq \overline{t}$. Since $\overline{\mu}$ is an i-v anti fuzzy ideal of R, we have $\overline{\mu}(y+x-y) \leq \overline{\mu}(x) \leq \overline{t}$. So $(y+x-y)_{\overline{t}} < \overline{\mu}$. Let $x, y \in R$ and $\overline{t} \in D[0,1]$ with $\overline{t} \neq \overline{1}$ such that $y_{\overline{t}} < \overline{\mu}$. Then $\overline{\mu}(y) \leq \overline{t}$. Thus $\overline{\mu}(xy) \leq \overline{\mu}(y) \leq \overline{t}$, because $\overline{\mu}$ is an i-v anti fuzzy ideal of R.So $(xy)_{\overline{t}} < \overline{\mu}$. Again let $x, y, z \in R$ and $\overline{t} \in D[0,1]$ with $\overline{t} \neq \overline{1}$ such that $z_{\overline{t}} < \overline{\mu}$. Then $\overline{\mu}(z) \leq \overline{t}$. Since $\overline{\mu}$ is an i-v anti fuzzy ideal of R, $\overline{\mu}((x+z)y - xy) \leq \overline{\mu}(z) \leq \overline{t}$. Thus $((x+z)y - xy)_{\overline{t}} < \overline{\mu}$. Hence $\overline{\mu}$ is an i-v $(<,<)$-fuzzy ideal of R.

Conversely, assume that $\overline{\mu}$ is an i-v $(<,<)$-fuzzy ideal of R. On the contrary, assume that there exist $x, y \in R$ such that $\overline{\mu}(x-y) > \max\{\overline{\mu}(x), \overline{\mu}(y)\}$. Choose $\overline{t}$ such that $\overline{\mu}(x-y) > \overline{t} > \max\{\overline{\mu}(x), \overline{\mu}(y)\}$. Then $x_{\overline{t}}, y_{\overline{t}} < \overline{\mu}$ and $(x-y)_{\overline{t}} \overline{<} \overline{\mu}$. This is a contradiction to our assumption that $\overline{\mu}$ is an i-v $(<,<)$-fuzzy ideal of R. Thus $\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y)\}$. Suppose that $\overline{\mu}(y+x-y) > \overline{\mu}(x)$, for some $x, y \in R$. Choose $\overline{t}$ such that $\overline{\mu}(y+x-y) > \overline{t} > \overline{\mu}(x)$. Then $x_{\overline{t}} < \overline{\mu}$ and $(y+x-y)_{\overline{t}} \overline{<} \overline{\mu}$, which is a contradiction and hence $\overline{\mu}(y+x-y) \leq \overline{\mu}(x)$. Let us assume that $\overline{\mu}(xy) > \overline{\mu}(y)$, for some $x, y \in R$. Then there exist $\bar{t}$ such that $\overline{\mu}(xy) > \bar{t} > \overline{\mu}(y)$. This implies that $y_{\bar{t}} < \overline{\mu}$ but $(xy)_{\bar{t}}\overline{<}\overline{\mu}$. This again contradicts our hypothesis. Thus $\overline{\mu}(xy) \leq \overline{\mu}(y)$. Again assume that there exist $x, y, z \in R$ such that $\overline{\mu}((x+z)y - xy) > \overline{\mu}(z)$. Let $\bar{t}$ be such that $\overline{\mu}((x+z)y - xy) > \bar{t} > \overline{\mu}(z)$. Then $z_{\bar{t}} < \overline{\mu}$ but $((x+z)y - xy)_{\bar{t}}\overline{<}\overline{\mu}$, which is a contradiction and so $\overline{\mu}((x+z)y - xy) \leq \overline{\mu}(z)$. Hence $\overline{\mu}$ is an i-v anti fuzzy ideal of R. □

Definition 3.7. An i-v fuzzy subset $\overline{\mu}$ of R is called an i-v $(<, < \vee\gamma)$-fuzzy subnear-ring of R, if for all $x, y \in R$ and $\bar{s}, \bar{t} \in D[0,1] \neq \bar{1}$,

(i) $x_{\bar{s}} < \overline{\mu}, y_{\bar{t}} < \overline{\mu} \implies (x-y)_{\max\{\bar{s},\bar{t}\}} < \vee\gamma\overline{\mu}$,

(ii) $x_{\bar{s}} < \overline{\mu}, y_{\bar{t}} < \overline{\mu} \implies (xy)_{\max\{\bar{s},\bar{t}\}} < \vee\gamma\overline{\mu}$.

Definition 3.8. An i-v fuzzy subset $\overline{\mu}$ of R is called an i-v $(<, < \vee\gamma)$-fuzzy left (right) ideal of R, if for all $x, y \in R$ and $\bar{s}, \bar{t} \in D[0,1] \neq \bar{1}$,

(i) $x_{\bar{s}} < \overline{\mu}, y_{\bar{t}} < \overline{\mu} \implies (x-y)_{\max\{\bar{s},\bar{t}\}} < \vee\gamma\overline{\mu}$,

(ii) $x_{\bar{s}} < \overline{\mu}, y \in R \implies (y+x-y)_{\bar{s}} < \vee\gamma\overline{\mu}$,

(iii) $y_{\bar{s}} < \overline{\mu}, x \in R \implies (xy)_{\bar{s}} < \vee\gamma\overline{\mu}$,

(resp. $a_{\bar{s}} < \overline{\mu}, x, y \in R \implies ((x+a)y - xy)_{\bar{s}} < \vee\gamma\overline{\mu}$).

An i-v fuzzy subset which is an i-v $(<, < \vee\gamma)$ fuzzy left and right ideal of R is called an i-v $(<, < \vee\gamma)$ fuzzy ideal of R.

Example 3.9. Consider the Example 3.5, it can be verified that $\overline{\mu}$ is an i-v $(<, < \vee\gamma)$-fuzzy ideal of R.

Theorem 3.10. *Let $\overline{\mu}$ be an i-v $(<, < \vee\gamma)$-fuzzy ideal (subnear-ring) of R. Then the set $R_1 = \{x \in R | \overline{\mu}(x) < \bar{1}\} \neq \varnothing$ is an ideal (subnear-ring) of R.*

Proof. Let $x, y \in R_1$. Then $\overline{\mu}(x) < \bar{1}$ and $\overline{\mu}(y) < \bar{1}$. Assume that $x - y \notin R_1$. Then $\overline{\mu}(x-y) = \bar{1}$. Thus $x_{\overline{\mu}(x)} < \overline{\mu}$ and $y_{\overline{\mu}(y)} < \overline{\mu}$ but $(x-y)_{\max\{\overline{\mu}(x),\overline{\mu}(y)\}}\overline{< \vee\gamma}\overline{\mu}$, a contradiction. So $x - y \in R_1$.

Let $x \in R_1$ and $y \in R$. Then $\overline{\mu}(x) < \bar{1}$. Suppose that $y + x - y \notin R_1$. Then $\overline{\mu}(y+x-y) = \bar{1}$. Thus $x_{\overline{\mu}(x)} < \overline{\mu}$ but $(y+x-y)_{\overline{\mu}(x)}\overline{< \vee\gamma}\overline{\mu}$, a contradiction. So $y + x - y \in R_1$.

Let $y \in R_1$ and $x \in R$. Then $\overline{\mu}(y) < \bar{1}$. Suppose that $xy \notin R_1$. Then $\overline{\mu}(xy) = \bar{1}$. Thus $y_{\overline{\mu}(y)} < \overline{\mu}$ but $(xy)_{\overline{\mu}(y)}\overline{< \vee\gamma}\overline{\mu}$, is a contradiction. So $xy \in R_1$ and R_1 is a left ideal of R.

Let $a \in R_1$ and $x, y \in R$. Then $\overline{\mu}(a) < \bar{1}$. Suppose that $((x+a)y - xy) \notin R_1$. Then $\overline{\mu}((x+a)y - xy) = \bar{1}$. Thus $a_{\overline{\mu}(a)} < \overline{\mu}$ but $((x+a)y - xy)_{\overline{\mu}(a)}\overline{< \vee\gamma}\overline{\mu}$, a contradiction. So $\overline{\mu}((x+a)y - xy) < \bar{1}$. Hence $((x+a)y - xy) \in R_1$ and R_1 is a right ideal. □

Theorem 3.11. *Let I be an ideal of R and $\overline{\mu}$ be an i-v fuzzy subset of R such that*

$$\overline{\mu}(x) = \begin{cases} \leq \overline{0.5} & \textit{for all } x \in I \\ \bar{1} & \textit{otherwise.} \end{cases}$$

Then $\overline{\mu}$ is an i-v $(<, < \vee\gamma)$-fuzzy ideal of R.

Proof. Let $x, y \in R$ and $\bar{s}, \bar{t} \in D[0,1] \neq \bar{1}$ be such that $x_{\bar{s}}, y_{\bar{t}} < \overline{\mu}$. Then $\overline{\mu}(x) \leq \bar{s}$ and $\overline{\mu}(y) \leq \bar{t}$. Thus $x, y \in I$ and so $x - y \in I$, i.e., $\overline{\mu}(x-y) \leq \overline{0.5}$. If $\max\{\bar{s}, \bar{t}\} \geq \overline{0.5}$, then $\overline{\mu}(x-y) \leq \overline{0.5} \leq \max\{\bar{s}, \bar{t}\}$. Hence $(x-y)_{\max\{\bar{s},\bar{t}\}} < \overline{\mu}$. If $\max\{\bar{s}, \bar{t}\} < \overline{0.5}$, then $\overline{\mu}(x-y)+\max\{\overline{s},\overline{t}\} < \overline{0.5}+\overline{0.5} = \overline{1}$. Hence $(x-y)_{\max\{\overline{s},\overline{t}\}}\gamma\overline{\mu}$. Therefore $(x-y)_{\max\{\overline{s},\overline{t}\}} < \vee\gamma\overline{\mu}$.

Let $x, y \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$ be such that $x_{\overline{s}} < \overline{\mu}$. Then $\overline{\mu}(x) \leq \overline{s}$. Thus $x \in I$ and $y \in R$ and so $y+x-y \in I$, since I is an ideal of R. Consequently $\overline{\mu}(y+x-y) \leq \overline{0.5}$. If $\overline{s} \geq \overline{0.5}$, then $\overline{\mu}(y+x-y) \leq \overline{0.5} \leq \overline{s}$ and so $(y+x-y)_{\overline{s}} < \overline{\mu}$. If $\overline{s} < \overline{0.5}$, then $\overline{\mu}(y+x-y)+\overline{s} < \overline{0.5}+\overline{0.5} = \overline{1}$ and so $(y+x-y)_{\overline{s}}\gamma\overline{\mu}$. Hence $(y+x-y)_{\overline{s}} < \vee\gamma\overline{\mu}$.

Let $x, y \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$ be such that $y_{\overline{s}} < \overline{\mu}$. Then $\overline{\mu}(y) \leq \overline{s}$. Clearly $y \in I$ and $x \in R$ and so $xy \in I$, since I is an ideal of R. Consequently $\overline{\mu}(xy) \leq \overline{0.5}$. If $\overline{s} \geq \overline{0.5}$, then $\overline{\mu}(xy) \leq \overline{0.5} \leq \overline{s}$ and so $(xy)_{\overline{s}} < \overline{\mu}$. If $\overline{s} < \overline{0.5}$ then $\overline{\mu}(xy)+\overline{s} < \overline{0.5}+\overline{0.5} = \overline{1}$ and so $(xy)_{\overline{s}}\gamma\overline{\mu}$. Hence $(xy)_{\overline{s}} < \vee\gamma\overline{\mu}$. Therefore $\overline{\mu}$ is an i-v $(<, <\vee\gamma)$-fuzzy left ideal of R.

Now let $a, x, y \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$ be such that $a_{\overline{s}} < \overline{\mu}$. Then $\overline{\mu}(a) \leq \overline{s}$. This means that $a \in I$ and $x, y \in R$ and so $((x+a)y-xy) \in I$. Consequently $\overline{\mu}((x+a)y-xy) \leq \overline{0.5}$. If $\overline{s} \geq \overline{0.5}$, then $\overline{\mu}((x+a)y-xy) \leq \overline{0.5} \leq \overline{s}$ and so $((x+a)y-xy)_{\overline{s}} < \overline{\mu}$. If $\overline{s} < \overline{0.5}$, then $\overline{\mu}((x+a)y-xy)+\overline{s} < \overline{0.5}+\overline{0.5} = \overline{1}$ and so $((x+a)y-xy)_{\overline{s}}\gamma\overline{\mu}$. Hence $((x+a)y-xy)_{\overline{s}} < \vee\gamma\overline{\mu}$. Therefore $\overline{\mu}$ is an i-v $(<, <\vee\gamma)$-fuzzy ideal of R. □

Lemma 3.12. *Let $\overline{\mu}$ be an i-v fuzzy subset of R. Then the following are equivalent:*

(1) $x_{\overline{s}}, y_{\overline{t}} < \overline{\mu} \implies (x-y)_{\max\{\overline{s},\overline{t}\}} < \vee\gamma\overline{\mu}$.

(2) $\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$ *for all* $x, y \in R$ *and* $\overline{s}, \overline{t} \in D[0,1] \neq \overline{1}$.

Proof. (1) $\implies$ (2): Suppose there exist $x, y \in R$ such that

$$\overline{\mu}(x-y) > \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}.$$

If $\max\{\overline{\mu}(x), \overline{\mu}(y)\} > \overline{0.5}$, then $\overline{\mu}(x-y) > \max\{\overline{\mu}(x), \overline{\mu}(y)\}$. Choose $\overline{s} \in D[0,1] \neq \overline{1}$ such that $\overline{\mu}(x-y) > \overline{s} > \max\{\overline{\mu}(x), \overline{\mu}(y)\}$. Then $x_{\overline{s}}, y_{\overline{s}} < \overline{\mu}$ but $(x-y)_{\overline{s}}\overline{< \vee\gamma}\overline{\mu}$, which is a contradiction.

If $\max\{\overline{\mu}(x), \overline{\mu}(y)\} \leq \overline{0.5}$, then $\overline{\mu}(x-y) > \overline{0.5}$. Thus $x_{\overline{0.5}} < \overline{\mu}$ and $y_{\overline{0.5}} < \overline{\mu}$ but $(x-y)_{\overline{0.5}}\overline{< \vee\gamma}\overline{\mu}$, which is a contradiction. So, $\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$.

(2) $\implies$ (1): Let $x_{\overline{s}}, y_{\overline{t}} < \overline{\mu}$. Then $\overline{\mu}(x) \leq \overline{s}$ and $\overline{\mu}(y) \leq \overline{t}$.

If $\max\{\overline{s}, \overline{t}\} \geq \overline{0.5}$, then $\overline{\mu}(x-y) \leq \max\{\overline{s}, \overline{t}\}$, that is, $(x-y)_{\max\{\overline{s},\overline{t}\}} < \overline{\mu}$.

If $\max\{\overline{s}, \overline{t}\} < \overline{0.5}$, then $\overline{\mu}(x-y) < \overline{0.5}$, which implies that, $\overline{\mu}(x-y)+\max\{\overline{s}, \overline{t}\} < \overline{0.5}+\overline{0.5} = \overline{1}$. Thus, $(x-y)_{\max\{\overline{s},\overline{t}\}}\gamma\overline{\mu}$. So $(x-y)_{\max\{\overline{s},\overline{t}\}} < \vee\gamma\overline{\mu}$. □

Lemma 3.13. *Let $\overline{\mu}$ be an i-v fuzzy subset of R. Then the following conditions are equivalent:*

(1) $x_{\overline{s}}, y_{\overline{t}} < \overline{\mu} \implies (xy)_{\max\{\overline{s},\overline{t}\}} < \vee\gamma\overline{\mu}$.

(2) $\overline{\mu}(xy) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$ *for all* $x, y \in R$ *and* $\overline{s}, \overline{t} \in D[0,1] \neq \overline{1}$.

Proof. (1) $\implies$ (2): Suppose that $\overline{\mu}(xy) > \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$ for some $x, y \in R$.

If $\max\{\overline{\mu}(x), \overline{\mu}(y)\} > \overline{0.5}$, then $\overline{\mu}(xy) > \max\{\overline{\mu}(x), \overline{\mu}(y)\}$. Choose $\overline{s} \in D[0,1] \neq \overline{1}$ such that $\overline{\mu}(xy) > \overline{s} > \max\{\overline{\mu}(x), \overline{\mu}(y)\}$. This implies that $x_{\overline{s}}, y_{\overline{s}} < \overline{\mu}$ but $(xy)_s\overline{< \vee\gamma}\overline{\mu}$, which is a contradiction to our assumption.

If $\max\{\overline{\mu}(x), \overline{\mu}(y)\} \leq \overline{0.5}$, then $\overline{\mu}(xy) > \overline{0.5}$. Thus $x_{\overline{0.5}}, y_{\overline{0.5}} < \overline{\mu}$ but $(xy)_{\overline{0.5}}\overline{< \vee\gamma}\overline{\mu}$, a contradiction. So, $\overline{\mu}(xy) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$.

(2) $\Longrightarrow$ (1): Let $x, y \in R$ and $\overline{s}, \overline{t} \in D[0,1]$ with $\overline{s}, \overline{t} \neq \overline{1}$ be such that $x_{\overline{s}}, y_{\overline{t}} < \overline{\mu}$. Then $\overline{\mu}(x) \leq \overline{s}$ and $\overline{\mu}(y) \leq \overline{t}$. By (2),

$$\overline{\mu}(xy) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\} \leq \max\{\overline{s}, \overline{t}, \overline{0.5}\}.$$

If $\max\{\overline{s}, \overline{t}\} \geq \overline{0.5}$, then $\overline{\mu}(x-y) \leq \max\{\overline{s}, \overline{t}\}$. Thus $(x-y)_{\max\{\overline{s},\overline{t}\}} < \overline{\mu}$.

If $\max\{\overline{s}, \overline{t}\} < \overline{0.5}$, then $\overline{\mu}(x-y) < \overline{0.5}$. Thus, $(x-y)_{\max\{\overline{s},\overline{t}\}} \gamma \overline{\mu}$.
So, $(x-y)_{\max\{\overline{s},\overline{t}\}} < \vee\gamma\overline{\mu}$. □

Lemma 3.14. *Let $\overline{\mu}$ be an i-v fuzzy subset of R. Then the following conditions are equivalent: for all $x, y \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$,*
(1) *(a) $x \in R$ and $y_{\overline{s}} < \overline{\mu} \Longrightarrow (xy)_{\overline{s}} < \vee\gamma\overline{\mu}$,*
(2) *$\overline{\mu}(xy) \leq \max\{\overline{\mu}(y), \overline{0.5}\}$.*

Proof. (1) $\Longrightarrow$ (2): Let $x, y \in R$ and suppose that $\overline{\mu}(xy) > \max\{\overline{\mu}(y), \overline{0.5}\}$.

If $\overline{\mu}(y) > \overline{0.5}$, then $\overline{\mu}(xy) > \overline{\mu}(y)$. Choose $\overline{s} \in D[0,1] \neq \overline{1}$ such that $\overline{\mu}(xy) > \overline{s} > \overline{\mu}(y)$. Then $y_{\overline{s}} < \overline{\mu}$ but $(xy)_{\overline{s}} \overline{< \vee\gamma\overline{\mu}}$, which is a contradiction to our assumption.

If $\overline{\mu}(y) \leq \overline{0.5}$, then $\overline{\mu}(xy) > \overline{0.5}$. This implies that $y_{\overline{0.5}} < \overline{\mu}$ but $(xy)_{\overline{0.5}} \overline{< \vee\gamma\overline{\mu}}$, a contradiction. Thus $\overline{\mu}(xy) \leq \max\{\overline{\mu}(y), \overline{0.5}\}$.

(2) $\Longrightarrow$ (1): Let $x, y \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$ be such that $y_{\overline{s}} < \overline{\mu}$. Then $\overline{\mu}(y) \leq \overline{s}$. By (2),

$$\overline{\mu}(xy) \leq \max\{\overline{\mu}(y), \overline{0.5}\} \leq \max\{\overline{s}, \overline{0.5}\}.$$

If $\overline{s} \geq \overline{0.5}$, then $\overline{\mu}(xy) \leq \overline{s}$, that is, $(xy)_{\overline{s}} < \overline{\mu}$.

If $\overline{s} < \overline{0.5}$, then $\overline{\mu}(xy) \leq \overline{0.5}$ implies $\overline{\mu}(xy) + \overline{s} < \overline{0.5} + \overline{0.5} = \overline{1}$. Thus $(xy)_{\overline{s}} \gamma \overline{\mu}$. So, $(xy)_{\overline{s}} < \vee\gamma\overline{\mu}$. Hence (1) holds. □

Lemma 3.15. *An i-v fuzzy subset $\overline{\mu}$ of R the following conditions are equivalent: for all $x, y \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$,*
(1) *$y \in R$ and $x_{\overline{s}} < \overline{\mu} \Longrightarrow (y + x - y)_{\overline{s}} < \vee\gamma\overline{\mu}$,*
(2) *$\overline{\mu}(y + x - y) \leq \max\{\overline{\mu}(x), \overline{0.5}\}$.*

Proof. (1) $\Longrightarrow$ (2): Let $x, y \in R$. Assume that $\overline{\mu}(y + x - y) > \max\{\overline{\mu}(x), \overline{0.5}\}$.

If $\overline{\mu}(x) > \overline{0.5}$, then $\overline{\mu}(y + x - y) > \overline{\mu}(x)$. Choose $\overline{s} \in D[0,1] \neq \overline{1}$ such that $\overline{\mu}(y + x - y) > \overline{s} > \overline{\mu}(x)$. Then $x_{\overline{s}} < \overline{\mu}$ but $(y + x - y)_{\overline{s}} \overline{< \vee\gamma\overline{\mu}}$, a contradiction.

If $\overline{\mu}(x) \leq \overline{0.5}$, then $\overline{\mu}(y + x - y) > \overline{0.5} \geq \overline{\mu}(x)$. This implies $x_{\overline{0.5}} < \overline{\mu}$ but $(y + x - y)_{\overline{0.5}} \overline{< \vee\gamma\overline{\mu}}$, a contradiction.

In both cases, it is clear that, $\overline{\mu}(y + x - y) \leq \max\{\overline{\mu}(x), \overline{0.5}\}$.

(2) $\Longrightarrow$ (1): Let $x_{\overline{s}} < \overline{\mu}$ and $y \in R$ such that $\overline{\mu}(x) \leq \overline{s}$. By our assumption, $\overline{\mu}(y + x - y) \leq \max\{\overline{\mu}(x), \overline{0.5}\} \leq \max\{\overline{s}, \overline{0.5}\}$.

If $\overline{s} \geq \overline{0.5}$, then $\overline{\mu}(y + x - y) \leq \overline{s}$. Thus $(y + x - y)_{\overline{s}} < \overline{\mu}$.

If $\overline{s} < \overline{0.5}$, then $\overline{\mu}(y + x - y) \leq \overline{0.5}$. Thus $\overline{\mu}(y + x - y) + \overline{s} < \overline{0.5} + \overline{0.5} = \overline{1}$, that is, $(y + x - y)_{\overline{s}} \gamma \overline{\mu}$. So $(y + x - y)_{\overline{s}} < \vee\gamma\overline{\mu}$. □

Lemma 3.16. *If $\overline{\mu}$ is an i-v fuzzy subset of R, then the following conditions are equivalent: for all $x, y, a \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$*
(1) *$x, y \in R$ and $a_{\overline{s}} < \overline{\mu} \Longrightarrow ((x + a)y - xy)_{\overline{s}} < \vee\gamma\overline{\mu}$,*
(2) *$\overline{\mu}((x + a)y - xy) \leq \max\{\overline{\mu}(a), \overline{0.5}\}$.*

Proof. (1) $\Longrightarrow$ (2): Let $x, y, a \in R$ and suppose that $\overline{\mu}((x+a)y-xy) > \max\{\overline{\mu}(a), \overline{0.5}\}$.

If $\overline{\mu}(a) > \overline{0.5}$, then $\overline{\mu}((x+a)y - xy) > \overline{\mu}(a)$. Choose $\overline{s} \in D[0,1] \neq \overline{1}$ such that

$$\overline{\mu}((x+a)y - xy) > \overline{s} > \overline{\mu}(a).$$

This implies that $a_{\overline{s}} < \overline{\mu}$ but $((x+a)y - xy)_{\overline{s}} \overline{< \vee \gamma} \overline{\mu}$, which is a contradiction.

If $\overline{\mu}(a) \leq \overline{0.5}$, then $\overline{\mu}((x+a)y - xy) > \overline{0.5}$. This implies that $a_{\overline{0.5}} < \overline{\mu}$ but $((x+a)y - xy)_{\overline{0.5}} \overline{< \vee \gamma} \overline{\mu}$, which is a contradiction. Thus $\overline{\mu}((x+a)y - xy) \leq \max\{\overline{\mu}(a), \overline{0.5}\}$.

(2) $\Longrightarrow$ (1): Let $x, y, a \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$ such that $a_{\overline{s}} < \overline{\mu}$. Then $\overline{\mu}(a) \leq \overline{s}$. On one hand,

$$\overline{\mu}((x+a)y - xy) \leq \max\{\overline{\mu}(a), \overline{0.5}\} \leq \max\{\overline{s}, \overline{0.5}\}.$$

If $\overline{s} \geq \overline{0.5}$, then $\overline{\mu}((x+a)y - xy) \leq \overline{s}$ and so $((x+a)y - xy)_{\overline{s}} < \overline{\mu}$.

If $\overline{s} < \overline{0.5}$, then $\overline{\mu}((x+a)y - xy) \leq \overline{0.5}$. Thus $\overline{\mu}((x+a)y - xy) + \overline{s} < \overline{0.5} + \overline{0.5} = \overline{1}$. So $((x+a)y - xy)_{\overline{s}} \gamma \overline{\mu}$. Hence $((x+a)y - xy)_{\overline{s}} < \vee \gamma \overline{\mu}$. □

Theorem 3.17. *Let $\overline{\mu}$ be an i-v fuzzy subset of R. Then $\overline{\mu}$ is an i-v $(<, < \vee\gamma)$-fuzzy subnear-ring if and only if*

(1) $\overline{\mu}(x - y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$.

(2)$\overline{\mu}(xy) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$, *for all* $x, y \in R$.

Proof. Straightforward from Lemma 3.12 and Lemma 3.13. □

Theorem 3.18. *Let $\overline{\mu}$ be an i-v fuzzy subset of R. Then $\overline{\mu}$ is an i-v $(<, < \vee\gamma)$-fuzzy ideal if and only if for all $x, y \in R$,*

(1) $\overline{\mu}(x - y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$,

(2) $\overline{\mu}(y + x - y) \leq \max\{\overline{\mu}(x), \overline{0.5}\}$,

(3) $\overline{\mu}(xy) \leq \max\{\overline{\mu}(y), \overline{0.5}\}$,

(4) $\overline{\mu}((x+a)y - xy) \leq \max\{\overline{\mu}(a), \overline{0.5}\}$.

Proof. The proof follows from Lemmas 3.12, 3.14, 3.15 and 3.16. □

Theorem 3.19. *An i-v fuzzy subset $\overline{\mu}$ of R is an i-v $(<, < \vee\gamma)$-fuzzy ideal (subnear-ring) of R if and only if the level subset $\overline{L}(\overline{\mu} : \overline{t})$ is an ideal of R, for all $\overline{0.5} \leq \overline{t} < \overline{1}$.*

Proof. Let $\overline{\mu}$ be an i-v $(<, < \vee\gamma)$-fuzzy ideal of R and $\overline{0.5} \leq \overline{t} < \overline{1}$. Let $x, y \in \overline{L}(\overline{\mu} : \overline{t})$. Then $\overline{\mu}(x) \leq \overline{t}$ and $\overline{\mu}(y) \leq \overline{t}$. By Theorem 3.18,

$$\overline{\mu}(x - y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\} \leq \max\{\overline{t}, \overline{0.5}\} = \overline{t},$$

that is $x - y \in \overline{L}(\overline{\mu} : \overline{t})$.

Let $x \in \overline{L}(\overline{\mu} : \overline{t})$ and $y \in R$. Then $\overline{\mu}(x) \leq \overline{t}$. Thus, $\overline{\mu}(y+x-y) \leq \max\{\overline{\mu}(x), \overline{0.5}\} \leq \max\{\overline{t}, \overline{0.5}\} = \overline{t}$. So, $(y + x - y) \in \overline{L}(\overline{\mu} : \overline{t})$. Let $x \in R$ and $y \in \overline{L}(\overline{\mu} : \overline{t})$ such that $\overline{\mu}(y) \leq \overline{t}$. Since $\overline{\mu}$ is an i-v $(<, < \vee\gamma)$-fuzzy ideal of R, we have

$$\overline{\mu}(xy) \leq \max\{\overline{\mu}(y), \overline{0.5}\} \leq \max\{\overline{t}, \overline{0.5}\} = \overline{t}.$$

Thus, $xy \in \overline{L}(\overline{\mu} : \overline{t})$.

Similarly, $((x+a)y - xy) \in \overline{L}(\overline{\mu} : \overline{t})$.

Conversely, assume that $\overline{L}(\overline{\mu} : \overline{t})$ is an ideal of R for all $\overline{0.5} \leq \overline{t} < \overline{1}$. If there exist $x, y \in R$ such that $\overline{\mu}(x - y) > \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$. Choose $\overline{t}$ such that

$$\overline{\mu}(x - y) > \overline{t} > \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}.$$

Then $x, y \in \overline{L}(\overline{\mu} : \overline{t})$. Since $\overline{L}(\overline{\mu} : \overline{t})$ is an ideal of R, $x - y \in \overline{L}(\overline{\mu} : \overline{t})$. Thus $\overline{\mu}(x-y) \leq \overline{t}$, a contradiction to our assumption. So $\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$, for all $x, y \in R$.

Assume that $\overline{\mu}(y + x - y) > \max\{\overline{\mu}(x), \overline{0.5}\}$, for some $x, y \in R$. Choose $\overline{t}$ such that

$$\overline{\mu}(y + x - y) > \overline{t} > \max\{\overline{\mu}(x), \overline{0.5}\}.$$

Then $x \in \overline{L}(\overline{\mu} : \overline{t})$ but $y + x - y \notin \overline{L}(\overline{\mu} : \overline{t})$, a contradiction to our assumption that $\overline{L}(\overline{\mu} : \overline{t})$ is an ideal of R. Thus $\overline{\mu}(y + x - y) \leq \max\{\overline{\mu}(x), \overline{0.5}\}$.

Assume that $\overline{\mu}(xy) > \max\{\overline{\mu}(y), \overline{0.5}\}$. Choose $\overline{t}$ such that

$$\overline{\mu}(xy) > \overline{t} > \max\{\overline{\mu}(y), \overline{0.5}\}.$$

Then $y \in \overline{L}(\overline{\mu} : \overline{t})$ but $xy \notin \overline{L}(\overline{\mu} : \overline{t})$, a contradiction to our assumption. Thus $\overline{\mu}(xy) \leq \max\{\overline{\mu}(y), \overline{0.5}\}$.

Similarly, we prove that $\overline{\mu}((x + a)y - xy) \leq \max\{\overline{\mu}(a), \overline{0.5}\}$. Thus $\overline{\mu}$ is an i-v $(<, < \vee\gamma)$-fuzzy ideal of R. □

Definition 3.20. Let I be a non empty subset of a near-ring R. The i-v anti characteristic function $\overline{f}_I : R \to D[0, 1]$ is defined such that,

$$\overline{f}_I(x) = \begin{cases} \overline{0} & \text{for all } x \in I \\ \overline{1} & \text{otherwise.} \end{cases}$$

Theorem 3.21. *A non empty subset I of R is an ideal of R if and only if $\overline{f}_I$ is an i-v $(<, < \vee\gamma)$- fuzzy ideal of R.*

Proof. Assume that I is an ideal of R. Suppose that

$$\overline{f}_I(x - y) > \max\{\overline{f}_I(x), \overline{f}_I(y), [0.5, 0.5]\}.$$

Then $\overline{f}_I(x - y) = \overline{1}$ and $\overline{f}_I(x) = \overline{f}_I(y) = \overline{0}$. This implies $x, y \in I$ but $x - y \notin I$, which is a contradiction. Thus $\overline{f}_I(x - y) \leq \max\{\overline{f}_I(x), \overline{f}_I(y), \overline{0.5}\}$.

Suppose that $\overline{f}_I(y + x - y) > \max\{\overline{f}_I(x), \overline{0.5}\}$. Then $\overline{f}_I(y + x - y) = \overline{1}$ and $\overline{f}_I(x) = \overline{0}$. This implies $x \in I$ but $y + x - y \notin I$, which is a contradiction to our assumption. Thus $\overline{f}_I(y + x - y) \leq \max\{\overline{f}_I(x), \overline{0.5}\}$.

Suppose that $\overline{f}_I(xy) > \max\{\overline{f}_I(y), \overline{0.5}\}$ for all $x, y \in R$, that is, $\overline{f}_I(xy) = \overline{1}$ and $\overline{f}_I(y) = \overline{0}$. Then this implies $y \in I$ but $xy \notin I$, which is a contradiction. Thus $\overline{f}_I(xy) \leq \max\{\overline{f}_I(y), \overline{0.5}\}$.

Suppose that $\overline{f}_I((x + a)y - xy) > \max\{\overline{f}_I(a), \overline{0.5}\}$. Then $\overline{f}_I((x + a)y - xy) = \overline{1}$ and $\overline{f}_I(a) = \overline{0}$. This implies $a \in I$ but $(x + a)y - xy \notin I$, which is a contradiction. Thus $\overline{f}_I((x + a)y - xy) \leq \max\{\overline{f}_I(a), \overline{0.5}\}$.

Conversely, let $\overline{f}_I$ be an i-v $(<, < \vee\gamma)$-fuzzy ideal of R. For any $x, y \in I$, we have

$$\begin{aligned} \overline{f}_I(x - y) &\leq \max\{\overline{f}(x), \overline{f}(y), \overline{0.5}\} \\ &= \max\{\overline{0}, \overline{0}, \overline{0.5}\} \\ &= \overline{0.5} \neq \overline{1}. \end{aligned}$$

Then $\overline{f}_I(x - y) = \overline{0}$. Thus $x - y \in I$.

Let $y \in R$ and $x \in I$. Then

$$\begin{aligned}\overline{f}_I(y+x-y) &\leq \max\{\overline{f}(x), \overline{0.5}\}\\ &= \max\{\overline{0}, \overline{0.5}\}\\ &= \overline{0.5} \neq \overline{1}.\end{aligned}$$

Thus $\overline{f}_I(y+x-y) = \overline{0}$. This shows that $y+x-y \in I$ and therefore $(I,+)$ is a normal subgroup of $(R,+)$.

Now let $x \in R$ and $y \in I$. Then

$$\begin{aligned}\overline{f}_I(xy) &\leq \max\{\overline{f}(y), \overline{0.5}\}\\ &= \max\{\overline{0}, \overline{0.5}\}\\ &= \overline{0.5} \neq \overline{1}.\end{aligned}$$

Thus $xy \in I$.

Finally, let $x, y \in R$ and $a \in I$. Then

$$\begin{aligned}\overline{f}_I((x+a)y-xy) &\leq \max\{\overline{f}(a), \overline{0.5}\}\\ &= \max\{\overline{0}, \overline{0.5}\}\\ &= \overline{0.5} \neq \overline{1}.\end{aligned}$$

Thus $(x+a)y - xy \in I$. So I is an ideal of R. □

Theorem 3.22. *Every i-v $(<,<)$-fuzzy ideal of R is an i-v $(<,<\vee\gamma)$-fuzzy ideal(subnear-ring) of R.*

Proof. The proof is straightforward. □

The converse of Theorem 3.22 is not true in general as shown in the following example.

Example 3.23. Let $R = \{0, a, b, c\}$ be a set with two binary operation $'+'$ and $'\cdot'$ defined as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	a	0	a
a	0	a	0	a
b	0	a	0	a
c	0	a	b	c

Then clearly $(R,+,\cdot)$ is a left near-ring. Let $\overline{\mu}: R \to D[0,1]$ be an i-v fuzzy subset of R and defined by $\overline{\mu}(0) = [0.2, 0.3], \overline{\mu}(a) = [0.6, 0.7] = \overline{\mu}(c)$ and $\overline{\mu}(b) = [0, 0.1]$. Then $\overline{\mu}$ is an i-v $(<,<\vee\gamma)$-fuzzy ideal of R, but not i-v $(<,<)$-fuzzy ideal of R, since $b_{[0.11,0.12]} < \overline{\mu} \implies (b-b)_{[0.11,0.12]} = 0_{[0.11,0.12]} \overline{<} \overline{\mu}$.

In next Theorem, we give a condition for an i-v $(<,<\vee\gamma)$-fuzzy ideal of R to be an i-v $(<,<)$-fuzzy ideal of R.

Theorem 3.24. *Let $\overline{\mu}$ be an i-v $(<,<\vee\gamma)$-fuzzy ideal of R such that $\overline{\mu}(x) > \overline{0.5}$ for all $x \in R$. Then $\overline{\mu}$ is an i-v $(<,<)$-fuzzy ideal of R.*

Proof. Let $\overline{\mu}$ be an i-v $(<, < \vee\gamma)$ fuzzy ideal of R such that $\overline{\mu}(x) > \overline{0.5}$, for all $x \in R$. Let $x, y \in R$ and $\overline{s}, \overline{t} \in D[0,1] \neq \overline{1}$ be such that $x_{\overline{s}}, y_{\overline{t}} < \overline{\mu}$. Then $\overline{\mu}(x) \leq \overline{s}$ and $\overline{\mu}(y) \leq \overline{t}$. Thus

$$\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\} = \max\{\overline{\mu}(x), \overline{\mu}(y)\} \leq \max\{\overline{s}, \overline{t}\}.$$

This implies that $(x-y)_{\max\{\overline{s},\overline{t}\}} < \overline{\mu}$.

Let $x, y \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$ be such that $x_{\overline{s}} < \overline{\mu}$. Then $\overline{\mu}(x) \leq \overline{s}$. Since $\overline{\mu}$ is an i-v $(<, < \vee q)$-fuzzy ideal of R, we have

$$\overline{\mu}(y + x - y) \leq \max\{\overline{\mu}(x), \overline{0.5}\} \leq \overline{s}.$$

Thus $(y + x - y)_{\overline{s}} < \overline{\mu}$.

Now let $x, y \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$ such that $y_{\overline{s}} < \overline{\mu}$. Then $\overline{\mu}(y) \leq \overline{s}$. By assumption, $\overline{\mu}(xy) \leq \max\{\overline{\mu}(y), \overline{0.5}\} \leq \overline{s}$ implies $(xy)_{\overline{s}} < \overline{\mu}$.

Let $x, y, a \in R$ and $\overline{s} \in D[0,1] \neq \overline{1}$ such that $a_{\overline{s}} < \overline{\mu}$. Then $\overline{\mu}(a) \leq \overline{s}$. Thus $\overline{\mu}((x+a)y - xy) \leq \max\{\overline{\mu}(a), \overline{0.5}\} \leq \overline{s}$. This implies that $((x+a)y - xy)_{\overline{s}} < \overline{\mu}$. So $\overline{\mu}$ is an i-v $(<, <)$-fuzzy ideal of R. □

Theorem 3.25. *The union of any family of i-v $(<, < \vee\gamma)$-fuzzy ideals of R is an i-v $(<, < \vee\gamma)$-fuzzy ideal of R.*

Proof. Let $\{\overline{\mu}_j\}_{j\in\Omega}$ be any family of i-v $(<, < \vee\gamma)$-fuzzy ideals of R and $\overline{\mu} = \bigcup_{j\in\Omega} \overline{\mu}_j$. Let $x, y, a \in R$. Then,

$$\begin{aligned}
\overline{\mu}(x-y) &= (\bigcup_{j\in\Omega} \overline{\mu}_j)(x-y) \\
&= \bigcup_{j\in\Omega} (\overline{\mu}_j(x-y)) \\
&\leq \bigcup_{j\in\Omega} (\max\{\overline{\mu}_j(x), \overline{\mu}_j(y), \overline{0.5}\}) \\
&= \max\{(\bigcup_{j\in\Omega} \overline{\mu}_j)(x), (\bigcup_{j\in\Omega} \overline{\mu}_j)(y), \overline{0.5}\}.
\end{aligned}$$

Thus $\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$.
Now,

$$\begin{aligned}
\overline{\mu}(y+x-y) &= (\bigcup_{j\in\Omega} \overline{\mu}_j)(y+x-y) \\
&= \bigcup_{j\in\Omega} (\overline{\mu}_j(y+x-y)) \\
&\leq \bigcup_{j\in\Omega} (\max\{\overline{\mu}_j(x), \overline{0.5}\}) \\
&= \max\{(\bigcup_{j\in\Omega} \overline{\mu}_j)(x), \overline{0.5}\} \\
&= \max\{\overline{\mu}(x), \overline{0.5}\}.
\end{aligned}$$

Next,

$$\begin{aligned}\overline{\mu}(xy) &= (\bigcup_{j\in\Omega}\overline{\mu}_j)(xy)\\ &= \bigcup_{j\in\Omega}(\overline{\mu}_j(xy))\\ &\leq \bigcup_{j\in\Omega}\{\max\{\overline{\mu}_j(y),\overline{0.5}\}\}\\ &= \max\{(\bigcup_{j\in\Omega}\overline{\mu}_j)(y),\overline{0.5}\}\\ &= \max\{\overline{\mu}(y),\overline{0.5}\}.\end{aligned}$$

Further,

$$\begin{aligned}\overline{\mu}((x+a)y-xy) &= (\bigcup_{j\in\Omega}\overline{\mu}_j)((x+a)y-xy)\\ &= \bigcup_{j\in\Omega}(\overline{\mu}_j((x+a)y-xy))\\ &\leq \bigcup_{j\in\Omega}\{\max\{\overline{\mu}_j(a),\overline{0.5}\}\}\\ &= \max\{(\bigcup_{j\in\Omega}\overline{\mu}_j)(a),\overline{0.5}\}\\ &= \max\{\overline{\mu}(a),\overline{0.5}\}.\end{aligned}$$

So $\overline{\mu}=\bigcup_{j\in\Omega}\overline{\mu}_j$ is an i-v $(<,<\vee\gamma)$-fuzzy ideal of R. □

For any i-v fuzzy subset $\overline{\mu}$ of R and $\overline{t}\in D[0,1]\neq\overline{1}$. Consider the sets
$\overline{Q}(\overline{\mu}:\overline{t})=\{x\in R|x_{\overline{t}}\gamma\overline{\mu}\}$
and
$[\overline{\mu}]_{\overline{t}}=\{x\in R|x_{\overline{t}}<\vee\gamma\overline{\mu}\}$. Clearly, $[\overline{\mu}]_{\overline{t}}=\overline{L}(\overline{\mu}:\overline{t})\cup\overline{Q}(\overline{\mu},\overline{t})$.
We call $[\overline{\mu}]_{\overline{t}}$ as a $(<\vee\gamma)-$ level set and $\overline{Q}(\overline{\mu}:\overline{t})$ a γ-level set of $\overline{\mu}$.

Lemma 3.26. *Every i-v fuzzy subset $\overline{\mu}$ of R satisfies the following $\overline{t}\in D[0,0.5],\overline{t}\neq\overline{0.5}$ implies $[\overline{\mu}]_{\overline{t}}=\overline{Q}(\overline{\mu}:\overline{t})$.*

Proof. Clearly, $\overline{Q}(\overline{\mu}:\overline{t})\subseteq[\overline{\mu}]_{\overline{t}}$, from the definition of $\overline{Q}(\overline{\mu}:\overline{t})$. Let $x\in[\overline{\mu}]_{\overline{t}}$ and $\overline{0}\leq\overline{t}<\overline{0.5}$. Then $x_{\overline{t}}<\overline{\mu}$ or $x_{\overline{t}}\gamma\overline{\mu}$. If $x_{\overline{t}}\gamma\overline{\mu}$, then there is nothing to prove. If $x_{\overline{t}}<\overline{\mu}$, then $\overline{\mu}(x)\leq\overline{t}$, that is, $\overline{\mu}(x)+\overline{t}<\overline{t}+\overline{t}<\overline{0.5}+\overline{0.5}=\overline{1}$. Thus $x\in\overline{Q}(\overline{\mu}:\overline{t})$. □

Lemma 3.27. *Every i-v fuzzy subset $\overline{\mu}$ of R satisfies the following $\overline{t}\in D[0.5,1]\neq\overline{1}$ implies $[\overline{\mu}]_{\overline{t}}=\overline{L}(\overline{\mu}:\overline{t})$.*

Proof. Clearly, $\overline{L}(\overline{\mu}:\overline{t})\subseteq[\overline{\mu}]_{\overline{t}}$, from the definition of $\overline{L}(\overline{\mu}:\overline{t})$. Let $x\in[\overline{\mu}]_{\overline{t}}$ and $\overline{t}\in D[0.5,1]$ with $\overline{t}\neq\overline{1}$ be such that $x_{\overline{t}}<\overline{\mu}$ or $x_{\overline{t}}\gamma\overline{\mu}$. If $x_{\overline{t}}<\overline{\mu}$, there is nothing to prove. If $x_{\overline{t}}\gamma\overline{\mu}$, then $\overline{\mu}(x)+\overline{t}<\overline{1}$, implies that, $\overline{\mu}(x)\leq\overline{t}$. Thus $x\in\overline{L}(\overline{\mu}:\overline{t})$. □

Theorem 3.28. *Let $\overline{\mu}$ be an i-v fuzzy subset of R. Then, $\overline{\mu}$ is an i-v $(<,<\vee\gamma)$-fuzzy ideal (subnear-ring) of R if and only if $[\overline{\mu}]_{\overline{t}}\neq\varnothing$ is an ideal (subnear-ring) of R.*

Proof. Let $\overline{\mu}$ be an i-v $(<,< \vee\gamma)$-fuzzy ideal of R and $\overline{t} \in D[0,1]$ with $\overline{t} \neq \overline{1}$. Let $x, y \in [\overline{\mu}]_{\overline{t}}$. Then $\overline{\mu}(x) \leq \overline{t}$ or $\overline{\mu}(x) + \overline{t} < \overline{1}$ and $\overline{\mu}(y) \leq \overline{t}$ or $\overline{\mu}(y) + \overline{t} < \overline{1}$. We first prove the condition (i) of Definition 3.4. Consider four cases:

(i) $\overline{\mu}(x) \leq \overline{t}$ and $\overline{\mu}(y) \leq \overline{t}$,
(ii) $\overline{\mu}(x) \leq \overline{t}$ and $\overline{\mu}(y) + \overline{t} < \overline{1}$,
(iii) $\overline{\mu}(x) + \overline{t} < \overline{1}$ and $\overline{\mu}(y) \leq \overline{t}$,
(iv) $\overline{\mu}(x) + \overline{t} < \overline{1}$ and $\overline{\mu}(y) + \overline{t} < \overline{1}$.

Case (i): Suppose $\overline{\mu}(x) \leq \overline{t}$ and $\overline{\mu}(y) \leq \overline{t}$. Then

$$\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\} \leq \max\{\overline{t}, \overline{0.5}\}.$$

If $\overline{t} < \overline{0.5}$, then $\overline{\mu}(x-y) < \overline{0.5}$, that is, $\overline{\mu}(x-y) + \overline{t} < \overline{0.5} + \overline{0.5} = \overline{1}$. This implies that $(x-y)_{\overline{t}}\gamma\overline{\mu}$. If $\overline{t} \geq \overline{0.5}$, then $\overline{\mu}(x-y) \leq \overline{t}$, that is, $(x-y)_{\overline{t}} < \overline{\mu}$. Thus $x - y \in [\overline{\mu}]_{\overline{t}}$.

Case (ii): Suppose $\overline{\mu}(x) \leq \overline{t}$ and $\overline{\mu}(y) + \overline{t} < \overline{1}$. Then,

$$\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\} \leq \max\{\overline{t}, \overline{1} - \overline{t}, \overline{0.5}\}.$$

If $\overline{t} < \overline{0.5}$, then $\overline{\mu}(x-y) \leq \overline{1} - \overline{t}$. This implies that $\overline{\mu}(x-y) + \overline{t} < \overline{1}$. Thus, $(x-y)_{\overline{t}}\gamma\overline{\mu}$. If $\overline{t} \geq \overline{0.5}$, then $\overline{\mu}(x-y) \leq \overline{t}$, that is, $(x-y)_{\overline{t}} < \overline{\mu}$.

Similarly, we prove case (iii).

Case (iv): Suppose $\overline{\mu}(x) + \overline{t} < \overline{1}$ and $\overline{\mu}(y) + \overline{t} < \overline{1}$. Then

$$\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\} \leq \max\{\overline{1} - \overline{t}, \overline{0.5}\}.$$

If $\overline{t} < \overline{0.5}$, then $\overline{\mu}(x-y) \leq \overline{1} - \overline{t}$. This implies $(x-y)_{\overline{t}}\gamma\overline{\mu}$. If $\overline{t} \geq \overline{0.5}$, then $\overline{\mu}(x-y) \leq \overline{0.5} \leq \overline{t}$. This implies $(x-y)_{\overline{t}} < \overline{\mu}$. Thus $x - y \in [\overline{\mu}]_{\overline{t}}$.

Let $x \in [\overline{\mu}]_{\overline{t}}$ and $y \in R$. Then $\overline{\mu}(x) \leq \overline{t}$ or $\overline{\mu}(x) + \overline{t} < \overline{1}$.

We now prove condition (ii) of Definition 3.4. There are two cases:

Case (i): Let $\overline{\mu}(x) \leq \overline{t}$. Then

$$\overline{\mu}(y + x - y) \leq \max\{\overline{\mu}(x), \overline{0.5}\} \leq \max\{\overline{t}, \overline{0.5}\}.$$

If $\overline{t} < \overline{0.5}$, then $\overline{\mu}(y+x-y) \leq \overline{0.5}$, that is, $\overline{\mu}(y+x-y) + \overline{t} < \overline{0.5} + \overline{0.5} = \overline{1}$. If $\overline{t} \geq \overline{0.5}$, then $\overline{\mu}(y + x - y) \leq \overline{t}$. Thus $(y + x - y)_{\overline{t}} < \vee\gamma\overline{\mu}$.

Case (ii): Let $\overline{\mu}(x) + \overline{t} < \overline{1}$. Then

$$\overline{\mu}(y + x - y) \leq \max\{\overline{\mu}(x), \overline{0.5}\} \leq \max\{\overline{1} - \overline{t}, \overline{0.5}\}.$$

If $\overline{t} < \overline{0.5}$, then $\overline{\mu}(y + x - y) \leq \overline{1} - \overline{t}$, that is, $\overline{\mu}(y + x - y) + \overline{t} \leq \overline{1}$. If $\overline{t} \geq \overline{0.5}$, then $\overline{\mu}(y + x - y) \leq \overline{0.5} \leq \overline{t}$. Thus $(y + x - y)_{\overline{t}} < \overline{\mu}$. So $y + x - y \in [\overline{\mu}]_{\overline{t}}$.

Let $x \in R$ and $y \in [\overline{\mu}]_{\overline{t}}$. Then $\overline{\mu}(y) \leq \overline{t}$ or $\overline{\mu}(y) + \overline{t} < \overline{1}$. Next we prove the condition for left ideal. There are two cases:

Case (i): Suppose $\overline{\mu}(y) \leq \overline{t}$. Then $\overline{\mu}(xy) \leq \max\{\overline{\mu}(y), \overline{0.5}\} \leq \max\{\overline{t}, \overline{0.5}\}$. If $\overline{t} < \overline{0.5}$, then $\overline{\mu}(xy) \leq \overline{0.5}$. This implies $\overline{\mu}(xy) + \overline{t} < \overline{0.5} + \overline{0.5} = \overline{1}$. Thus $(xy)_{\overline{t}}\gamma\overline{\mu}$. If $\overline{t} \geq \overline{0.5}$, then $\overline{\mu}(xy) \leq \overline{t}$, that is, $(xy)_{\overline{t}} \leq \overline{\mu}$. So $(xy)_{\overline{t}} < \vee\gamma\overline{\mu}$.

Case (ii): Suppose $\overline{\mu}(y) + \overline{t} < \overline{1}$. Then

$$\overline{\mu}(xy) \leq \max\{\overline{\mu}(y), \overline{0.5}\} \leq \max\{\overline{1} - \overline{t}, \overline{0.5}\}.$$

If $\overline{t} < \overline{0.5}$, then $\overline{\mu}(xy) \leq \overline{1} - \overline{t}$. This implies $\overline{\mu}(xy) + \overline{t} \leq \overline{1}$. If $\overline{t} \geq \overline{0.5}$, then $\overline{\mu}(xy) \leq \overline{0.5} \leq \overline{t}$. Thus $(xy)_{\overline{t}} < \vee\gamma\overline{\mu}$. So $xy \in [\overline{\mu}]_{\overline{t}}$.

Similarly, $(x + a)y - xy \in [\overline{\mu}]_{\overline{t}}$ for all $x, y \in R$ and $a_{\overline{t}} \in [\overline{\mu}]_{\overline{t}}$. Hence, $[\overline{\mu}]_{\overline{t}}$ is an ideal of R.

Conversely, assume that $[\overline{\mu}]_{\overline{t}}$ is an ideal of R, for all $\overline{0} \leq \overline{t} < \overline{1}$. If possible, let

$$\overline{\mu}(x-y) > \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}.$$

Choose $\overline{t}$ such that

$$\overline{\mu}(x-y) > \overline{t} > \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}.$$

Then, $\overline{0.5} \leq \overline{t} < \overline{1}$ and $x, y \in \overline{L}(\overline{\mu} : \overline{t}) \subseteq [\overline{\mu}]_{\overline{t}}$, but $(x-y) \notin \overline{L}(\overline{\mu} : \overline{t})$, which is a contradiction. Thus, $\overline{\mu}(x-y) \leq \max\{\overline{\mu}(x), \overline{\mu}(y), \overline{0.5}\}$.

Suppose that $\overline{\mu}(y+x-y) > \max\{\overline{\mu}(x), \overline{0.5}\}$. Choose $\overline{t}$ such that

$$\overline{\mu}(y+x-y) > \overline{t} > \max\{\overline{\mu}(x), \overline{0.5}\}.$$

Then $\overline{0.5} \leq \overline{t} < \overline{1}$, $x \in \overline{L}(\overline{\mu} : \overline{t})$ but $(y+x-y) \notin \overline{L}(\overline{\mu} : \overline{t})$, a contradiction. Thus $\overline{\mu}(y+x-y) \leq \max\{\overline{\mu}(x), \overline{0.5}\}$.

In a similar way, we can prove that $\overline{\mu}(xy) \leq \max\{\overline{\mu}(y), \overline{0.5}\}$ and $\overline{\mu}((x+a)y-xy) \leq \max\{\overline{\mu}(a), \overline{0.5}\}$, for all $x, y, a \in R$. So $\overline{\mu}$ is an i-v $(<, < \vee\gamma)$- fuzzy ideal of R. □

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