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On product fuzzy graphs

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Abstract. In this paper, we provide three new products on product fuzzy graphs, we give sufficient conditions for each one of them to be strong and we show that if any of these products is complete, then at least one factor is strong. Moreover, we introduce and study the notions of balanced and cobalanced product fuzzy graphs and give necessary and sufficient conditions for the product of two balanced (resp., cobalanced) product fuzzy graphs to be balanced (resp., cobalanced). Finally, we prove that these notions are preserved under isomorphism.

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1. Background

A fuzzy subset of a non-empty set V is a mapping $\sigma : V \to [0,1]$ and a fuzzy relation μ on a fuzzy subset σ, is a fuzzy subset of $V \times V$. All throughout this paper, we assume that σ is reflexive, μ is symmetric and V is finite.

Definition 1.1 ([11]). A fuzzy graph, with V as the underlying set, is a pair $G : (\sigma, \mu)$, where

$\sigma : V \to [0,1]$ is a fuzzy subset

and

$\mu : V \times V \to [0,1]$ is a fuzzy relation on σ such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum.

The underlying crisp graph of G is denoted by $G^* : (\sigma^*, \mu^*)$, where

$\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$

and

$\mu^* = \sup p(\mu) = \{(x,y) \in V \times V : \mu(x,y) > 0\}$.

$H = (\sigma', \mu')$ is a fuzzy subgraph of G, if there exists $X \subseteq V$ such that

$\sigma' : X \to [0, 1]$ is a fuzzy subset

and

$\mu' : X \times X \to [0, 1]$ is a fuzzy relation on σ' such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in X$.

Definition 1.2 ([11])**.** Two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ are isomorphic, if there exists a bijection $h : V_1 \to V_2$ such that

$\sigma_1(x) = \sigma_2(h(x))$ for all $x \in V_1$

and

$\mu_1(x, y) = \mu_2(h(x), h(y))$ for all $(x, y) \in E_1$.

Then we write $G_1 \simeq G_2$ and h is called an isomorphism. If $G_1 = G_2$, h is called an automorphism.

Product fuzzy graphs were introduced by Ramaswamy and Poornima in [10], where they used the operation of product instead of minimum.

Definition 1.3 ([10])**.** Let $G^* : (V, E)$ be a graph, σ be a fuzzy subset of V and μ be a fuzzy subset of $V \times V$. We call $G : (\sigma, \mu)$ product fuzzy graph, if $\mu(x, y) \leq \sigma(x)\sigma(y)$ for all $x, y \in V$.

The following is an immediate result.

Lemma 1.4. *Every product fuzzy graph is a fuzzy graph, but the converse need not be true.*

Definition 1.5 ([10])**.** A product fuzzy graph $G : (\sigma, \mu)$ with underlying graph $G^* : (V, E)$ is said to be complete if $\mu(x, y) = \sigma(x)\sigma(y)$ for all $x, y \in V$.

Definition 1.6 ([10])**.** A product fuzzy graph $G : (\sigma, \mu)$ with underlying graph $G^* : (V, E)$ is said to be strong if $\mu(x, y) = \sigma(x)\sigma(y)$ for all $(x, y) \in E$.

Definition 1.7 ([10])**.** The complement of a product fuzzy graph $G : (\sigma, \mu)$ is $G^c : (\sigma^c, \mu^c)$ where $\sigma^c = \sigma$ and

$$\begin{aligned} \mu^c(x, y) &= \sigma^c(x)\sigma^c(y) - \mu(x, y) \\ &= \sigma(x)\sigma(y) - \mu(x, y). \end{aligned}$$

Graph theory has several interesting applications in system analysis, operations research and economics. Since most of the time the aspects of graph problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. The concept of fuzzy relation which has a widespread application in pattern recognition was introduced by Zadeh [15] in his landmark paper "Fuzzy sets" in 1965. Fuzzy graph and several fuzzy analogs of graph theoretic concepts were first introduced by Rosenfeld [11] in 1975. Sense then, fuzzy graph theory is finding an increasing number of applications in modelling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems.

Since the notions of degree, complement, completeness, regularity, connectedness and many others play very important rules in the crisp graph case, the idea is to find what corresponds to these notions in the case of fuzzy graphs. Several authors introduced and studied product fuzzy graphs, see for example [1, 2, 3, 10, 12, 13]. AL-Hawary [4] introduced the concept of balanced fuzzy graphs. He defined three new operations on fuzzy graphs and explored what classes of fuzzy graphs are balanced. Sense then, many authors have studied the idea of balanced on distinct kinds of fuzzy graphs, see for example [5, 6, 7, 8, 9, 14]. Our aim in this paper is to study the notions of complete, strong, balanced and cobalanced product fuzzy graphs. Moreover, several relatively new operations on product fuzzy graphs are provided and properties are explored.

We remark that the results in this paper were done in Bayan Hourani masters thesis titled "ON COMPLETE AND BALANCED FUZZY GRAPHS" under the supervision of Talal Al-Hawary at Yarmouk University in 2015.

2. Complete product fuzzy graphs

In this section, we provide relatively new definitions and operations on product fuzzy graphs. We start by presenting some results on self-complementary product fuzzy graphs.

Lemma 2.1. *If $G:(\sigma,\mu)$ with underlying graph $G^*:(V,E)$ is a self-complementary product fuzzy graph, then*

$$\sum_{(x,y)\in E}\mu(x,y)=\frac{1}{2}\sum_{(x,y)\in E}\sigma(x)\sigma(y).$$

Proof. Let $G:(\sigma,\mu)$ be a self-complementary product fuzzy graph. Then by Definition 1.2, there exist a bijection $h:V\to V$ such that $\sigma^c(h(x))=\sigma(x)$ for all $x\in V$ and $\mu^c(h(x),h(y))=\mu(x,y)$ for all $(x,y)\in E$. But by Definition 1.7, we get

$$\mu^c(h(x),h(y))=\sigma^c(h(x))\sigma^c(h(y))-\mu(h(x),h(y))$$

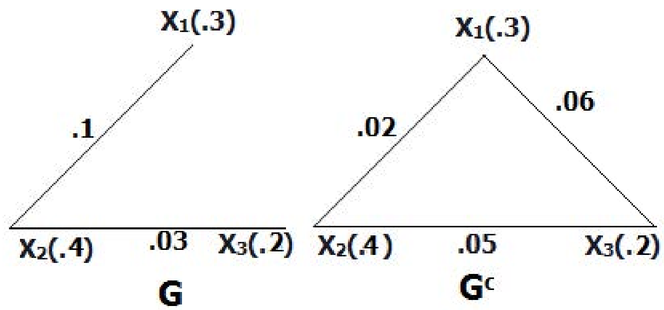
and then

$$\mu(x,y)=\sigma(x)\sigma(y)-\mu(h(x),h(y)).$$

Thus $\mu(x,y)+\mu(h(x),h(y))=\sigma(x)\sigma(y)$. So
$\sum_{(x,y)\in E}\mu(x,y)+\sum_{(x,y)\in E}\mu(h(x),h(y))=\sum_{x,y\in V}\sigma(x)\sigma(y)$.
Furthermore, $\sum_{(x,y)\in E}\mu(x,y)=\sum_{(x,y)\in E}\mu(h(x),h(y))$.
Hence $2\sum_{(x,y)\in E}\mu(x,y)=\sum_{x,y\in V}\sigma(x)\sigma(y)$.
Therefore $\sum_{(x,y)\in E}\mu(x,y)=\frac{1}{2}\sum_{x,y\in V}\sigma(x)\sigma(y)$. □

We now give an example to show that the converse of the above result need not be true.

Example 2.2. Consider the following graph G. Then
$\sum_{(x,y)\in E}\mu(x,y)=\frac{1}{2}\sum_{x,y\in V}\sigma(x)\sigma(y)=0.13$,
but G is not self-complementary.



Lemma 2.3. *Let $G : (\sigma, \mu)$ be a product fuzzy graph with underlying graph $G^* : (V, E)$ such that$\mu(x, y) = \frac{1}{2}\sigma(x)\sigma(y)$ for all $x, y \in V$. Then $G$is self-complementary.*

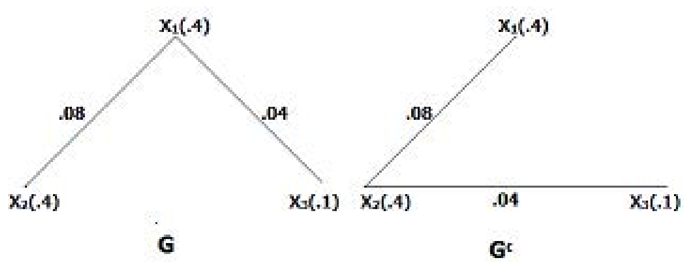
Proof. Define $h : V \to V$ by $\sigma(x) = \sigma(h(x))$ for all $x \in V$. Then for all $x, y \in V$,

$$\begin{aligned} \mu^c(x, y) &= \sigma(x)\sigma(y) - \mu(x, y) \\ &= \sigma(x)\sigma(y) - \frac{1}{2}(\sigma(x)\sigma(y)) \\ &= \frac{1}{2}(\sigma(x)\sigma(y)) \\ &= \mu(x, y). \end{aligned}$$

Thus $G \simeq G^c$. □

We now give an example to show the converse of the above result need not be true.

Example 2.4. The following graph G is self-complementary, but $\mu(x_1, x_3) = 0.04$ and $\frac{1}{2}(\sigma(x_1)\sigma(x_3)) = 0.02$.



3. Operations on product fuzzy graphs

In this section, we define relatively new operations on product fuzzy graphs that are similar to those of fuzzy graphs in [4]. We first start by recalling the following definition from [10].

Definition 3.1 ([10]). Assume that $V_1 \cap V_2 = \varnothing$. Then The direct product (simply, product) of two product fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ is defined to be the product fuzzy graph $G_1 \boxdot G_2 : (\sigma_1 \boxdot \sigma_2, \mu_1 \boxdot \mu_2)$ with underlying graph $G^* : (V_1 \times V_2, E)$, where

$$E = \{(x_1, y_1)(x_2, y_2) : (x_1, x_2) \in E_1, (y_1, y_2) \in E_2\},$$
$$(\sigma_{1\boxdot}\sigma_2)(x, y) = \sigma_1(x)\sigma_2(y), \text{ for all } (x, y) \in V_1 \times V_2$$

and

$$(\mu_{1\boxdot}\mu_2)((x_1, y_1)(x_2, y_2)) = \mu_1(x_1, x_2)\mu_2(y_1, y_2),$$

for all $(x_1, x_2) \in E_1$ and $(y_1, y_2) \in E_2$.

Definition 3.2. Assume that $V_1 \cap V_2 = \varnothing$. Then the semi-strong product of two product fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ is defined to be the product fuzzy graph $G_1 \cdot G_2 : (\sigma_1 \cdot \sigma_2, \mu_1 \cdot \mu_2)$ with underlying graph $G^* : (V_1 \times V_2, E)$, where

$$E = \{(x, y_1)(x, y_2) : x \in V_1, (y_1, y_2) \in E_2\}$$
$$\cup\{(x_1, y_1)(x_2, y_2) : (x_1, x_2) \in E_1, (y_1, y_2) \in E_2\},$$
$$(\sigma_1 \cdot \sigma_2)(x, y) = \sigma_1(x)\sigma_2(y) \text{ for all } (x, y) \in V_1 \times V_2,$$
$$(\mu_1 \cdot \mu_2)((x, y_1)(x, y_2)) = (\sigma_1(x))^2\mu_2(y_1, y_2)$$

and

$$(\mu_1 \cdot \mu_2)((x_1, y_1)(x_2, y_2)) = \mu_1(x_1, x_2)\mu_2(y_1, y_2),$$

for all $x, x_1, x_2 \in V_1$ and $y_1, y_2 \in V_2$.

Definition 3.3. Assume that $V_1 \cap V_2 = \varnothing$. Then the strong product of two product fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ is defined to be the product fuzzy graph $G_1 \otimes G_2 : (\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2)$ with underlying graph $G^* : (V_1 \times V_2, E)$, where

$$E = \{(x, y_1)(x, y_2) : x \in V_1, (y_1, y_2) \in E_2\}$$
$$\cup\{(x_1, y)(x_2, y) : (x_1, x_2) \in E_1, y \in V_2\}$$
$$\cup\{(x_1, y_1)(x_2, y_2) : (x_1, x_2) \in E_1, (y_1, y_2) \in E_2\},$$
$$(\sigma_1 \otimes \sigma_2)(x, y) = \sigma_1(x)\sigma_2(y), \text{ for all } (x, y) \in V_1 \times V_2,$$
$$(\mu_{1\otimes}\mu_2)((x, y_1)(x, y_2)) = (\sigma_1(x))^2\mu_2(y_1, y_2),$$
$$(\mu_{1\otimes}\mu_2)((x_1, y)(x_2, y)) = (\sigma_2(y))^2\mu_1(x_1, x_2)$$

and

$$(\mu_{1\otimes}\mu_2)((x_1, y_1)(x_2, y_2)) = \mu_1(x_1, x_2)\mu_2(y_1, y_2),$$

for all $x, x_1, x_2 \in V_1$ and $y, y_1, y_2 \in V_2$.

Next, we study which of these operations preserves the strong and complete notions.

Theorem 3.4. *If $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ are strong product fuzzy graphs, then $G_1 \boxdot G_2$ is strong.*

Proof. If $(x_1, y_1)(x_2, y_2) \in E$, then since G_1 and G_2 are strong,

$$\begin{aligned}(\mu_1 \boxdot \mu_2)((x_1, y_1)(x_2, y_2)) &= \mu_1(x_1, x_2)\mu_2(y_1, y_2) \\ &= \sigma_1(x_1)\sigma_1(x_2)\sigma_2(y_1)\sigma_2(y_2) \\ &= (\sigma_1 \boxdot \sigma_2)(x_1, y_1)(\sigma_1 \boxdot \sigma_2)(x_2, y_2).\end{aligned}$$

Thus $G_1 \boxdot G_2$ is strong. □

The following result comes from the fact that every complete product fuzzy graph is strong.

Corollary 3.5. *If $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ are complete product fuzzy graphs, then $G_1 \boxdot G_2$ is strong.*

Theorem 3.6. *If $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ are strong product fuzzy graphs, then $G_1 \odot G_2$ is strong.*

Proof. If $(x, y_1)(x, y_2) \in E$, then

$$\begin{aligned}(\mu_1 \odot \mu_2)((x, y_1)(x, y_2)) &= (\sigma_1(x))^2 \mu_2(y_1, y_2)\\ &= \sigma_1(x)\sigma_1(x)\sigma_2(y_1)\sigma_2(y_2)\\ &= (\sigma_1 \odot \sigma_2)(x, y_1)(\sigma_1 \odot \sigma_2)(x, y_2).\end{aligned}$$

If $(x_1, y_1)(x_2, y_2) \in E$, then, since G_1 and G_2 are strong,

$$\begin{aligned}(\mu_1 \odot \mu_2)((x_1, y_1)(x_2, y_2)) &= \mu_1(x_1, x_2)\mu_2(y_1, y_2)\\ &= \sigma_1(x_1)\sigma_1(x_2)\sigma_2(y_1)\sigma_2(y_2)\\ &= (\sigma_1 \odot \sigma_2)((x_1, y_1))(\sigma_1 \odot \sigma_2)((x_2, y_2)).\end{aligned}$$

Thus $G_1 \odot G_2$ is strong. □

Corollary 3.7. *If $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ are complete product fuzzy graphs, then $G_1 \odot G_2$ is strong.*

Theorem 3.8. *If $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ are product strong fuzzy graphs, then $G_1 \otimes G_2$ is strong.*

Proof. If $(x, y_1)(x, y_2) \in E$, then

$$\begin{aligned}(\mu_{1\otimes}\mu_2)((x, y_1)(x, y_2)) &= (\sigma_1(x))^2 \mu_2(y_1, y_2)\\ &= \sigma_1(x)\sigma_1(x)\sigma_2(y_1)\sigma_2(y_2)\\ &= (\sigma_1 \otimes \sigma_2)(x, y_1)(\sigma_1 \otimes \sigma_2)(x_2, y_2).\end{aligned}$$

If $(x_1, y)(x_2, y) \in E$, then

$$\begin{aligned}(\mu_{1\otimes}\mu_2)((x_1, y)(x_2, y)) &= (\sigma_2(y))^2 \mu_1(x_1, x_2)\\ &= \sigma_1(x_1)\sigma_1(x_2)\sigma_2(y)\sigma_2(y)\\ &= (\sigma_1 \otimes \sigma_2)(x_1, y)(\sigma_1 \otimes \sigma_2)(x_2, y).\end{aligned}$$

If $(x_1, y_1)(x_2, y_2) \in E$, then since G_1 and G_2 are strong

$$\begin{aligned}(\mu_1 \otimes \mu_2)((x_1, y_1)(x_2, y_2)) &= \mu_1(x_1, x_2)\mu_2(y_1, y_2)\\ &= \sigma_1(x_1)\sigma_1(x_2)\sigma_2(y_1)\sigma_2(y_2)\\ &= (\sigma_1 \otimes \sigma_2)(x_1, y_1)(\sigma_1 \otimes \sigma_2)(x_2, y_2).\end{aligned}$$

Thus $G_1 \otimes G_2$ is strong. □

Corollary 3.9. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete product fuzzy graphs, then $G_1 \otimes G_2$ is strong.*

We can easily generalize the previous result to get a complete strong product fuzzy graph instead of strong. We remark that it can not be generalized in the cases of direct product and semi-strong product since these products are never complete.

Lemma 3.10. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete product fuzzy graphs, then $G_1 \otimes G_2$ is complete.*

Next we prove that if the direct product, semi-strong product or strong product of two product fuzzy graphs is strong, then at least one of them is strong. We only prove the case of semi-strong product since the other cases are similar.

Theorem 3.11. *If $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ are product fuzzy graphs such that $G_1 \odot G_2$ ($G_1 \boxdot G_2$ or $G_1 \otimes G_2$) is strong, then at least G_1 or G_2 must be strong.*

Proof. Suppose that both G_1 and G_2 are not strong. Since G_1 is not strong, there exist $(x_1, y_1) \in E_1$ such that $\mu_1(x_1, y_1) < \sigma_1(x_1)\sigma_1(y_1)$. Since G_2 is not strong, then there exists $(x_2, y_2) \in E_2$ such that $\mu_2(x_2, y_2) < \sigma_2(x_2)\sigma_2(y_2)$. Now,

$$(\mu_1 \odot \mu_2)((x_1, y_1)(x_2, y_2)) = \mu_1(x_1, x_2)\mu_2(y_1, y_2) < \sigma_1(x_1)\sigma_1(y_1)\sigma_2(x_2)\sigma_2(y_2).$$

But $(\sigma_1 \odot \sigma_2)(x_1, y_1) = \sigma_1(x_1)\sigma_2(y_1)$ and $(\sigma_1 \odot \sigma_2)(x_2, y_2) = \sigma_1(x_2)\sigma_2(y_2)$. Thus

$$\begin{aligned} d_(\sigma_1 \odot \sigma_2)(x_1, y_1)(\sigma_1 \odot \sigma_2)(x_2, y_2) &= \sigma_1(x_1)\sigma_1(x_2)\sigma_2(y_1)\sigma_2(y_2) \\ &> (\mu_1 \odot \mu_2)((x_1, y_1)(x_2, y_2)). \end{aligned}$$

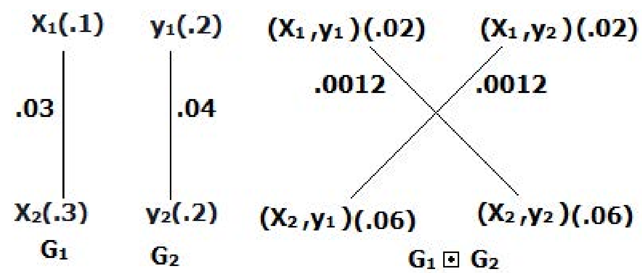
So $G_1 \odot G_2$ is not strong. □

Theorem 3.12. *If $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ are complete product fuzzy graphs, then $\overline{G_1 \otimes G_2} \simeq \bar{G}_1 \otimes \bar{G}_2$.*

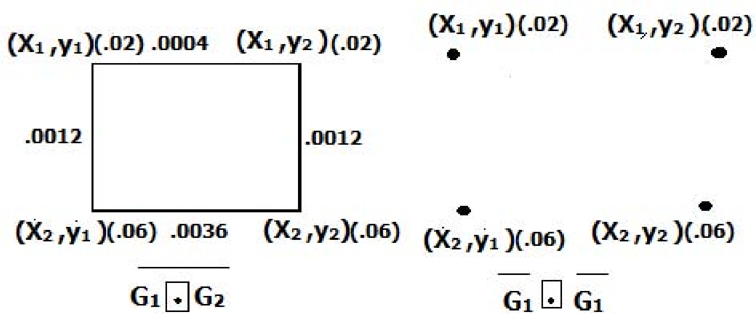
Proof. Let $\overline{G_1 \otimes G_2} = (\sigma_1 \otimes \sigma_2, \overline{\mu_1 \otimes \mu_2})$. We only need to show that $\overline{\mu_1 \otimes \mu_2}(x, y) = \bar{\mu}_1 \otimes \bar{\mu}_2(x, y)$ for all $x, y \in V$. Since G_1 and G_2 are two complete product fuzzy graphs, then by Lemma 3.10, $G_1 \otimes G_2$ is complete. Hence $\overline{\mu_1 \otimes \mu_2}(x, y) = 0$ for all $x, y \in V$. Since G_1 and G_2 are complete product fuzzy graphs, then their complements are empty fuzzy graphs and the strong product of two product empty fuzzy graphs is empty. So $(\bar{\mu}_1 \otimes \bar{\mu}_2)(x, y) = 0$ for all $x, y \in V$. □

The preceding result need not be true in the cases of direct product and semi-strong product. See the following example.

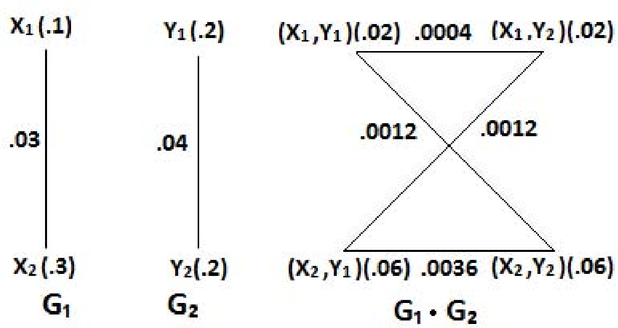
Example 3.13.



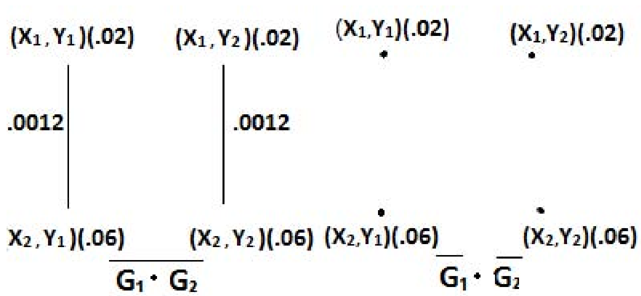
X1(.1)
.03
X2(.3)
G1
y1(.2)
.04
y2(.2)
G2
(X1 ,y1)(.02)
(X1 ,y2)(.02)
.0012
.0012
(X2 ,y1)(.06)
(X2 ,y2)(.06)
G1 ⊡ G2



(X1 ,y1)(.02) .0004 (X1 ,y2)(.02)
.0012
.0012
(X2 ,y1)(.06) .0036 (X2 ,y2)(.06)
G1 ⊡ G2
(X1 ,y1)(.02)
(X1 ,y2)(.02)
(X2 ,y1)(.06)
(X2 ,y2)(.06)
G1 ⊡ G1



X1 (.1)
.03
X2 (.3)
G1
Y1 (.2)
.04
Y2(.2)
G2
(X1,Y1)(.02) .0004 (X1,Y2)(.02)
.0012
.0012
(X2,Y1)(.06) .0036 (X2,Y2)(.06)
G1 • G2



(X1, Y1)(.02)
.0012
X2, Y1)(.06)
(X1, Y2)(.02)
.0012
(X2, Y2)(.06)
G1 • G2
(X1,Y1)(.02)
(X1,Y2)(.02)
(X2,Y1)(.06)
(X2,Y2)(.06)
G1 • G2

Theorem 3.14. *If $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ are complete product fuzzy graphs, then $\bar{G}_1 \otimes \bar{G}_2 \simeq \bar{G}_1 \odot \bar{G}_2 \simeq \bar{G}_1 \boxdot \bar{G}_2$.*

Proof. We only need to show $(\bar{\mu}_1 \otimes \bar{\mu}_2)(x, y) = (\bar{\mu}_1 \odot \bar{\mu}_2)(x, y) = (\bar{\mu}_1 \boxdot \bar{\mu}_2)(x, y)$. Since G_1 and G_2 are complete product fuzzy graphs, then their complements are empty fuzzy graphs and the strong product of two product empty fuzzy graph is empty. Thus $(\bar{\mu}_1 \otimes \bar{\mu}_2)(x, y) = 0$ for all $x, y \in V$. Since G_1 and G_2 are complete fuzzy graphs, then their complements are empty and the semi-strong product of two product empty fuzzy graph is empty. So $(\bar{\mu}_1 \odot \bar{\mu}_2)(x, y) = 0$ for all $x, y \in V$. Since G_1 and G_2 are complete fuzzy graphs, then their complements are empty and the direct product of two product empty fuzzy graph is empty. Hence $(\bar{\mu}_1 \boxdot \bar{\mu}_2)(x, y) = 0$ for all $x, y \in V$. Therefore $(\bar{\mu}_1 \otimes \bar{\mu}_2)(x, y) = (\bar{\mu}_1 \odot \bar{\mu}_2)(x, y) = (\bar{\mu}_1 \boxdot \bar{\mu}_2)(x, y) = 0$ for all $x, y \in V$. □

4. Balanced product fuzzy graphs

Analogous to the idea of balanced fuzzy graphs in [4], we introduce the notion of balanced product fuzzy graphs and prove several results related to them.

Definition 4.1. The density of a product fuzzy graph is $D(G) = \dfrac{2 \sum\limits_{(x,y) \in E} (\mu(x, y))}{\sum\limits_{x,y \in V} (\sigma(x) \wedge \sigma(y))}$. G is balanced if $D(H) \leq D(G)$ for any non-empty product fuzzy subgraphs H of G.

Theorem 4.2. *The density of a complete product fuzzy graph is less than or equal to 2.*

Proof. Let G be any complete product fuzzy graph. Then $\mu(x, y) = \sigma(x)\sigma(y)$ for all $x, y \in V$. Thus $\sum_{(x,y)\in E} \mu(x, y) = \sum_{x,y \in V} \sigma(x)\sigma(y)$ and as $\sigma(x)\sigma(y) \leq \sigma(x) \wedge \sigma(y)$,

$$\sum_{x,y \in V} \sigma(x)\sigma(y) \leq \sum_{x,y \in V} \sigma(x) \wedge \sigma(y).$$

So $\sum_{(x,y)\in E}(\mu(x, y)) \leq \sum_{x,y\in V}(\sigma(x) \wedge \sigma(y))$. Hence $D(G) \leq 2$. □

Theorem 4.3. *Let G be any complete product fuzzy graph and H be a non-empty product fuzzy subgraph of G such that H has less edges than G. Then G is balanced.*

Proof. If H has less edges than G, then

$$\sum_{(x,y)\in E(H)} (\mu(x, y)) \leq \sum_{(x,y)\in E} (\mu(x, y))$$

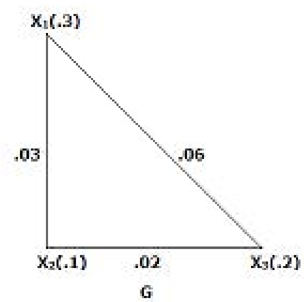
and

$$\sum_{x,y\in V(H)} \sigma(x) \wedge \sigma(y) = \sum_{x,y\in V} \sigma(x) \wedge \sigma(y).$$

Thus $D(H) \leq D(G)$. □

We now give an example to show that if H only has vertices less than G, then G need not be balanced.

Example 4.4. Consider the following graph G.



Then G is not balanced since $D(G) = 0.55$. But if we take $H = (x_1, x_3)$, then $D(H) = 0.6$. Thus it is clear that a complete product fuzzy graph is not necessary balanced.

Theorem 4.5. *Every self-complementary product fuzzy graph has density less than or equal to* 1.

Proof. Let G be self-complementary product fuzzy graph. Then, by Lemma 2,

$$\sum_{(x,y)\in E} \mu(x,y) = \frac{1}{2} \sum_{x,y\in V} (\sigma(x)\sigma(y)).$$

Since

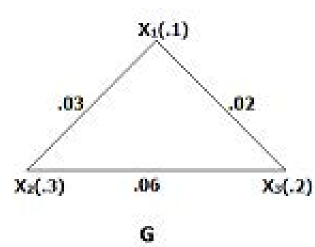
$$\tfrac{1}{2}\sum_{x,y\in V}(\sigma(x)\sigma(y)) \leq \tfrac{1}{2}\sum_{x,y\in V}(\sigma(x)\wedge\sigma(y)),$$

$$D(G) = \frac{2\sum\limits_{(x,y)\in E}(\mu(x,y))}{\sum\limits_{x,y\in V}(\sigma(x)\wedge\sigma(y))} = \frac{\sum\limits_{x,y\in V}(\sigma(x)\sigma(y))}{\sum\limits_{x,y\in V}(\sigma(x)\wedge\sigma(y))} \leq \frac{\sum\limits_{x,y\in V}(\sigma(x)\sigma(y))}{\sum\limits_{x,y\in V}(\sigma(x)\sigma(y))} \leq 1.$$

Thus the result hods. □

The converse of the above results need not be true.

Example 4.6. Consider the following graph G. Then $D(G) = 0.55$, but G is not



self-complementary.

Theorem 4.7. *Let $G:(\sigma,\mu)$ be a product fuzzy graph such that $\mu(x,y)=\frac{1}{2}(\sigma(x)\sigma(y))$ for all $x,y\in V$. Then $D(G)\leq 1$.*

Proof. By Lemma 1.4, G is self-complementary and by Theorem 4.5, $D(G)\leq 1$.

Let G_1 and G_2 be two complete product fuzzy graphs. Then $D(G_1\boxdot G_2)\geq D(G_i)$ for $i=1,2$ if and only if $D(G_1)=D(G_2)=D(G_1\boxdot G_2)$.

$$\begin{aligned}
D(G_1) &= \frac{2\sum\limits_{(x_1,x_2)\in E}(\mu_1(x_1,x_2))}{\sum\limits_{x_1,x_2\in V}(\sigma(x_1)\wedge\sigma(x_2))}\\
&\geq \frac{2\sum\limits_{(x_1,x_2)\in E}(\mu_1(x_1,x_2))(\sigma_2(y_1)\wedge\sigma_2(y_2))}{\sum\limits_{x_1,x_2\in V}(\sigma(x_1)\wedge\sigma(x_2))(\sigma_2(y_1)\wedge\sigma_2(y_2))}\\
&\geq \frac{2\sum\limits_{(x_1,x_2)\in E}(\mu_1(x_1,x_2))(\sigma_2(y_1)\sigma_2(y_2))}{\sum\limits_{x_1,x_2\in V}(\sigma(x_1)\wedge\sigma(x_2))(\sigma_2(y_1)\wedge\sigma_2(y_2))}\\
&\geq \frac{2\sum\limits_{(x_1,x_2)\in E}(\mu_2(x_1,x_2))(\mu_2(y_1,y_2))}{\sum\limits_{(x,y)\in V_1\times V_2}(\sigma(x)\wedge\sigma(y))}\\
&= D(G_1\boxdot G_2).
\end{aligned}$$

□

Theorem 4.8. *Let G_1 and G_2 be isomorphic product fuzzy graphs. If one of them is balanced, then the other is balanced.*

Proof. Suppose G_2 is balanced and let $h:V_1\rightarrow V_2$ be a bijection such that $\sigma_1(x)=\sigma_2(h(x))$ and $\mu_1(x,y)=\mu_2(h(x),h(y))$ for all $x,y\in V_1$. Now $\sum_{x\in V_1}\sigma_1(x)=\sum_{x\in V_2}\sigma_2(x)$ and $\sum_{(x,y)\in E_1}\mu_1(x,y)=\sum_{(x,y)\in E_2}\mu_2(x,y)$. If $H_1=(\acute{\sigma}_1,\acute{\mu}_1)$ is a product fuzzy subgraph of G_1 with underlying set W, then $H_2=(\acute{\sigma}_2,\acute{\mu}_2)$ is a product fuzzy subgraph of G_2 with underlying set $h(W)$ where $\acute{\sigma}_2(h(x))=\acute{\sigma}_1(x)$ and $\acute{\mu}_2(h(x),h(y))=\acute{\mu}_1(x,y)$ for all $x,y\in W$. Since G_2 is balanced, $D(H_1)\leq D(G_2)$ and so $2\frac{\sum_{(x,y)\in E_1}\mu_2(h(x),h(y))}{\sum_{x,y\in V_1}(\acute{\sigma}_2(x)\wedge\acute{\sigma}_2(y))}\leq 2\frac{\sum_{(x,y)\in E_1}\mu_2(x,y)}{\sum_{x,y\in V_1}(\sigma_2(x)\wedge\sigma_2(y))}$. Hence

$$2\frac{\sum_{(x,y)\in E_1}\mu_1(x,y)}{\sum_{x,y\in V_1}(\acute{\sigma}_2(x)\wedge\acute{\sigma}_2(y))}\leq 2\frac{\sum_{(x,y)\in E_1}\mu_1(x,y)}{\sum_{x,y\in V_1}(\sigma_2(x)\wedge\sigma_2(y))}.$$

Thus, G_1 is balanced. □

5. Cobalanced product fuzzy graphs

In this section, we introduce the relatively new notion of cobalanced. We note that using this notion, we get better results than using balanced notion.

Definition 5.1. The codensity of a product fuzzy graph G is $CD(G) = 2\frac{\sum\limits_{(x,y)\in E}\mu(x,y))}{\sum\limits_{x,y\in V}\sigma(x)\sigma(y)}$. G is Cobalanced if $CD(H) \le CD(G)$ for all fuzzy non-empty subgraphs H of G.

Theorem 5.2. *Let G be a product fuzzy graph. Then $CDG) = 2$ iff and only if G is complete.*

Proof. Let G be a complete product fuzzy graph. Then $CD(G) = \frac{2\sum\limits_{x,y\in V}\sigma(x)\sigma(y)}{\sum\limits_{x,y\in V}\sigma(x)\sigma(y)} = 2$. Conversely, suppose G is not complete with codensity equals 2. Then $CD(G) = \frac{2\sum\limits_{(x,y)\in E}\mu(x,y))}{\sum\limits_{x,y\in V}\sigma(x)\sigma(y)} = 2$. Thus $\sum_{(x,y)\in E}\mu(x,y)) = \sum_{x,y\in V}\sigma(x)\sigma(y)$. Since G is not complete, $\mu(x,y) < \sigma(x)\sigma(y)$ for some $x, y \in V$. That means $\mu(\acute{x},\acute{y}) > \sigma(\acute{x})\sigma(\acute{y})$ for some $\acute{x},\acute{y} \in V - \{x,y\}$, a contradiction. □

Theorem 5.3. *Any complete product fuzzy graph is cobalanced.*

Proof. Let G be a complete product fuzzy graph. Then by Theorem 5.2, $CD(G) = 2$. If H is a non-empty product fuzzy subgraph of G, then we have two cases:

Case I If H has less edges than G, then $\sum_{(x,y)\in E(H)}\mu(x,y)) \le \sum_{(x,y)\in E}\mu(x,y))$ and $\sum_{x,y\in V(H)}\sigma(x)\sigma(y) = \sum_{x,y\in V}\sigma(x)\sigma(y)$. Thus

$$CD(H) = \frac{2\sum\limits_{(x,y)\in E(H)}(\mu(x,y))}{\sum\limits_{x,y\in V(H)}(\sigma(x)\sigma(y))} = \frac{2\sum\limits_{(x,y)\in E(H)}(\mu(x,y))}{\sum\limits_{x,y\in V}(\sigma(x)\sigma(y))}$$
$$\le \frac{2\sum\limits_{(x,y)\in E}(\mu(x,y))}{\sum\limits_{x,y\in V}(\sigma(x)\sigma(y))} = 2 = CD(G).$$

Case II If H has vertices lass than G, then it is clear that H is a complete product fuzzy graph. We conclude that $CD(H) = CD(G)$.

Thus G is cobalanced product fuzzy graph. □

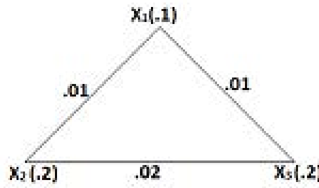
The converse of preceding result need not be true.

Example 5.4. Consider the following graph G. Then G is a cobalanced product fuzzy graph that is not complete.

Corollary 5.5. *A strong product fuzzy graph that is not complete is cobalanced.*

Proof. Let G be a strong product fuzzy graph. Then by Theorem 5.2, we conclude that $CD(G) < 2$ and it is clear that it has a complete subgraph H. By Theorem 5.2 again, $CD(H) = 2 > CD(G)$. □

Theorem 5.6. *Every self-complementary product fuzzy graph has codensity equal 1.*



G

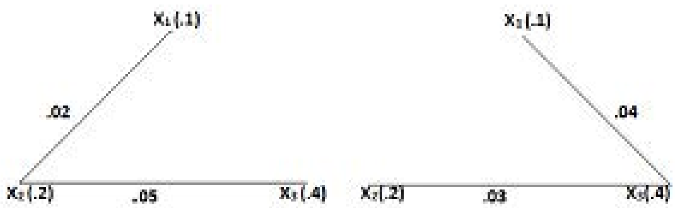
Proof. Let G be self-complementary product fuzzy graph. Then

$$CD(G) = \frac{2\sum\limits_{(x,y)\in E}(\mu(x,y))}{\sum\limits_{x,y\in V}(\sigma(x)\sigma(y))} = \frac{2\frac{1}{2}\sum\limits_{x,y\in V}(\sigma(x)\sigma(y))}{\sum\limits_{x,y\in V}(\sigma(x)\sigma(y))} = \frac{\sum\limits_{x,y\in V}(\sigma(x)\sigma(y))}{\sum\limits_{x,y\in V}(\sigma(x)\sigma(y))} = 1.$$

□

The converse of the above result need not be true.

Example 5.7. Consider the following graph G where $CD(G) = 1$, but G is not self-complementary. Then G is a cobalanced product fuzzy graph that is not complete.



G $\qquad$ G^c

Theorem 5.8. *Let $G : (\sigma, \mu)$ be a product fuzzy graph such that $\mu(x, y) = \frac{1}{2}(\sigma(x)\sigma(y))$ for all $x, y \in V$. Then $CD(G) = 1$.*

Proof. By Lemma 4, G is self-complementary and by Theorem 5.6, $CD(G_1) = 1$.

Let G_1 and G_2 be two complete product fuzzy graphs. Then $CD(G_i) \leq CD(G_1 \boxdot G_2)$ for $i = 1, 2$ if and only if $CD(G_1) = CD(G_2) = CD(G_1 \boxdot G_2)$. If $D(G_i) \leq D(G_1 \boxdot G_2)$ for $i = 1, 2$, then since G_1 and G_2 are complete product fuzzy graphs, by Theorem 5.8,

$$CD(G_1) = CD(G_2) = 2.$$

By Corollary 3.5, $G_1 \boxdot G_2$ is strong and hence by Theorem 5.8, $CD(G_1 \boxdot G_2) < 2$. Thus $CD(G_i) \geq CD(G_1 \boxdot G_2)$ for $i = 1, 2$. So $CD(G_1) = CD(G_2) = CD(G_1 \boxdot G_2)$.

The converse is trivial. □

Theorem 5.9. *Let G_1 and G_2 be two cobalanced product fuzzy graphs. Then $G_1 \boxdot G_2$ is cobalanced if and only if $CD(G_1) = CD(G_2) = CD(G_1 \boxdot G_2)$.*

Proof. If $G_1 \boxdot G_2$ is cobalanced, then $CD(G_i) \le CD(G_1 \boxdot G_2)$ for $i = 1, 2$. Thus, by Lemma 5,

$$CD(G_1) = CD(G_2) = CD(G_1 \boxdot G_2).$$

Conversely, If $CD(G_1) = CD(G_2) = CD(G_1 \boxdot G_2)$ and H is a product fuzzy subgraph of $G_1 \boxdot G_2$, then there exist product fuzzy subgraph H_1 of G_1 and H_2 of G_2 such that $H \simeq H_1 \boxdot H_2$. As G_1 and G_2 are cobalanced and say $CD(G_1) = CD(G_2) = \frac{n_1}{r_1}$, then $CD(H_1) = \frac{a_1}{b_1} \le \frac{n_1}{r_1}$ and $CD(H_2) = \frac{a_2}{b_2} \le \frac{n_1}{r_1}$. Thus $a_1 r_1 + a_2 r_1 \le b_1 n_1 + b_2 n_1$ and hence $CD(H) \le \frac{a_1+a_2}{b_1+b_2} \le \frac{n_1}{r_1} = CD(G_1 \boxdot G_2)$. Therefore $G_1 \boxdot G_2$ is cobalanced. □

The next results can be proved in a similar manor to the preceding one.

Theorem 5.10. *Let G_1 and G_2 be complete product fuzzy graphs. Then $G_1 \odot G_2$ (resp., $G_1 \otimes G_2$) is cobalanced iff $CD(G_1) = CD(G_2) = CD(G_1 \odot G_2)$ (resp., $CD(G_1 \otimes G_2)$).*

Theorem 5.11. *Let G_1 and G_2 be isomorphic product fuzzy graphs. If one of them is cobalanced, then the other is cobalanced.*

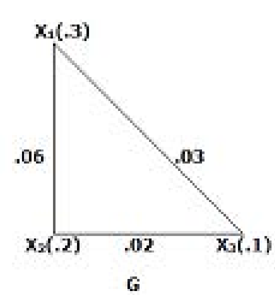
Proof. Suppose G_2 is cobalanced and let $h : V_1 \to V_2$ be a bijection such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x, y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$. Thus $\sum_{x \in V_1} \sigma_1(x) = \sum_{x \in V_2} \sigma_2(x)$ and $\sum_{(x,y) \in E_1} \mu_1(x, y) = \sum_{(x,y) \in E_2} \mu_2(x, y)$. If $H_1 = (\acute{\sigma}_1, \acute{\mu}_1)$ is a product fuzzy subgraph of G_1 with underlying set W, then $H_2 = (\acute{\sigma}_2, \acute{\mu}_2)$ is a fuzzy subgraph of G_2 with underlying set $h(W)$ where $\acute{\sigma}_2(h(x)) = \acute{\sigma}_1(x)$ and $\acute{\mu}_2(h(x), h(y)) = \acute{\mu}_1(x, y)$ for all $x, y \in W$. Since G_2 is cobalanced, $CD(H_1) \le CD(G_2)$. So

$$2\frac{\sum_{(x,y)\in E_1} \mu_2(h(x), h(y))}{\sum(\acute{\sigma}_2(x)\acute{\sigma}_2(y))} \le 2\frac{\sum_{(x,y)\in E_1} \mu_2(x, y)}{\sum(\sigma_2(x)\sigma_2(y))} \le 2\frac{\sum_{(x,y)\in E_1} \mu_1(x, y)}{\sum(\sigma_2(x)\sigma_2(y))}.$$

Hence, $G_1$is cobalanced. □

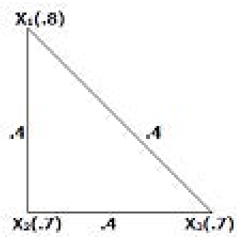
Next, we show that the notions of balanced and cobalanced are independent.

Example 5.12. The following graph G is cobalanced, but is not balanced since $D(G) = .055$, but if we take $H = (x_1, x_2)$, then $D(H) = 0.6$. Then G is a cobalanced



product fuzzy graph that is not complete.

Example 5.13. The following graph G graph is balanced, but is not cobalanced since if we take $H = (X_2, X_3)$, then $CD(H) = 1.6326530612$ while $CD(G) = 1.4906832298$. Then G is a cobalanced product fuzzy graph that is not complete.



G

Theorem 5.14. *Every balanced complete product fuzzy graph is cobalanced.*

Proof. Let G be a complete product fuzzy graph and H be a non-empty product fuzzy subgraph of G. Then as G is balanced , $D(H) \leq D(G)$. Since G is complete, G is cobalanced. □

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