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Anti fuzzy quasi-ideals of near-rings

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Abstract. In this paper, we introduce the notion of anti fuzzy quasi-ideals of near-rings. We have discussed some of their theoretical properties in detail and obtained some characterizations.

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1. Introduction

Zadeh[10] introduced the concept of fuzzy sets in 1965. Rosenfeld[9] initiated the study of fuzzy subgroup and investigated some of its properties. In [1], Abou Zaid introduced the concept of fuzzy subnear-rings and ideals of near-rings. In 1990, Biwas[2] introduced the notion of anti fuzzy subgroups. Kim, Jun and Yon[5] have discussed the notion of anti fuzzy ideals of near-ring. Iwao Yakabe[4] initiated the idea of quasi-ideals in right near-rings. Kim and Jun[6] introduced the concept of anti fuzzy R-subgroups of near-rings. Narayanan[7] has studied the notion of fuzzy quasi-ideals in near-rings. Chinnadurai and Kadalarasi[3] has studied the concept of interval valued fuzzy quasi ideals of near-rings. In this paper, we introduce the notion of anti fuzzy quasi-ideals of a near-ring. We investigate some of their theoretical properties and provide examples. It is shown that every anti fuzzy ideal (R-subgroup) of a near-ring is an anti fuzzy quasi-ideal, but the converse is not true in general.

2. Preliminaries

Throughout this paper R will denote a left near-ring. In this section, we present some basic definitions and results used in this paper.

Definition 2.1 ([8])**.** A near-ring is an algebraic system $(R, +, \cdot)$ consisting of a non empty set R together with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group, not necessarily abelian and $(R, \cdot)$ is a semigroup in which the distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$ holds for all $x, y, z \in R$. We will use the word 'near-ring 'to mean 'left near-ring '. We denote xy instead of $x \cdot y$.

An ideal I of a near-ring R is a subset of R such that
(i) $(I, +)$ is a normal subgroup of $(R, +)$,
(ii) $RI \subseteq I$,
(iii) $(x + a)y - xy \in I$, for any $a \in I$ and $x, y \in R$.
Note that I is a left ideal of R, if I satisfies (i) and (ii), and a right ideal of R if it satisfies (i) and (iii).

Definition 2.2 ([6])**.** A nonempty subset H of R is said to be a two sided R-subgroup of R if
(i) $(H, +)$ is a subgroup of $(R, +)$,
(ii) $RH \subset H$,
(iii) $HR \subset H$.
If H satisfies (i) and (ii), it is called a left R-subgroup of R. If H satisfies (i) and (iii), it is called a right R-subgroup of R.

Definition 2.3 ([8])**.** Let A and B be any two non-empty subsets of R. We define

$$AB = \{ab \,|\, a \in A, b \in B\}$$

and

$$A * B = \{(a + c)b - ab \,|\, a, b \in A, c \in B\}.$$

A near-ring R is called zero-symmetric, if $0x = 0$ for all $x \in R$.

Definition 2.4 ([8])**.** An additive subgroup Q of $(R, +)$ is said to be a quasi-ideal of R if $QR \cap RQ \cap Q * R \subseteq Q$.

Definition 2.5 ([6])**.** A fuzzy subset μ of R is a function $\mu : R \to [0, 1]$. For $t \in [0, 1]$, the set $\mu_t = \{x \in R | \mu(x) \leq t\}$ is called a t lower t-level set of μ.

Definition 2.6 ([8])**.** The characteristic function of R is denoted by $\mathbf{R}$, that is $\mathbf{R}(x) = 1$ for all $x \in R$.

Definition 2.7 ([8])**.** Let μ and λ be any two fuzzy subsets of R. Then sum $\mu + \lambda$, product $\mu \cdot \lambda$, and product $\mu * \lambda$ are fuzzy subsets of R defined by

$$(\mu + \lambda)(x) = \begin{cases} \sup\limits_{x=y+z} \min\{\mu(y), \lambda(z)\} & \text{if } x \text{ can be expressed as } x = y + z \\ 0 & \text{otherwise,} \end{cases}$$

$$(\mu \cdot \lambda)(x) = \begin{cases} \sup\limits_{x=yz} \min\{\mu(y), \lambda(z)\} & \text{if } x \text{ can be expressed as } x = yz \\ 0 & \text{otherwise,} \end{cases}$$

$$(\mu * \lambda)(x) = \begin{cases} \sup\limits_{x=(a+c)b-ab} \min\{\mu(c), \lambda(b)\} & \text{if } x \text{ can be expressed as } x = (a + c)b - ab \\ 0 & \text{otherwise,} \end{cases}$$

where $x \in R$.

Definition 2.8 ([7]). A fuzzy subgroup μ of R is called a fuzzy quasi-ideal of R, if

$$(\mu \cdot \mathbf{R}) \cap (\mathbf{R} \cdot \mu) \cap (\mu * \mathbf{R}) \subseteq \mu.$$

Definition 2.9 ([6]). A fuzzy subset μ of R is called an anti fuzzy left(resp. right)R-subgroup of R, if
(i) $\mu(x-y) \leq \max\{\mu(x), \mu(y)\}$,
(ii) $\mu(xy) \leq \mu(y)$ (resp. $\mu(xy) \leq \mu(x)$) for all $x, y \in R$.

Definition 2.10 ([5]). A fuzzy subset μ of R is called an anti fuzzy left(resp. right)ideal of R, if
(i) $\mu(x-y) \leq \max\{\mu(x), \mu(y)\}$,
(ii) $\mu(y+x-y) \leq \mu(x)$,
(iii) $\mu(xy) \leq \mu(y)$ (resp. $\mu((x+z)y - xy) \leq \mu(z)$) for all $x, y, z \in R$.

3. Anti fuzzy quasi-ideals of near-rings

In this section, we introduce the notion of anti fuzzy quasi-ideals of near-rings and establish some of their properties and characterizations.

Definition 3.1. Let μ and λ be any two fuzzy subsets of R. Then $\mu \cup \lambda, \mu \cap \lambda$, anti sum $\mu +_a \lambda$, anti product $\mu \cdot_a \lambda$, and anti $*_a$ product $\mu *_a \lambda$ are fuzzy subsets of R defined by
$(\mu \cup \lambda)(x) = \max\{\mu(x), \lambda(x)\}$,
$(\mu \cap \lambda)(x) = \min\{\mu(x), \lambda(x)\}$,

$$(\mu +_a \lambda)(x) = \begin{cases} \inf\limits_{x=y+z} \max\{\mu(y), \lambda(z)\} & \text{if } x \text{ can be expressed as } x = y+z \\ 1 & \text{otherwise,} \end{cases}$$

$$(\mu \cdot_a \lambda)(x) = \begin{cases} \inf\limits_{x=yz} \max\{\mu(y), \lambda(z)\} & \text{if } x \text{ can be expressed as } x = yz \\ 1 & \text{otherwise,} \end{cases}$$

$$(\mu *_a \lambda)(x) = \begin{cases} \inf\limits_{x=(a+c)b-ab} \max\{\mu(c), \lambda(b)\} & \text{if } x \text{ can be expressed as } x = (a+c)b-ab \\ 1 & \text{otherwise,} \end{cases}$$

where $x \in R$.

Definition 3.2. The anti-characteristic function of R is denoted by $\mathcal{R}$, that is, $\mathcal{R}(x) = 0$ for all $x \in R$.

Definition 3.3. A fuzzy subset μ of R is called an anti fuzzy subgroup of R, if

$$\mu(x-y) \leq \max\{\mu(x), \mu(y)\} \text{ for all } x, y \in R.$$

Definition 3.4. An anti fuzzy subgroup μ of R is called an anti fuzzy quasi-ideal of R, if

$$(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}) \supseteq \mu.$$

Note that if R is a zero-symmetric near-ring then $(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \supseteq \mu$.

Lemma 3.5. *Every anti fuzzy quasi-ideal of a zero-symmetric near-ring R is an anti fuzzy subnear-ring of R.*

Proof. Let μ be an anti fuzzy quasi-ideal of a zero-symmetric near-ring R. Choose $a, b, c, x, y, z \in R$ such that $a = bc = (x+z)y - xy$. Then

$$\begin{aligned}
\mu(bc) = \mu(a) &\leq \max\{(\mu \cdot_a \mathcal{R})(a), (\mathcal{R} \cdot_a \mu)(a)(\mu *_a \mathcal{R})(a)\} \\
&= \max\{\inf_{a=bc} \max\{\mu(b), \mathcal{R}(c)\}, \inf_{a=bc} \max\{\mathcal{R}(b), \mu(c)\}, \\
&\qquad \inf_{a=(x+z)y-xy} \max\{\mu(z), \mathcal{R}(y)\}\} \\
&\leq \max\{\inf_{a=bc} \max\{\mu(b), \mathcal{R}(c)\}, \inf_{a=bc} \max\{\mathcal{R}(b), \mu(c)\}, \\
&\qquad \inf_{a=(0+b)c-0c} \max\{\mu(b), \mathcal{R}(c)\}\} \\
&= \max\{\mu(b), \mu(c), \mu(b)\} \\
&= \max\{\mu(b), \mu(c)\}.
\end{aligned}$$

Thus $\mu(bc) \leq \max\{\mu(b), \mu(c)\}$. Since μ is an anti fuzzy quasi-ideal of a zero-symmetric near-ring R, we have $\mu(b-c) \leq \max\{\mu(b), \mu(c)\}$ for all $b, c \in R$. So μ is an anti fuzzy subnear-ring of R. □

Lemma 3.6. *Every anti fuzzy left ideal of R is an anti fuzzy quasi-ideal of R.*

Proof. Let μ be anti fuzzy left ideal of R. For $x' \in R$, let $a, b, x, y, z \in R$ such that $x' = ab = (x+z)y - xy$. Then

$$\begin{aligned}
&((\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}))(x') \\
&= \max\{(\mu \cdot_a \mathcal{R})(x'), (\mathcal{R} \cdot_a \mu)(x'), (\mu *_a \mathcal{R})(x')\} \\
&= \max\{\inf_{x'=ab} \max\{\mu(a), \mathcal{R}(b)\}, \inf_{x'=ab} \max\{\mathcal{R}(a), \mu(b)\}, \\
&\qquad \inf_{x'=(x+z)y-xy} \max\{\mu(z), \mathcal{R}(y)\}\} \\
&= \max\{\inf\{\mu(a)\}, \inf\{\mu(b)\}, \inf\{\mu(z)\}\}, \\
&\qquad [\text{Since } \mu \text{ is an anti fuzzy left ideal of } R, \ \mu(ab) \leq \mu(b).] \\
&\geq \max\{\mathcal{R}(a), \mu(ab), \mathcal{R}(z)\} \\
&= \max\{0, \mu(ab), 0\} = \mu(ab) = \mu(x').
\end{aligned}$$

If x' is not expressible as $x' = ab = (x+z)y - xy$, then

$$(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R})(x') = 1 \geq \mu(x').$$

Thus $(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}) \supseteq \mu$. So μ is an anti fuzzy quasi-ideal of R. □

Lemma 3.7. *Every anti fuzzy right ideal of R is an anti fuzzy quasi-ideal of R.*

Proof. Let μ is an anti fuzzy right ideal of R. For $x' \in R$, let $x' = ab = (x+z)y - xy$, where $a, b, x, y, z \in R$. Then,

$$\begin{aligned}
&((\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}))(x') \\
&= \max\{(\mu \cdot_a \mathcal{R})(x'), (\mathcal{R} \cdot_a \mu)(x'), (\mu *_a \mathcal{R})(x')\} \\
&= \max\{\inf_{x'=ab} \max\{\mu(a), \mathcal{R}(b)\}, \inf_{x'=ab} \max\{\mathcal{R}(a), \mu(b)\}, \\
&\quad \inf_{x'=(x+z)y-xy} \max\{\mu(z), \mathcal{R}(y)\}\} \\
&= \max\{\inf \mu(a), \inf \mu(b), \inf \mu(z)\}, \\
&\quad [\text{Since } \mu \text{ is an anti fuzzy right ideal of } R,\ \mu((x+z)y - xy) \leq \mu(z).] \\
&\geq \max\{\mathcal{R}(a), \mathcal{R}(b), \mu((x+z)y - xy)\} \\
&= \max\{0, 0, \mu((x+z)y - xy)\} \\
&= \mu((x+z)y - xy) = \mu(x').
\end{aligned}$$

If x' is not expressible as $x' = ab = (x+z)y - xy$, then

$$(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R})(x') = 1 \geq \mu(x').$$

Thus $(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}) \supseteq \mu$. So μ is an anti fuzzy quasi-ideal of R. □

Theorem 3.8. *Every anti fuzzy ideal of R is an anti fuzzy quasi-ideal of R.*

Proof. The proof is straight forward from Lemmas 3.6 and 3.7. □

Lemma 3.9. *Every anti fuzzy left R-subgroup of R is an anti fuzzy quasi-ideal of R.*

Proof. Let μ be an anti fuzzy left R-subgroup of R. Let $a, b, c, x, y, z \in R$ such that $x = ab = (y+c)z - yz$. Then

$$\begin{aligned}
&((\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}))(x) \\
&= \max\{(\mu \cdot_a \mathcal{R})(x), (\mathcal{R} \cdot_a \mu)(x), (\mu *_a \mathcal{R})(x)\} \\
&= \max\{\inf_{x=ab} \max\{\mu(a), \mathcal{R}(b)\}, \inf_{x=ab} \max\{\mathcal{R}(a), \mu(b)\}, \\
&\quad (\mu *_a \mathcal{R})((y+c)z - yz)\} \\
&= \max\{\inf_{x=ab} \mu(a), \inf_{x=ab} \mu(b), (\mu *_a \mathcal{R})((y+c)z - yz)\}, \\
&\quad [\text{Since } \mu \text{ is an anti fuzzy left } R\text{-subgroup of} R,\ \mu(ab) \leq \mu(b).] \\
&\geq \max\{\mathcal{R}(a), \mu(ab), \mathcal{R}((y+c)z - yz)\} \\
&= \max\{0, \mu(ab), 0\} = \mu(ab) = \mu(x).
\end{aligned}$$

Thus $(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}) \supseteq \mu$. So μ is an anti fuzzy quasi-ideal of R. □

Lemma 3.10. *Every anti fuzzy right R-subgroup of R is an anti fuzzy quasi-ideal of R.*

Proof. Let μ be an anti fuzzy right R-subgroup of R. Let $a, b, c, x, y, z \in R$ such that $x = ab = (y+c)z - yz$. Then

$$\begin{aligned}
&((\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}))(x)\\
&= \max\{(\mu \cdot_a \mathcal{R})(x), (\mathcal{R} \cdot_a \mu)(x), (\mu *_a \mathcal{R})(x)\}\\
&= \max\{\inf_{x=ab} \max\{\mu(a), \mathcal{R}(b)\}, \inf_{x=ab} \max\{\mathcal{R}(a), \mu(b)\},\\
&\qquad (\mu *_a \mathcal{R})((y+c)z - yz)\}\\
&= \max\{\inf_{x=ab} \mu(a), \inf_{x=ab} \mu(b), (\mu *_a \mathcal{R})((y+c)z - yz)\},\\
&[\text{Since } \mu \text{ is an anti fuzzy right } R\text{-subgroup of} R,\ \mu(ab) \leq \mu(a).]\\
&\geq \max\{\mu(ab), \mathcal{R}(b), \mathcal{R}((y+c)z - yz)\}\\
&= \max\{\mu(ab), 0, 0\} = \mu(ab) = \mu(x).
\end{aligned}$$

Thus $(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}) \supseteq \mu$. So μ is an anti fuzzy quasi-ideal of R. □

Theorem 3.11. *Every anti fuzzy two-sided R-subgroup of R is an anti fuzzy quasi-ideal of R.*

Proof. The proof is straightforward from Lemma 3.9 and Lemma 3.10. □

The converses of Theorem 3.8 and Theorem 3.11 are not true in general as shown by the following example .

Example 3.12. Let $R = \{0, a, b, c\}$ be a set with two binary operations '+' and '·' defined as

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	a	0	a
a	0	a	0	a
b	0	a	b	c
c	0	a	b	c

Let $\mu : R \to [0,1]$ be a fuzzy set defined by $\mu(0) = 0.2, \mu(a) = \mu(b) = 0.7$ and $\mu(c) = 0.3$. Then

$(\mu \cdot_a \mathcal{R})(0) = 0.2$ $(\mathcal{R} \cdot_a \mu)(0) = 0.2$ $(\mu *_a \mathcal{R})(0) = 0.2$
$(\mu \cdot_a \mathcal{R})(a) = 0.2$ $(\mathcal{R} \cdot_a \mu)(a) = 0.3$ $(\mu *_a \mathcal{R})(a) = 1$
$(\mu \cdot_a \mathcal{R})(b) = 0.3$ $(\mathcal{R} \cdot_a \mu)(b) = 0.7$ $(\mu *_a \mathcal{R})(b) = 0.3$
$(\mu \cdot_a \mathcal{R})(c) = 0.3$ $(\mathcal{R} \cdot_a \mu)(c) = 0.3$ $(\mu *_a \mathcal{R})(c) = 1.$

Thus μ is an anti fuzzy quasi-ideal of R. But μ is not an anti fuzzy ideal of R, since $\mu(0c) = \mu(a) = 0.7 \nleq 0.3 = \max\{\mu(0), \mu(c)\}$. Also μ is not an anti fuzzy R-subgroup of R, since $\mu(0c) = \mu(a) = 0.7 \nleq 0.3 = \mu(c)$ and $\mu(0c) = \mu(a) = 0.7 \nleq 0.2 = \mu(0)$.

Note 3.13. The conditions $(\mu \cdot_a \mathcal{R})^c = \mu^c \cdot \mathbf{R}$; $(\mathcal{R} \cdot_a \mu)^c = \mathbf{R} \cdot \mu^c$; $(\mu *_a \mathcal{R})^c = \mu^c * \mathbf{R}$ are true.

Consider

$$\begin{aligned}(\mu \cdot_a \mathcal{R})^c(x) &= 1 - (\mu \cdot_a \mathcal{R})(x)\\ &= 1 - \inf_{x=ab} \max\{\mu(a), \mathcal{R}(b)\}\\ &= \sup_{x=ab} \min\{1-\mu(a), 1-\mathcal{R}(b)\}\\ &= \sup_{x=ab} \min\{\mu^c(a), \mathbf{R}(b)\}\\ &= (\mu^c \cdot \mathbf{R})(x).\end{aligned}$$

Similarly, $(\mathcal{R} \cdot_a \mu)^c = \mathbf{R} \cdot \mu^c$; $(\mu *_a \mathcal{R})^c = \mu^c * \mathbf{R}$.

Theorem 3.14. *A fuzzy subset μ of R is an anti fuzzy quasi-ideal of R if and only if its complement μ^c is a fuzzy quasi-ideal of R.*

Proof. Assume that μ is an anti fuzzy quasi-ideal of R. Let $x, y \in R$. Then

$$\begin{aligned}\mu^c(x-y) &= 1-\mu(x-y)\\ &\geq 1-\max\{\mu(x), \mu(y)\}\\ &= \min\{1-\mu(x), 1-\mu(y)\}\\ &= \min\{\mu^c(x), \mu^c(y)\}.\end{aligned}$$

Let $x \in R$. Then

$$\begin{aligned}&((\mu^c \cdot R) \cap (R \cdot \mu^c) \cap (\mu^c * R))(x)\\ &= ((\mu \cdot_a \mathcal{R})^c \cap (\mathcal{R} \cdot_a \mu)^c \cap (\mu *_a \mathcal{R})^c)(x)\\ &= \min\{(\mu \cdot_a \mathcal{R})^c(x), (\mathcal{R} \cdot_a \mu)^c(x), (\mu *_a \mathcal{R})^c(x)\}\\ &= \min\{1-(\mu \cdot_a \mathcal{R})(x), 1-(\mathcal{R} \cdot_a \mu)(x), 1-(\mu *_a \mathcal{R})(x)\}\\ &= 1-\max\{(\mu \cdot_a \mathcal{R})(x), (\mathcal{R} \cdot_a \mu)(x), (\mu *_a \mathcal{R})(x)\}\\ &\leq 1-\mu(x) \leq \mu^c(x).\end{aligned}$$

Thus $(\mu \cdot_a \mathcal{R})^c \cap (\mathcal{R} \cdot_a \mu)^c \cap (\mu *_a \mathcal{R})^c \subseteq \mu^c$. So μ^c is a fuzzy quasi-ideal of R.

Conversely, assume that μ^c is a fuzzy quasi-ideal of R. Let $x, y \in R$. Then

$$\begin{aligned}1-\mu(x-y) = \mu^c(x-y) &\geq \min\{\mu^c(x), \mu^c(y)\}\\ &= 1-\max\{1-\mu^c(x), 1-\mu^c(y)\}\\ &= 1-\max\{\mu(x), \mu(y)\}.\end{aligned}$$

Thus $\mu(x-y) \leq \max\{\mu(x), \mu(y)\}$. On the other hand

$$\begin{aligned}&1-\max\{(\mu \cdot_a \mathcal{R})(x), (\mathcal{R} \cdot_a \mu)(x), (\mu *_a \mathcal{R})(x)\}\\ &= 1-\max\{1-(\mu \cdot_a \mathcal{R})^c(x), 1-(\mathcal{R} \cdot_a \mu)^c(x), 1-(\mu *_a R)^c(x)\}\\ &= \min\{(\mu \cdot_a \mathcal{R})^c(x), (\mathcal{R} \cdot_a \mu)^c(x), (\mu *_a \mathcal{R})^c(x)\}\\ &= \min\{(\mu^c \cdot R)(x), (R \cdot \mu^c)(x), (\mu^c * R)(x)\}\\ &\leq \mu^c(x)\\ &= 1-\mu(x).\end{aligned}$$

So, $\max\{(\mu \cdot \mathcal{R})(x), (\mathcal{R} \cdot_a \mu)(x), (\mu *_a \mathcal{R})(x)\} \geq \mu(x)$.
Hence, $(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}) \supseteq \mu$. □

Theorem 3.15. *A nonempty subset Q of R is a quasi-ideal of R if and only if the complement of the characteristic function f_Q is an anti fuzzy quasi-ideal of R.*

Proof. Assume that Q is a quasi-ideal of R. By Theorem 5.4.5[7], f_Q is a fuzzy quasi-ideal of R. By Theorem 3.14 ${f_Q}^c$ is an anti fuzzy quasi-ideal of R.

Conversely, assume that ${f_Q}^c$ is a fuzzy quasi-ideal of R. Then, by Theorem 5.4.5 [7], Q is quasi-ideal of R. □

Theorem 3.16. *Let μ be a fuzzy subset of R. Then μ is an anti fuzzy quasi-ideal of R if and only if the lower level subset $L(\mu : t)$ is a quasi-ideal of R, for all $t \in (0, 1]$.*

Proof. Assume that μ is an anti fuzzy quasi-ideal of R. Let $t \in (0, 1]$ and $x, y \in L(\mu : t)$. Then $\mu(x) \leq t$ and $\mu(y) \leq t$. By hypothesis, μ is an anti fuzzy quasi-ideal of R, and so $\mu(x - y) \leq \max\{\mu(x), \mu(y)\} \leq \max\{t, t\} = t$, that is, $x - y \in L(\mu : t)$. Let $x \in R$ and $x \in (L(\mu : t) \cdot_a \mathcal{R}) \cap (\mathcal{R} \cdot_a L(\mu : t)) \cap (L(\mu : t) *_a \mathcal{R})$. Then there exist $a, b_1, c \in L(\mu : t)$ and $a_1, b, y, z \in R$ such that $x = ab = a_1b_1 = (y + c)z - yz$, which implies that $\mu(a) \leq t, \mu(b_1) \leq t$ and $\mu(c) \leq t$. Consider

$$\begin{aligned}
\mu(x) &\leq ((\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}))(x) \\
&= \max\{(\mu \cdot_a \mathcal{R})(x), (\mathcal{R} \cdot_a \mu)(x), (\mu *_a \mathcal{R})(x)\} \\
&= \max\{\inf_{x=ab} \max\{\mu(a), \mathcal{R}(b)\}, \inf_{x=a_1b_1} \max\{\mathcal{R}(a_1), \mu(b_1)\}, \\
&\qquad \inf_{x=(y+c)z-yz} \max\{\mu(c), \mathcal{R}(z)\}\} \\
&= \max\{\inf_{x=ab} \mu(a), \inf_{x=a_1b_1} \mu(b_1), \inf_{x=(y+c)z-yz} \mu(c)\} \leq t.
\end{aligned}$$

Then $x \in L(\mu : t)$. Thus $(L(\mu : t) \cdot_a \mathcal{R}) \cap (\mathcal{R} \cdot_a L(\mu : t)) \cap (L(\mu : t) *_a \mathcal{R}) \subseteq L(\mu : t)$. Thus $L(\mu : t)$ is a quasi ideal of R.

Conversely, assume that $L(\mu : t)$ is a quasi-ideal of R, for all $t \in (0, 1]$. Let $x \in R$ and assume that $(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R})(x) < \mu(x)$. Choose $t \in (0, 1]$ such that $((\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}))(x) < t < \mu(x)$. Then, $(\mu \cdot_a \mathcal{R})(x) < t, (\mathcal{R} \cdot_a \mu)(x) < t$ and $(\mu *_a \mathcal{R})(x) < t$. Then

$$\begin{aligned}
(\mu \cdot_a \mathcal{R})(x) &= \inf_{x=ab} \max\{\mu(a), \mathcal{R}(b)\} = \inf_{x=ab} \max\{\mu(a)\} < t \\
(\mathcal{R} \cdot_a \mu)(x) &= \inf_{x=a_1b_1} \max\{\mathcal{R}(a_1), \mu(b_1)\} = \inf_{x=a_1b_1} \max\{\mu(b_1)\} < t \text{ and} \\
(\mu *_a \mathcal{R})(x) &= \inf_{x=(y+c)z-yz} \max\{\mu(c), \mathcal{R}(z)\} = \inf_{x=(y+c)z-yz} \mu(c) < t.
\end{aligned}$$

Thus $a, b_1, c \in L(\mu : t)$. So $x = ab \in L(\mu : t) \cdot_a \mathcal{R}$, $x = a_1b_1 \in \mathcal{R} \cdot_a L(\mu : t)$ and $x = (y + c)z - yz \in L(\mu : t) *_a \mathcal{R}$, since $L(\mu : t)$ is a quasi-ideal of R. So $x \in (L(\mu : t) \cdot_a \mathcal{R}) \cap (\mathcal{R} \cdot_a L(\mu : t)) \cap (L(\mu : t) *_a \mathcal{R})$, which implies that $x \in L(\mu : t)$ that is, $\mu(x) \leq t$, which is a contradiction.
Hence, $(\mu \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a \mu) \cup (\mu *_a \mathcal{R}) \supseteq \mu$.
Therefore μ is an anti fuzzy quasi-ideal of R. □

Theorem 3.17. *Let μ and λ be any two anti fuzzy quasi ideals of R. Then $\mu \cup \lambda$ is also an anti fuzzy quasi ideal of R.*

Proof. Assume that μ and λ are any two anti fuzzy quasi-ideal of R. For $x, y \in R$, we have

$$\begin{aligned}(\mu \cup \lambda)(x-y) &= \max\{\mu(x-y), \lambda(x-y)\} \\ &\leq \max\{\max\{\mu(x), \mu(y)\}, \max\{\lambda(x), \lambda(y)\}\} \\ &= \max\{\max\{\mu(x), \lambda(x)\}, \max\{\mu(y), \lambda(y)\}\} \\ &= \max\{(\mu \cup \lambda)(x), (\mu \cup \lambda)(y)\}.\end{aligned}$$

Let $x \in R$ and select $a, b, c, y, z \in R$ such that $x = ab = (y+c)z - yz$. Then

$$\max\{\inf_{x=ab} \max\{\mu(a), \mathcal{R}(b)\},$$
$$\inf_{x=ab} \max\{\mathcal{R}(a), \mu(b)\}$$

and

$$\inf_{x=(y+c)z-yz} \max\{\mu(c), \mathcal{R}(z)\}\} \geq \mu(x).$$

Thus

$$\max\{\inf_{x=ab} \mu(a), \inf_{x=ab} \mu(b), \inf_{x=(y+c)z-yz} \mu(c)\} \geq \mu(x). \tag{3.1}$$

Similarly,

$$\max\{\inf_{x=ab} \lambda(a), \inf_{x=ab} \lambda(b), \inf_{x=(y+c)z-yz} \lambda(c)\} \geq \lambda(x). \tag{3.2}$$

So

$$\begin{aligned}&(((\mu \cup \lambda) \cdot_a \mathcal{R}) \cup (\mathcal{R} \cdot_a (\mu \cup \lambda)) \cup ((\mu \cup \lambda) *_a \mathcal{R}))(x) \\ &= \max\{((\mu \cup \lambda) \cdot_a \mathcal{R})(x), (\mathcal{R} \cdot_a (\mu \cup \lambda))(x), ((\mu \cup \lambda) *_a \mathcal{R})(x)\} \\ &= \max\{\inf_{x=ab} (\mu \cup \lambda)(a), \inf_{x=ab} (\mu \cup \lambda)(b), \inf_{x=(y+c)z-yz} (\mu \cup \lambda)(c)\} \\ &= \max\{\inf_{x=ab} \max\{\mu(a), \lambda(a)\}, \inf_{x=ab} \max\{\mu(b), \lambda(b)\}, \\ &\qquad \inf_{x=(y+c)z-yz} \max\{\mu(c), \lambda(c)\}\} \\ &\geq \max\{\max\{\inf_{x=ab} \mu(a), \inf_{x=ab} \mu(b), \inf_{x=(y+c)z-yz} \mu(c)\}, \\ &\qquad \max\{\inf_{x=ab} \lambda(a), \inf_{x=ab} \lambda(b), \inf_{x=(y+c)z-yz} \lambda(c)\}\} \\ &\geq \max\{\mu(x), \lambda(x)\}, \text{from (3.1) and (3.2)} \\ &= (\mu \cup \lambda)(x).\end{aligned}$$

Hence $\mu \cup \lambda$ is an anti fuzzy quasi-ideal of R. □

Intersection of any two anti fuzzy quasi ideals of near-ring need not be an anti fuzzy quasi ideal as shown in the following example.

Example 3.18. consider Example 3.12. In this near-ring R, let $\mu : R \to [0, 1]$ be a fuzzy set defined by $\mu(0) = 0.2, \mu(a) = \mu(c) = 0.6, \mu(b) = 0.4$. Then

$(\mu \cdot_a \mathcal{R})(0) = 0.2$ $\quad(\mathcal{R} \cdot_a \mu)(0) = 0.2$ $\quad(\mu *_a \mathcal{R})(0) = 0.2$
$(\mu \cdot_a \mathcal{R})(a) = 0.2$ $\quad(\mathcal{R} \cdot_a \mu)(a) = 0.6$ $\quad(\mu *_a \mathcal{R})(a) = 1$
$(\mu \cdot_a \mathcal{R})(b) = 0.4$ $\quad(\mathcal{R} \cdot_a \mu)(b) = 0.4$ $\quad(\mu *_a \mathcal{R})(b) = 0.4$

$(\mu \cdot_a \mathcal{R})(c) = 0.4 \quad (\mathcal{R} \cdot_a \mu)(c) = 0.6 \quad (\mu *_a \mathcal{R})(c) = 1.$

Thus μ is an anti fuzzy quasi ideal of R. Similarly, let $\lambda : R \to [0,1]$ be a fuzzy set defined by $\lambda(0) = 0.1, \lambda(a) = 0.5, \lambda(b) = \lambda(c) = 0.8$. Then

$(\lambda \cdot_a \mathcal{R})(0) = 0.1 \quad (\mathcal{R} \cdot_a \lambda)(0) = 0.1 \quad (\lambda *_a \mathcal{R})(0) = 0.1$

$(\lambda \cdot_a \mathcal{R})(a) = 0.1 \quad (\mathcal{R} \cdot_a \lambda)(a) = 0.5 \quad (\lambda *_a \mathcal{R})(a) = 1$

$(\lambda \cdot_a \mathcal{R})(b) = 0.8 \quad (\mathcal{R} \cdot_a \lambda)(b) = 0.8 \quad (\lambda *_a \mathcal{R})(b) = 0.8$

$(\lambda \cdot_a \mathcal{R})(c) = 0.8 \quad (\mathcal{R} \cdot_a \lambda)(c) = 0.8 \quad (\lambda *_a \mathcal{R})(c) = 0.8.$

Thus λ is an anti fuzzy quasi ideal of R. But $\mu \cap \lambda$ is not an anti fuzzy quasi ideal, since $(\mu \cap \lambda)(a - b) = (\mu \cap \lambda)(c) = .6 \nleq 0.5 = \max\{0.5, 0.4\} = \max\{(\mu \cap \lambda)(a), (\mu \cap \lambda)(b)\}$.

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