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Monotonic soft sets and its applications

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Abstract. In this paper, the concept of monotonic soft set is introduced and some of its structural properties are studied. Then some applications are expressed.

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1. Introduction

In science, engineering, economics and environmental sciences, many scientists seeks to develop a mathematical model to analyze the uncertainty. But we cannot successfully use classical mathematical methods for those models. Firstly, L.A. Zadeh [15] proposed fuzzy set theory which is important tool to solve problems that contains vagueness. This theory has been studied by many scientists over the years.

Soft set theory, which is a completely new approach for modelling uncertainty, is introduced by Molodtsov [11] in 1999. He established the fundamental results of this theory and applied in analysis, game theory and probability theory. Set-theoretical operations such as soft subset, soft union, soft intersection etc. [2, 9], soft algebraic structures which are parametrized family of substructures of an algebraic structure [1, 7], soft relation and its properties [4] were studied extensively. As Molodtsov pointed out, soft set theory can be applied to many areas. In [13] and [6], it was applied in information systems and decision making problems. Kharal and Ahmad defined the concept of a mapping classes of soft sets and studied the properties of soft image and soft inverse image of soft sets in [8].

Our decisions guide our life and decisions depends on the attributes or parameters in life. Of course, the relationship among parameters affect our decisions. So we need to choose appropriate parameters, and this choice is made according to particular order of preferences of individual (or decision maker). Hence ordering of the preferences according to attributes or parameters is very important in decision making. Right here, soft set theory lend a helping hand to our decision making problems. Although this theory is very suitable, it does not order our preference attributes or parameters for our decisions. Therefore, we need to build a new tool ordering the parameters or according to ordered parameters.

In this paper, we have defined the concept of monotonic soft set to cope with the above-mentioned problems and have studied its structural properties. In Section 4., we give specific examples and some results for monotonic soft sets. In the last section, we suggest an algorithm of decision making very simply.

2. Preliminaries

In this section, we give in detail the materials to be used in following sections. Throughout this paper U will be an initial universe, E will be the set of all possible parameters which are attributes, characteristic or properties of the objects in U, and the set of all subsets of U will be denoted by $\mathcal{P}(U)$.

Definition 2.1 ([11]). Let A be a subset of E. A pair (F, A) is called a soft set over U where $F : A \longrightarrow \mathcal{P}(U)$ is a set-valued function.

As mentioned in [9], a soft set (F, A) can be viewed $(F, A) = \{a = F(a) \mid a \in A\}$ where the symbol "$a = F(a)$" indicates that the approximation for $a \in A$ is $F(a)$.

Definition 2.2 ([13]). For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B), denoted by $(F, A)\widetilde{\subset}(G, B)$, if

(i) $A \subset B$ and
(ii) $\forall a \in A$, $F(a) \subset G(a)$.

Definition 2.3 ([13]). Two soft sets (F, A) and (G, B) over a common universe U are said soft equal, denoted by $(F, A) = (G, B)$, if $(F, A)\widetilde{\subset}(G, B)$ and $(G, B)\widetilde{\subset}(F, A)$.

Definition 2.4 ([13]). Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \varnothing$. The soft intersection of (F, A) and (G, B) is denoted by $(F, A)\widetilde{\cap}(G, B)$, and is defined as $(F, A)\widetilde{\cap}(G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.

We will use this definition of intersection given in [13] instead of given in [9], because generally $F(c)$ and $G(c)$ are not necessarily equal for $c \in C$. So this definition is more applicable to soft sets.

Definition 2.5 ([9]). The soft union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C) , denoted by $(F, A)\widetilde{\cup}(G, B) = (H, C)$, where $C = A \cup B$, and $\forall c \in C$,

$$H(c) = \begin{cases} F(c) & ,\text{if } c \in A - B \\ G(c) & ,\text{if } c \in B - A \\ F(c) \cup G(c) & ,\text{if } c \in A \cap B. \end{cases}$$

Definition 2.6 ([2]). Let U be an initial universe set, E be the universe set of parameters and $A \subset E$.

(i) (F, A) is called a relative null soft set (with respect to the parameter set A), denoted by Φ_A, if $F(a) = \varnothing$ for all $a \in A$.

(ii) (F, A) called a relative whole soft set (with respect to the parameter set A), denoted by $\mathcal{U}_A$, if $F(a) = U$ for all $a \in A$.

The relative whole soft set $\mathcal{U}_E$ with respect to the universe set of parameters E and relative null soft set Φ_E are called the absolute soft set and null soft set over U, respectively.

Definition 2.7 ([9])**.** Let (F, A) and (G, B) be two soft sets over the common universe U. Then (F, A) AND (G, B) denoted by $(F, A) \wedge (G, B)$ and is defined by $(F, A) \wedge (G, B) = (H, A \times B)$, where $H((a, b)) = F(a) \cap G(b)$, for all $(a, b) \in A \times B$.

Definition 2.8 ([9])**.** Let (F, A) and (G, B) be two soft sets over the common universe U. Then (F, A) OR (G, B) denoted by $(F, A) \vee (G, B)$ and is defined by $(F, A) \vee (G, B) = (H, A \times B)$, where $H((a, b)) = F(a) \cup G(b)$, for all $(a, b) \in A \times B$.

Definition 2.9 ([13])**.** The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathcal{P}(U)$ is a mapping given by $F^c(a) = U - F(a)$ for all $a \in A$.

Example 2.10. Let $U = \{a, b, c\}$ be universe, $E = \{1, 2, 3\}$ be parameter set and $A = \{1, 3\} \subset E$. From Definition 2.1, $(F, A) = \{1 = \{a, b\},\ 3 = \{b, c\}\}$ is a soft set over U and its complement is $(F, A)^c = \{1 = \{c\}, 3 = \{a\}\}$ which is a soft set over U.

Definition 2.11 ([4])**.** Let (F, A) and (G, B) be two soft set over U. Then the cartesian product of (F, A) and (G, B) is defined as, $(F, A) \times (G, B) = (H, A \times B)$, where $H : A \times B \to \mathcal{P}(U \times U)$ and $H(a, b) = F(a) \times G(b)$, where $(a, b) \in A \times B$.

Let U be an initial universe and E be a parameters set, then the collection of all soft sets over U via E is denoted by $S(U, E)$. Now we can define the soft function that given function between universes and between parameters sets.

Kharal and Ahmad [8] defined the concept of soft function as the follows. We have modified appropriately.

Definition 2.12 ([8], Soft Mapping)**.** Let U_1, U_2 be initial universes, E_1, E_2 be parameters sets, φ be a function from U_1 to U_2 and ψ be a function from E_1 to E_2. Then the pair (φ, ψ) is called a soft function from $S(U_1, E_1)$ to $S(U_2, E_2)$.

The image of each $(F, A) \in S(U_1, E_1)$ under the soft function (φ, ψ) is denoted by $(\varphi, \psi)(F, A) = (\varphi F, \psi(A))$ and is defined as following:

$$(\varphi F)(\beta) = \begin{cases} \varphi\left(\bigcup_{\alpha \in \psi^{-1}(\beta) \cap A} F(\alpha)\right) & \text{, if } \psi^{-1}(\beta) \cap A \neq \varnothing \\ \varnothing & \text{, otherwise} \end{cases}$$

for each $\beta \in \psi(A)$.

Similarly, the inverse image of each $(G, B) \in S(U_2, E_2)$ under the soft function (φ, ψ) is denoted by $(\varphi, \psi)^{-1}(G, B) = (\varphi^{-1}G, \psi^{-1}(B))$ and is defined as following;

$$(\varphi^{-1}G)(\alpha) = \begin{cases} \varphi^{-1}(G(\psi(\alpha))) & \text{, if } \psi(\alpha) \in B \\ \varnothing & \text{, otherwise} \end{cases}$$

for each $\alpha \in \psi^{-1}(B)$.

Min described the similarity in soft set theory and gave some results in [10]. He gave the definition of similarity between two soft sets as follows.

Definition 2.13 ([10]). Let (F, A) and (G, B) be soft sets over a common universe set U. Then (F, A) is similar to (G, B) (simply $(F, A) \cong (G, B)$) if there exists a bijective function $\phi : A \to B$ such that $F(\alpha) = (G \circ \phi)(\alpha)$ for every $\alpha \in A$, where $(G \circ \phi)(\alpha) = G(\phi(\alpha))$.

Now, we can give the definition of generalized form of similarity between soft sets over different universes as follows;

Definition 2.14. Let E be a set of parameters, U and V be two universes and (F, A) and (G, B) be soft sets over U and V respectively, where $A, B \subseteq E$. We called that (F, A) is similar to (G, B) if there exist bijective functions $f : U \to V$ and $\phi : A \to B$ such that $(f \circ F)(\alpha) = (G \circ \phi)(\alpha)$ for every $\alpha \in A$.

Note that, the given functions in the above definition should not be confused with the soft functions.

3. Monotonic soft sets

Ali et al. defined the concept of lattice ordered soft sets in [3]. They also discussed some soft set-theoretical operations among them. But in their works, the parameter set is taken as the lattice. So, this situation is very restricted.

We prefer to call monotonic soft set instead of lattice ordered soft sets. Because, our definition is more general than their definition due to fact that we take a partial ordered set for the parameter set. Now, we can define the monotonic soft set as follows.

Definition 3.1. Let U be an initial universe, E be a parameters set and $\leq \subseteq E \times E$ be a partial order relation on E and let (F, E) be a soft set over U. Then (F, E) is called monotonic (increasing) soft set over U if and only if for $x, y \in E$ if $x \leq y$, then $F(x) \subseteq F(y)$.

The dual of this definition is for monotonic (decreasing) soft set over an initial universe U, i.e. (F, E) is monotonic (decreasing) soft set over U if and only if for $x, y \in E$ if $x \leq y$, then $F(x) \supseteq F(y)$.

Note that a monotonic soft set over a universe may be increasing or decreasing. It depends on how it is defined. Ignoring the increasing or decreasing, we would prefer to call monotonic soft set in general. But we would emphasize if it is important.

Example 3.2. Let $U = \{a, b, c\}$ be an initial universe, $E = \{1, 2, 3\}$ and $\leq = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$ be a partial order relation on E. Then $(F, E) = \{1 = \{a\}, 2 = \{a, b\}, 3 = \{a, b\}\}$ is a monotonic soft set over U (of course it is increasing) .

Aktaş et al. in [1] proved that every fuzzy set may be considered a soft set, i.e. if F is a fuzzy set over U, then $(F, [0, 1])$ is a soft set over U. Now the interval $[0, 1]$ is total ordered set with respect to ordinary partial order relation. Then we obtain the following example;

Example 3.3. Every fuzzy set may be considered a monotonic soft set. Actually, for $\alpha_1, \alpha_2 \in [0, 1]$, either $\alpha_1 \leq \alpha_2$ or $\alpha_2 \leq \alpha_1$, and if $\alpha_1 \leq \alpha_2$ then $F(\alpha_1) \supset F(\alpha_2)$. Thus, $(F, [0, 1])$ is monotonic (decreasing) soft set over U.

Example 3.4. Monotone sequences of subsets of a set can be regarded as a monotonic soft set. So, let U be a set and $\{A_i\}_{i\in I}$ be a monotone sequence of subsets of U. If we take the index set I as a parameter set and define $F : I \to \mathcal{P}(U)$ mapping such that $F(i) = A_i$, then we obtain the monotonic soft set (F, I) over U.

Soft set-theoretic operations between two monotonic soft sets can be given as follows;

Theorem 3.5. *If (F, E) and (G, E) are monotonic soft sets over U, then $(F, E)\widetilde{\cap}(G, E)$ is also monotonic soft set over U.*

Proof. Suppose that $(F, E)\widetilde{\cap}(G, E) = (H, E)$ and (H, E) is soft set over U. We take $x, y \in E$ and $x \leq y$. Since (F, E) and (G, E) are monotonic soft sets,

$$H(x) = F(x) \cap G(x) \subseteq F(y) \cap G(y) = H(y).$$

Thus (H, E) is monotonic soft set over U. □

Theorem 3.6. *If (F, E) and (G, E) are monotonic soft sets over U, then $(F, E)\widetilde{\cup}(G, E)$ is also monotonic soft set over U.*

Proof. Suppose that $(F, E)\widetilde{\cup}(G, E) = (H, E)$. As a result of Definition 2.5, for all $x \in E$, $H(x) = F(x) \cup G(x)$. Let $x, y \in E$ and $x \leq y$. Since (F, E) and (G, E) are monotonic soft sets,

$$H(x) = F(x) \cup G(x) \subseteq F(y) \cup G(y) = H(y).$$

Thus (H, E) is a monotonic soft set over U. □

Theorem 3.7. *The null soft set Φ_E and the absolute soft set $\mathcal{U}_E$ are monotonic soft sets over U.*

Proof. Since $\Phi_E = (F, E)$ is null soft set, then $F(x) = \varnothing$ for all $x \in E$. Let $x \leq y$ for $x, y \in E$. Then $F(x) = \varnothing \subseteq \varnothing = F(y)$. Thus Φ_E is monotonic soft set over U.

Similarly, since $\mathcal{U}_E = (F, E)$ is absolute soft set, $F(x) = U$ for all $x \in E$. Let $x \leq y$ for $x, y \in E$. Then $F(x) = U \subseteq U = F(y)$. Thus $\mathcal{U}_E$ is monotonic soft set over U. □

Theorem 3.8. *If (F, E) and (G, E) are monotonic soft sets over U, then $(F, E) \wedge (G, E)$ is also monotonic soft set over U.*

Proof. From Definition 2.7, suppose that $(F, E) \wedge (G, E) = (H, E \times E)$ such that for all $(x, y) \in E \times E$, $H(x, y) = F(x) \cap G(y)$. Now we take $(x_1, y_1), (x_2, y_2) \in E \times E$ and $(x_1, y_1) \leq (x_2, y_2)$ which is defined as $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. Since (F, E) and (G, E) are monotonic soft sets, then

$$H(x_1, y_1) = F(x_1) \cap G(y_1) \subseteq F(x_2) \cap G(y_2) = H(x_2, y_2).$$

Thus $(H, E \times E)$ is a monotonic set over U. □

Theorem 3.9. *If (F, E) and (G, E) are monotonic soft sets over U, then $(F, E) \vee (G, E)$ is also monotonic soft set over U.*

Proof. From Definition 2.8, suppose that $(F,E) \vee (G,E) = (H, E \times E)$ such that for all $(x,y) \in E \times E$, $H(x,y) = F(x) \cup G(y)$. Now we take $(x_1,y_1),(x_2,y_2) \in E \times E$ and $(x_1,y_1) \leq (x_2,y_2)$. Since (F,E) and (G,E) are monotonic soft sets,

$$H(x_1,y_1) = F(x_1) \cup G(y_1) \subseteq F(x_2) \cup G(y_2) = H(x_2,y_2).$$

Then $(H, E \times E)$ is a monotonic set over U. □

Theorem 3.10. *If (F,E) is monotonic increasing (respectively decreasing) soft set over U, then its complement $(F,E)^c$ is also monotonic decreasing (respectively increasing) soft set over U.*

Proof. From Definition 2.7 for all $x \in E$, $F^c(x) = U - F(x)$. We take $x, y \in E$ and $x \leq y$. Since (F,E) is monotonic increasing soft set, we obtain

$$F^c(x) = U - F(x) \supseteq U - F(y) = F^c(y).$$

Then $(F,E)^c$ is monotonic decreasing soft set over U.

Take the (F,E) is monotonic decreasing. Then $x \leq y$. Thus $F(x) \supseteq F(y)$. So for its complement, we obtain that

$$F^c(x) = U - F(x) \subseteq U - F(y) = F^c(y).$$

Hence $(F,E)^c$ is a monotonic increasing soft set over U. □

Theorem 3.11. *Let (F,E) be a monotonic soft set over U and $(G,A)\widetilde{\subset}(F,E)$. Then (G,A) is monotonic soft set if and only if $F\mid_A = G$.*

Proof. Obvious. □

Theorem 3.12. *If (F,E) and (G,E) are monotonic soft sets over U, then their cartesian product $(F,E) \times (G,E)$ is a monotonic soft set over $U \times U$.*

Proof. From Definition 2.11, $(F,E) \times (G,E) = (H, E \times E)$ such that $H(x,y) = F(x) \times G(y)$ for each $(x,y) \in E \times E$. Since (F,E) and (G,B) are monotonic soft sets, we obtain

$$H(x_1,y_1) = F(x_1) \times G(y_1) \subseteq F(x_2) \times G(y_2) = H(x_2,y_2)$$

for each $(x_1,y_1),(x_2,y_2) \in E \times E$ which satisfy $(x_1,y_1) \leq (x_2,y_2)$. □

We can give a relation between similarity of soft sets and monotonicity of soft sets by following theorem.

Theorem 3.13. *Let (F,E) and (G,E) be similar soft sets over U. If ϕ is an order preserving bijection on E and (F,E) is monotonic soft set, then (G,E) is also monotonic soft set.*

Proof. Let $e_1 \leq e_2$. Since ϕ is order preserving bijection and (F,E) is monotonic, we obtain

$$G(e_1) = F \circ \phi(e_1) \subseteq F \circ \phi(e_2) = G(e_2).$$

Then (G,E) is a monotonic soft set. □

We have emphasized the importance of ordering among parameters in the introduction. Note that, in our daily lives, there are some parameters that take priority over other parameters. Such parameters are more effective problem solving decision-making process. Moreover, priority of a parameter makes valuable to be related to the set. The mentioned structure is expressed in a mathematically by following definitions.

Definition 3.14. Let E be a parameters set, $\leq$ be a partial order relation on E and $x, y \in E$. We call that the y parameter is superior than the x parameter if and only if $x \leq y$.

We know that, in our world, if the elements or phenomenon that satisfy a certain parameter or feature are few, then that parameter or feature is more precious. The following definition gives the notion of worthiness depends on superiority of the parameters.

Definition 3.15. If the parameter x is superior than y and $F(x) \subseteq F(y)$, then we call that x is valuable. Otherwise, x is valueless.

We can characterize monotonic soft sets using this approach as follows.

Definition 3.16. Let U be an initial universe, E be a parameters set, $\leq$ be a partial order relation on E, $A \subseteq E$ and (F, A) be a monotonic soft set over U. If A has a maximum element, then (F, A) is called maximum approximated soft set and $\max A$ is called dominant parameter.

Dually, If A has a minimum element, then (F, A) is called minimum approximated soft set and $\min A$ is called recessive parameter.

Example 3.17. In Example 3.2, (F, E) is both maximum approximated and minimum approximated soft set. 3 is a dominant parameter, and 1 is a recessive parameter.

Corollary 3.18. *If A is a chain and bounded subset of E, then (F, A) is both maximum approximated and minimum approximated soft set.* □

Note that E be a totally ordered set and A be a bounded subset of E, then (F, A) is a both maximum and minimum approximated soft set over U.

Example 3.19. From Example 3.3, since $[0, 1]$ is a chain and bounded, then every fuzzy set F over the universe U is both maximum and minimum approximated soft set.

Theorem 3.20. *Let E be a totally ordered set, A, B be bounded subsets of E. Then (F, A) and (G, B) are maximum (or minimum) approximated soft sets over U. Then $(F, A)\widetilde{\cap}(G, B)$ is also maximum (or minimum) approximated soft set over U.*

Proof. Since A and B are totally ordered and bounded subsets of E, $A \cap B$ is a totally ordered and bounded subset of E. Let $(F, A)\widetilde{\cap}(G, B) = (H, C)$. Since $C = A \cap B$ is totally ordered and bounded, (H, C) is maximum (or minimum) approximated soft set over U. □

Theorem 3.21. *Let E be a totally ordered set, A, B be bounded subsets of E. Then (F, A) and (G, B) are maximum (or minimum) approximated soft sets over U. Then $(F, A)\widetilde{\cup}(G, B)$ is also maximum (or minimum) approximated soft set over U.*

Proof. Since $C = A \cup B$ is totally ordered and bounded, $(F, A)\widetilde{\cup}(G, B)$ is maximum (or minimum) approximated soft set. $\square$

Recall that, let $(E_1, \leq_1)$ and $(E_2, \leq_2)$ be partially ordered sets and ψ be a function from E_1 to E_2. Then ψ is called order preserving function if and only if $\alpha_1 \leq_1 \alpha_2$ implies $\psi(\alpha_1) \leq_2 \psi(\alpha_2)$ for $\alpha_1, \alpha_2 \in E_1$. Now, if we have a order preserving function between partially ordered parameters sets, we obtain following theorems.

Theorem 3.22. *Let U_1, U_2 be initial universes, E_1, E_2 be parameters set and $\leq_1, \leq_2$ be partial order relations on E_1, E_2, respectively. $\varphi : U_1 \to U_2$ be a function and $\psi : E_1 \to E_2$ be order preserving function. If (F, E_1) is a monotonic soft set over U_1, then $(\varphi, \psi)(F, E_1)$ is also monotonic soft set over U_2.*

Proof. From Definition 2.12, $(\varphi, \psi)(F, E_1) = (\varphi F, \psi(E_1))$. Suppose that, let $\alpha_1 \leq_1 \alpha_2$ for $\alpha_1, \alpha_2 \in E_1$. Since ψ is a order preserving function, $\psi(\alpha_1) \leq_2 \psi(\alpha_2)$ and, say $\beta_1 = \psi(\alpha_1)$ and $\beta_2 = \psi(\alpha_2)$. Since (F, E_1) is a monotonic soft set, we obtain

$$\varphi F(\beta_1) = \varphi \left(\bigcup F(\alpha_1) \right) \subseteq \varphi \left(\bigcup F(\alpha_2) \right) = \varphi F(\beta_2).$$

Thus $(\varphi, \psi)(F, E_1)$ is a monotonic soft set over U_2. $\square$ $\square$

Theorem 3.23. *Let U_1, U_2 be initial universes, E_1, E_2 be parameters set and $\leq_1, \leq_2$ be partial order relations on E_1, E_2, respectively. $\varphi : U_1 \to U_2$ be a function and $\psi : E_1 \to E_2$ be order preserving function. If (G, E_2) is a monotonic soft set over U_2, then $(\varphi, \psi)^{-1}(G, E_2)$ is also monotonic soft set over U_1.*

Proof. Let $\alpha_1, \alpha_2 \in \psi^{-1}(E_2)$ and $\alpha_1 \leq_1 \alpha_2$. Since ψ is an order preserving function, $\psi(\alpha_1) \leq_2 \psi(\alpha_2)$. Since (G, E_2) is a monotonic soft set over U_2, from Definition 2.12, we obtain

$$(\varphi^{-1}G)(\alpha_1) = \varphi^{-1}(G(\psi(\alpha_1))) \subseteq \varphi^{-1}(G(\psi(\alpha_2))) = (\varphi^{-1}G)(\alpha_2)$$

for $\alpha_1, \alpha_2 \in \psi^{-1}(E_2)$. Thus $(\varphi, \psi)^{-1}(G, E_2)$ is a monotonic soft set over U_1. $\square$

As we understand that, of course, each parameter is not necesseraly compared with each other. However, there are always superior or less superior than these parameters which can not be compared. This leads us to the lattice structure of parameters ordered by superiority.

Let the parameter set E be a lattice. Then we obtain special monotonic soft sets over U.

Definition 3.24. Let (F, E) be a monotonic soft set over U, $e_1, e_2 \in E$ but they are not compared. Since E is a lattice, there exist unique parameters $\epsilon, \varepsilon \in E$ such that $\epsilon = e_1 \wedge e_2$ and $\varepsilon = e_1 \vee e_2$. Then we called that (F, E) is a supre-infimal soft set over U such that $F(\epsilon) \subset F(e_1) \subset F(\varepsilon)$ and $F(\epsilon) \subset F(e_2) \subset F(\varepsilon)$.

Note that, every supre-infimal soft set over U is a monotonic soft set. Clearly, if E is a chain, then every both maximum and minimum approximated soft sets are supre-infimal soft set.

4. Some specifical examples of monotonic soft sets and some applications

4.1. **In posets.** Let $(X, \leq)$ be a partially ordered set. $\downarrow x$ is called downset for the element $x \in X$ and defined $\downarrow x = \{y \mid y \leq x\}$. If we define the mapping $F : X \to \mathcal{P}(X)$ such that $F(x) =\downarrow x$ for all $x \in X$, then we obtain a soft set (F, X) over X.

Theorem 4.1. *Let $(X, \leq)$ be a partially ordered set. If we define the soft set (F, X) over X as above, then (F, X) is a monotonic soft set over X.*

Proof. Let $x_1, x_2 \in X$. If $x_1 \leq x_2$. Then we obtain

$$F(x_1) =\downarrow x_1 = \{y \mid y \leq x_1\} \subseteq \{y \mid y \leq x_2\} =\downarrow x_2 = F(x_2).$$

Thus (F, X) is a monotonic (increasing) soft set over X. □

Similarly, we obtain monotonic (decreasing) soft set over X with the upsets $\uparrow x = \{y \mid x \leq y\}$.

Example 4.2. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $|$ denote a division relation on X. So, the pair $(X, |)$ is a partially ordered set. Thus we obtain monotonic soft set with respect to downset over X as follows:

$$\begin{aligned}(F, X) = \{&1 = \{1\}, 2 = \{1, 2\}, 3 = \{1, 3\},\\ &4 = \{1, 2, 4\}, 5 = \{1, 5\}, 6 = \{1, 2, 3, 6\}\}.\end{aligned}$$

Theorem 4.3. *Let $(X, \leq_1)$ and $(Y, \leq_2)$ be two partially ordered sets and f be a mapping from X to Y. Then, f is an order isomorphism if and only if $(F, X) \cong (G, Y)$.*

Proof. Suppose that f is an order isomorphism. Then f is bijective and order preserving function. Thus we have

$$(f \circ F)(x) = f(F(x)) = f(\downarrow x) =\downarrow f(x) = G(f(x)) = (G \circ f)(x)$$

for all $x \in X$. So $(F, X) \cong (G, Y)$ from Definition 2.14.

On the other hand, assume that $(F, X) \cong (G, Y)$. Then there exists a bijective function $f : X \to Y$ such that $(f \circ F)(x) = (G \circ f)(x)$ for all $x \in X$. Thus we have $f(\downarrow x) =\downarrow f(x)$ for all $x \in X$. Now, we take $x_1 \leq_1 x_2$ for $x_1, x_2 \in X$. Then we have $\downarrow x_1 \subseteq\downarrow x_2$. Thus we have $f(\downarrow x_1) \subseteq f(\downarrow x_2)$. Since (F, X) similar to (G, Y), we have $\downarrow f(x_1) \subseteq\downarrow f(x_2)$. So $f(x_1) \leq_2 f(x_2)$.

Conversely, if $f(x_1) \leq_2 f(x_2)$, then we can obtain $x_1 \leq_1 x_2$ by using same way. Consequently, f is an order isomorphism. □

4.2. **In nested sets.** We can express the hierarchical models by using monotonic soft sets, such as nested sets. In [12], Nguyen and Kreinovich state the definition of nested sets in generalized form as follows.

Definition 4.4 ([12]). Let an integer d be fixed. This number will be called the number of experts. Let a finite lattice A be fixed, with elements α_0 and α_{d+1} that are 0 and 1 (i.e., for which $\alpha_0 \vee \alpha = \alpha_0, \alpha_{d+1} \vee \alpha = \alpha, \alpha_0 \wedge \alpha = \alpha$, and $\alpha_{d+1} \wedge \alpha = \alpha_{d+1}$). Elements of A will be called degrees of belief. Let a set U be given. This set will be called the universal set, or, the universe of discourse. By a piece of information (or, nested set), we mean a non-increasing mapping X from A to the set $\mathcal{P}(U)$ of all subsets of U, i.e., a mapping for which if $\alpha \leq \beta$, then $X(\alpha) \supseteq X(\beta)$. Let u be an element of U, and $\alpha \in A$ be degree of belief. We say that u belongs to the nested set X with a degree of belief $\leq \alpha$ if $u \in X(\alpha)$.

Using the above definition, we obtain a soft set over the universal set U such that A is a parameters set and X is a mapping from A to $\mathcal{P}(U)$, i.e. the pair (X, A) is a soft set over U. Thus, we obtain the following theorem with this characterization.

Theorem 4.5. *Each nested set over the universal set U is a monotonic (decreasing) soft set.*

Proof. From Definition 4.4, it is obvious. □

Note that, this theorem is not true vice versa, in general.

By this way, we can state other hierarchical data models with using monotonic soft sets.

4.3. **In the von Neumann universe.** [14]

The von Neumann universe also known as cumulative hierarchy V is the hierarchy of all set-theoretic sets. Hence, with von Neumann, a natural number n is conveniently represented as $\{0, 1, 2, \ldots, n-1\}$. All sets in V are constructed from nothing. The von Neumann universe is constructed inductively, starting from $\varnothing$ and successively applying the Powerset operation $\mathcal{P}$. That is, $V_0 = \varnothing$, $V_1 = \mathcal{P}(\varnothing) = \{\varnothing\}$, $\cdots$, $V_{n+1} = \mathcal{P}(V_n)$ and $V_n \subset V_{n+1}$.

In other words, The cumulative hierarchy is a collection of sets V_n indexed by the class of ordinal numbers, in particular, V_n is the set of all sets having ranks less than n.

Thereby we obtain a monotonic soft set with the von Neumann universe. If we take the parameters set as the set of all ordinals, then for each ordinal n we have the set V_n. That is, we define the mapping $F : \mathcal{O} \to V$ such that $F(n) = V_n$ for each ordinal n, $(F, \mathcal{O})$ is a monotonic soft set.

4.4. **In connection structures.** In [5], Biacino and Gerla defined connection structures using mereological relations between objects. Connection structures are defined as follows.

Definition 4.6 ([5])**.** Let U be a nonempty set and $\mathcal{C}$ a binary relation on U, set $\mathcal{C}(x) = \{y \in U \mid (x, y) \in \mathcal{C}\}$ and suppose the following axioms are true for every $x, y \in U$:

(i) $(x, x) \in \mathcal{C}$,

(ii) $(x, y) \in \mathcal{C} \Rightarrow (y, x) \in \mathcal{C}$,

(iii) $\mathcal{C}(x) = \mathcal{C}(y) \Rightarrow x = y$.

We call regions the elements of U and, if $x, y \in U$ and $(x, y) \in \mathcal{C}$, we say that x is connected with y. If X is a nonempty subset of U, we say that x is the fusion of X just in case for every $y \in U$, $(x, y) \in \mathcal{C}$ iff for some $z \in X$, $(z, y) \in \mathcal{C}$; in other words, x is the fusion of X provided that $\mathcal{C}(x) = \bigcup\{\mathcal{C}(z) \mid z \in X\}$. The fusion of the nonempty subsets of U is assured by the following axiom.

(iv) $X \subseteq U$ and $X \neq \varnothing$ imply there exists $x \in U$ such that x is the fusion of X.

If (i)-(iv) are satisfied, we say that $\mathbf{C} = (U, \mathcal{C})$ is connection structure.

The axiom (iii) implies that the relation $\leq$ define in U by

$$x \leq y \Leftrightarrow \mathcal{C}(x) \subseteq \mathcal{C}(y)$$

is a partial ordering. Because of that we obtain a monotonic soft set over U, i.e. the connection structure $\mathbf{C}$ is a monotonic soft set.

5. Decision algorithm via monotonic soft sets

As is known, there are many applications of soft sets in many fields [6, 13]. Many researchers have developed decision-making techniques using the soft set theory. Here, we will give a new decision algorithm using monotonic soft sets.

Algorithm 1. Construct a monotonic soft set with respect to valueness of the parameters. So, this monotonic soft set is decreasing.

Algorithm 2. Choose maximal parameters and related approximated set with respect to superiority.

Algorithm 3. Intersect these maximal approximated set.

As a result, decision maker will choose an element in this set.

This decision algorithm let us to decide rapidly by choosing the most valuable parameters among to parameters ordered by priority.

We can discuss the Molodtsov's famous example to see how to work these decision making algorithms.

Example 5.1. A soft set (F, E) describes attractiveness of the houses which Mrs. Kandemir is going to buy. Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be a set of six houses under consideration. $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ be a set of parameters such that

- e_1 = expensive,
- e_2 = beautiful,
- e_3 = wooden,
- e_4 = cheap,
- e_5 = in the green surroundings,
- e_6 = modern,
- e_7 = in good repair.

Hasse diagram of the order of the parameters according to prefences of Mrs. Kandemir is as follows; Then, the (decreasing) monotonic soft set is built according

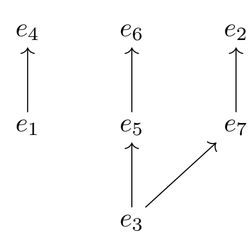

Figure 1. Order of The Parameters

to preferences of Mrs. Kandemir as follows;

$$(F,E) = \{e_1 = \{h_1, h_3, h_6\}, e_2 = \{h_3\}, e_3 = \{h_2, h_3, h_5\}, e_4 = \{h_3\},$$
$$e_5 = \{h_2, h_3, h_5\}, e_6 = \{h_3, h_5\}, e_7 = \{h_3, h_5\}\}.$$

Thus we obtain that maximal parameters are e_2, e_4, e_6. So their approximated sets are $F(e_2) = \{h_3\}$, $F(e_4) = \{h_3\}$ and $F(e_6) = \{h_3, h_5\}$. Taking the intersection of these sets:

$$F(e_2) \cap F(e_4) \cap F(e_6) = \{h_3\}.$$

Hence, Mrs. Kandemir will select h_3.

6. Conclusion

Ordering plays an important role in our daily life where many problems are caused by the parameters. Therefore, we need to identify the relationship better among parameters that we can solve such problems, that is, parameters can be ordered according to our preferences. In this paper, we have built monotonic soft set which parameter set is a poset then we have studied its properties. We gave some special examples of monotonic soft sets related other mathematical structures. We have showed that partially ordered sets, nested sets, The von Neumann Universe, connection structures are specifical monotonic soft sets. We have proved a theorem (Theorem 4.3) that can be important for partially ordered set theory. Finally, we constructed a decision algorithm using monotonic soft sets and gave a simple example.

As given in [13], there exist compact connections between soft sets and information systems. Relationship between monotonic soft sets and information systems, maybe ordered information systems, can be considered for future studies. The author hope that this article sheds light on a way of working in this field.

References

[1] H. Aktaş and N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.
[2] M. I. Ali, F. Feng, X. Y. Liu, W. K. Min and M. Shabir, On Some New Operations in Soft Set Theory, Comput. Math. Appl. 57 (2009) 1547–1553.
[3] M. I. Ali, T. Mahmood, M. M. U. Rehman and M. F. Aslam, On lattice ordered soft sets, Appl. Soft Comput. 36 (2015) 499–505.
[4] K. V. Babitha and J. J. Sunil, Soft set relations and functions, Comput. Math. Appl. 60 (2010) 1840–1849.
[5] L. Biacino and G. Gerla, Connection Structures, Notre Dame J. Form. Log. 32 (2) (1991) 242–247.
[6] D. Chen, E. C. C. Tsang, D. S. Yeung and X. Wong, The parametrization reduction of soft sets and its applications, Comput. Math. Appl. 49 (2005) 757–763.
[7] F. Feng, Y. B. Jun and X. Zhao, Soft Semirings, Comput. Math. Appl. 56 (2008) 2621–2628.
[8] A. Kharal, B. Ahmad, Mappings on Soft Classes, New Math. and Nat. Computation 7 (3) (2011) 471–481.
[9] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[10] W. K. Min, Similarity in soft set theory, Appl. Math. Lett. 25 (2012) 310–314.
[11] D. Molodtsov, Soft Set Theory-First Results, Comput. Math. Appl. 37 (1999) 19–31.
[12] H. T. Nguyen and V. Kreinovich, Nested intervals and sets: Concepts, relations to fuzzy sets, and applications, Applications of Interval Computations, Series: Applied Optimization Vol.3 Springer (1996) 1–46.

[13] D. Pei and D. Miao, From Soft Sets to Information Systems, Proc. IEEE Int. Confer. Granular Comput. 2 (2005) 617–621.
[14] D. Singh, J.N. Singh, von Neumann Universe: A Perspective, Int. J. Contemp. Math. Sci. 2 (10) (2007) 475–478.
[15] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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