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# Cubic BRK-ideal of BRK-algebra

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Abstract. In this paper the notion of cubic BRK-ideal of BRK-algebra is introduced. Several theorems are presented in this regard.

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## 1. Introduction

Y. Imai and K. Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [5]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [4], Q. P. Hu and X. Li introduced a wide class of abstract: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [10], J. Neggers, S. S. Ahn and H. S. Kim introduced Q-algebras which is a generalization of BCK / BCI-algebras and obtained several results. In 2002, Neggers and Kim [9] introduced a new notion, called a B-algebra, and obtained several results. In 2007, Walendziak [11] introduced a new notion, called a BF-algebra, which is a generalization of B-algebra. In [8], C. B. Kim and H.S. Kim introduced BG-algebra as a generalization of B-algebra. In 2012, R. K. Bandaru [1] introduces a new notion, called BRK-algebra which is a generalization of BCK / BCI / BCH / Q / QS / BM-algebras. Fuzzy sets, which were introduced by Zadeh [12], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. In [3] we introduce the notion of fuzzy BRK-ideal of BRK-algebra. Several basic properties which are related to fuzzy BRK-ideals are investigated, also see [2]. Based on the interval-valued fuzzy sets, Jun et al. [7] introduced the notion of cubic subalgebra / ideals in BCK/BCI-algebras, and they investigated several properties. They discussed relationship between a cubic subalgebra and a cubic ideal. Also, they provided characterizations of a cubic subalgebra / ideal, and considered a method to make a new cubic subalgebra from old one also see [6].

In this paper we introduce the notion of cubic BRK-ideal of BRK-algebras and then we study the homomorphic image and inverse image of cubic BRK-ideal.

## 2. Preliminaries

In this section we site the fundamental definitions that will be used in the sequel.

**Definition 2.1** ([1]). A BRK-algebra is a non-empty set X with a constant 0 and a binary operation $(*)$ satisfying the following conditions:
($BRK_1$) $x * 0 = x$,
($BRK_2$) $(x * y) * x = 0 * y$, for all $x,\ y \in X$.
In a BRK-algebra X, a partially ordered relation $\leq$ can be defined by $x \leq y$ if and only if $x * y = 0$.

**Definition 2.2** ([1]). If $(X;\ *,\ 0)$ is a BRK-algebra, the following conditions hold:
(i) $x * x = 0$,
(ii) $x * y = 0 \Rightarrow 0 * x = 0 * y$,
(iii) $0 * (x * y) = (0 * x) * (0 * y)$, for all $x, y\ \in X$.

**Definition 2.3** ([1]). A non empty subset $S$ of a BRK-algebra $X$ is said to be BRK-subalgebra of $X$, if $x, y \in S$, implies $x * y \in S$.

**Definition 2.4** (BRK-ideal of BRK-algebra). A non empty subset $I$ of a BRK-algebra $X$ is said to be a BRK-ideal of $X$ if it satisfies:
($I_1$) $0 \in I$,
($I_2$) $0 * (x * y) \in I$and $0 * y \in I$ imply $0 * x \in I$ for all $x, y\ \in X$.

**Example 2.5.** Let $X = \{0, a, b, c\}$. Define $*$ on $X$ as the following tabl e:

| $*$ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | 0 | a |
| a | a | 0 | a | 0 |
| b | b | a | 0 | a |
| c | c | b | c | 0 |

Then $(X;\ *\ , 0\ )$ is a BRK-algebra.

**Example 2.6.** Let $X = \{0, a, b, c\}$. Define $*$ on $X$ as the following table :

| $*$ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | b | b | 0 |
| a | a | 0 | 0 | b |
| b | b | 0 | 0 | b |
| c | c | a | a | 0 |

Then $(X;\ * , 0\ )$ is a BRK-algebra, and $A = \{0$, b, c$\}$ is a BRK-ideal of BRK-algebra $X$.

**Definition 2.7** (Homomorphism of BRK-algebra). Let $(X; *, 0)$ and $(Y; *', 0')$ be BRK-algebras. A mapping $f : X \to Y$ is said to be a homomorphism if

$$f(x * y) = f(x) *' f(y), \text{ for all } x, y \in X.$$

**Proposition 2.8** ([3]). *Let* $(X, *, 0)$ *and* $(Y, *', 0')$ *be BRK-algebras, and a mapping* $f : X \to Y$ *be a homomorphism of BRK-algebras, then the* $\ker(f)$ *is a BRK-ideal.*

**An interval–valued fuzzy set (briefly i-v fuzzy set)**

An i-v fuzzy set $A$ on the set $X$ ($\neq \phi$) is given by $A = \{(x, [\mu_A^L(x), \mu_A^U(x)]), x \in X\}$. (briefly, it is denoted by $A = [\mu_A^L, \mu_A^U]$) where $\mu_A^L$ and $\mu_A^U$ are any two fuzzy sets in $X$ such that $\mu_A^L \leq \mu_A^U$. Let $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ , and let$D[0,1]$be the family of all closed sub-interval of $[0, 1]$. It is clear if $\mu_A^L(x)$=$\mu_A^U(x)$=$c$, where $0 \leq c \leq 1$, then $\tilde{\mu}_A(x) = [c, c]$is in$D[0, 1]$. Thus $\tilde{\mu}_A(x) \in [0, 1]$, for all$x \in X$.Then the i-v fuzzy set $A$ is given by $A = \{(x, \tilde{\mu}_A(x)), x \in X\}$,where $\tilde{\mu}_A : X \to D[0, 1]$.Now we define the refined minimum (briefly $r$ min) and order "$\leq$" on the subintervals $D_1 = [a_1, b_1]$ and$D_2 = [a_2, b_2]$of $D[0, 1]$ as follows: $r\min(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$,$D_1 \leq D_2 \Leftrightarrow a_1 \leq a_2$and$b_1 \leq b_2$. Similarly we can define $\geq$ and $=$ .

## 3. Cubic BRK-ideal

**Definition 3.1** ([7])**.** Let $X$ be a nonempty set. A cubic set $\Psi$ in a set $X$ is a structure $\Psi = \{\langle x, A(x), \lambda(x)\rangle : x \in X\}$ which is briefly denoted by $\Psi = \langle A, \lambda\rangle$ where $A = [\mu_A^L, \mu_A^U]$ is an i-v fuzzy set in $X$ and $\lambda$ is a fuzzy set in $X$.

**Definition 3.2** ([7])**.** A cubic set $\Psi = \langle A, \lambda\rangle$ in a BCK\BCI-algebra $X$ is called a cubic ideal of $X$ if it satisfies:

(i) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$
(ii) $\lambda(0) \leq \lambda(x)$,
(iii) $\tilde{\mu}_A(x) \geq r\min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$,
(iv) $\lambda(x) \leq \max\{\lambda(x * y), \lambda(y)\}$, for all $x, y \in X$.

**Definition 3.3.** A cubic set $\Psi = \langle A, \lambda\rangle$ in $X$ is called a cubic subalgebra of a BRK-algebra $X$ if it satisfies:

(i) $\tilde{\mu}_A((x * y)) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$,
(ii) $\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\}$, for all $x, y \in X$.

**Definition 3.4.** Let $(X; *, 0)$ be a BRK-algebra. A cubic set $\Psi = \langle A, \lambda\rangle$ in $X$ is called cubic BRK-ideal of $X$ if it satisfies the following conditions:

($B_1$) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$,
($B_2$) $\lambda(0) \leq \lambda(x)$,
($B_3$) $\tilde{\mu}_A(0 * x) \geq r\min\{\tilde{\mu}_A(0 * (x * y)), \tilde{\mu}_A(0 * y)\}$,
($B_4$) $\lambda(0 * x) \leq \max\{\lambda(0 * (x * y)), \lambda(0 * y)\}$, for all $x, y \in X$.

**Example 3.5.** Consider a BRK-algebra $X$ = {0, a, b, c} in which the *-operation is given by Example 2.5. Define $A = [\mu_A^L, \mu_A^U]$ and $\lambda$ by

$$A = \begin{pmatrix} 0 & a & b & c \\ [0.5, 0.9] & [0.1, 0.3] & [0.2, 0.4] & [0.2, 0.4] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 0 & a & b & c \\ 0.2 & 0.3 & 0.5 & 0.6 \end{pmatrix}.$$

Then $\Psi = \langle A, \lambda\rangle$ is a cubic BRK-ideal of $X$.

**Example 3.6.** Consider a BRK-algebra $X$ = {0, a, b, c} in which the *-operation is given by Example 2.6. Define $A = [\mu_A^L, \mu_A^U]$ and $\lambda$ by

$$A = \begin{pmatrix} 0 & a & b & c \\ [0.4, 0.8] & [0.4, 0.8] & [0.1, 0.3] & [0.1, 0.3] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 0 & a & b & c \\ 0.2 & 0.2 & 0.6 & 0.6 \end{pmatrix}.$$

Then $\Psi = \langle A, \lambda \rangle$ is a cubic BRK-ideal of $X$.

**Lemma 3.7.** *Let $\Psi = \langle A, \lambda \rangle$ be a cubic BRK-ideal of BRK-algebra $X$. If $y * x \leq x$ holds in $X$, then $\tilde{\mu}_A(0 * y) \geq \tilde{\mu}_A(0 * x)$ and $\lambda(0 * y) \leq \lambda(0 * x)$.*

*Proof.* Assume that $y * x \leq x$holds in $X$. Then $(y * x) * x = 0$. By ($B_3$),

$$\tilde{\mu}_A(0 * x) \geq r\min\{\tilde{\mu}_A(0 * (x * y)), \tilde{\mu}_A(0 * y)\}.$$

Thus

$$\tilde{\mu}_A(0 * y) \geq r\min\{\tilde{\mu}_A(0 * (y * x)), \tilde{\mu}_A(0 * x)\}. \tag{3.1}$$

But

$$\begin{aligned} \tilde{\mu}_A(0 * (y * x)) &\geq r\min\{\tilde{\mu}_A(0 * ((y * x) * x), \tilde{\mu}_A(0 * x)\} \\ &= r\min\{\tilde{\mu}_A(0), \tilde{\mu}_A(0 * x)\} \text{ by } (B_1) \\ &= \tilde{\mu}_A(0 * x). \end{aligned}$$

So

$$\tilde{\mu}_A(0 * (y * x)) \geq \tilde{\mu}_A(0 * x). \tag{3.2}$$

From (3.1) and (3.2), we get $\tilde{\mu}_A(0 * y) \geq \tilde{\mu}_A(0 * x)$.
Similarly, by ($B_4$), $\lambda(0 * x) \leq \max\{\lambda(0 * (x * y)), \lambda(0 * y)\}$. Thus

$$\lambda(0 * y) \leq \max\{\lambda(0 * (y * x)), \lambda(0 * x)\}. \tag{3.3}$$

But

$$\begin{aligned} \lambda(0 * (y * x)) &\leq \max\{\lambda(0 * ((y * x) * x), \lambda(0 * x)\} \\ &= \max\{\lambda(0), \lambda(0 * x)\} \text{ by } (B_2) \\ &= \lambda(0 * x). \end{aligned}$$

So

$$\lambda(0 * (y * x)) \leq \lambda(0 * x). \tag{3.4}$$

From (3.3) and (3.4), we get $\lambda(0 * y) \leq \lambda(0 * x)$.
This completes the proof. □

**Lemma 3.8.** *Let $\Psi = \langle A, \lambda \rangle$ be a cubic BRK-ideal of BRK-algebra $X$. If $x \leq y$ holds in $X$, then $\tilde{\mu}_A(0 * x) \geq \tilde{\mu}_A(0 * y)$ and $\lambda(0 * x) \leq \lambda(0 * y)$.*

*Proof.* If $x \leq y$, then $x * y = 0$. This together with $x * 0 = x$, $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\lambda(0) \leq \lambda(x)$. Thus

$$\begin{aligned} \tilde{\mu}_A(0 * x) &\geq r\min\{\tilde{\mu}_A(0 * (x * y)), \tilde{\mu}_A(0 * y)\} \\ &= r\min\{\tilde{\mu}_A(0 * 0), \tilde{\mu}_A(0 * y)\} \\ &= r\min\{\tilde{\mu}_A(0), \tilde{\mu}_A(0 * y)\} \\ &= \tilde{\mu}_A(0 * y). \end{aligned}$$

Also

$$\begin{aligned}\lambda(0*x) &\leq \max\{\lambda(0*(x*y)), \lambda(0*y)\} \\ &= \max\{\lambda(0*0), \lambda(0*y)\} \\ &= \max\{\lambda(0), \lambda(0*y)\} \\ &= \lambda(0*y).\end{aligned}$$

This completes the proof. □

**Note** Let $\Psi = \langle A, \lambda\rangle$ and $\Omega = \langle R, \alpha\rangle$ be two cubic sets in a BRK-algebra $X$. Then

$$\begin{aligned}\Psi \bigcap \Omega &= \{\langle x, r\min\{\mu_A(x), \mu_R(x)\}, \max\{\lambda(x), \alpha(x)\}\rangle : x \in X\} \\ &= \{\left\langle x, \mu_A(x) \bigcap \mu_R(x), \lambda(x) \bigcup \alpha(x)\}\right\rangle : x \in X\}.\end{aligned}$$

**Proposition 3.9.** *Let* $\{\Psi_j\}_{j\in J}$ *be a family of cubic BRK-ideals of a BRK-algebra* $X$*. Then* $\bigcap\limits_{j\in J} \Psi_j$ *is a cubic BRK-ideal of* $X$*.*

*Proof.* Let $\{\Psi_j\}_{j\in J}$ be a family of cubic BRK-ideals of a BRK-algebra $X$. Then for any $x, y \in X$,

$$\left(\bigcap \tilde{\mu}_{A_j}\right)\ (0) = \inf\ \left(\tilde{\mu}_{A_j}(0)\right) \geq \inf\ \left(\tilde{\mu}_{A_j}(x)\right) = \left(\bigcap \tilde{\mu}_{A_j}\right)\ (x)$$

and

$$\begin{aligned}\left(\bigcap \tilde{\mu}_{A_j}\right)(0*x) &= \inf\left(\tilde{\mu}_{A_j}(0*x)\right) \\ &\geq \inf\left(\ r\min\{\tilde{\mu}_{A_j}(0*(x*y)), \tilde{\mu}_{A_j}(0*y)\}\right) \\ &= r\min\left(\ \inf\{\tilde{\mu}_{A_j}(0*(x*y)), \tilde{\mu}_{A_j}(0*y)\}\ \right) \\ &= r\min\left\{\ \inf\left(\tilde{\mu}_{A_j}(0*(x*y))\right), \inf\left(\tilde{\mu}_{A_j}(0*y)\right)\ \right\} \\ &= r\min\left\{\ \left(\bigcap \tilde{\mu}_{A_j}\right)\ \left(\tilde{\mu}_{A_j}(0*(x*y))\right), \left(\bigcap \tilde{\mu}_{A_j}\right)\ (0*y)\ \right\}.\end{aligned}$$

Also

$$\left(\bigcup \lambda_j\right)\ (0) = \sup\ \left(\lambda_j(0)\right) \leq \sup\ \left(\lambda(x)\right) = \left(\bigcup \lambda_j\right)\ (x)$$

and

$$\begin{aligned}\left(\bigcup \lambda_j\right)\ (0*x)\ ) &= \sup\ \left(\lambda_j(0*x)\right) \\ &\leq \sup\ \left(\max\{\lambda(0*(x*y))\ ,\ \lambda(0*y)\}\right) \\ &= \max\left(\sup\{\ \lambda(0*(x*y))\ ,\ \lambda(0*y)\ \}\ \right) \\ &= \max\left\{\sup\ \left(\lambda(0*(x*y))\right)\ ,\ \sup\ (0*y)\ \right\} \\ &= \max\ \left\{\ \left(\bigcup \lambda_j\right)\ (0*(x*y))\ ,\ \left(\bigcup \lambda_j\right)\ (0*y)\right\}\ .\end{aligned}$$

This completes the proof. □

**Definition 3.10.** For any $i \in D[0,\ 1]$ and $s \in [0,\ 1]$, let $\Psi = \langle A, \lambda\rangle$ be a cubic set of BRK-algebra $X$. Then the set $\nu\ (\Psi\ ;\ i\ ,\ s\ ) = \{x \in X : \mu_A(x) \geq i,\ \ \lambda(x) \leq s\}$ is called the cubic level set of $\Psi = \langle A, \lambda\rangle$.

**Theorem 3.11.** *Let* $\Psi = \langle A, \lambda\rangle$ *be a cubic subset in* $X$*. If* $\Psi = \langle A, \lambda\rangle$ *is a cubic BRK-ideal of* $X$ *then for all* $i \in D\ [0,\ 1]$ *and* $s \in [0,\ 1]$*, the set* $\nu\ (\Psi\ ;\ i\ ,\ s\ )$ *is either empty or a BRK-ideal of* $X$*.*

*Proof.* Let $\Psi = \langle A, \lambda\rangle$ be a cubic BRK-ideal of $X$. And let $i \in D\,[0,\ 1]$ , $s \in [0,\ 1]$ be such that $\nu\ (\Psi\ ;\ i\ ,\ s\ ) \neq \phi$. And let $x, y \in X$be such that $x \in \nu\ (\Psi\ ;\ i\ ,\ s\ )$. Then $\mu_A(x) \geq i$ and $\lambda(x) \leq s$. Thus, by Definition 2.2 (1), we get

$$\begin{aligned}\tilde{\mu}_A(0) &= \tilde{\mu}_A(x * x)\\ &\geq r\min\{\tilde{\mu}_A(x * (x * x)), \tilde{\mu}_A(x * x)\}\\ &= r\min\{\tilde{\mu}_A(x * 0), \tilde{\mu}_A(0)\}\\ &= r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(0)\} \geq i.\end{aligned}$$

And

$$\begin{aligned}\lambda(0) &= \lambda(x * x)\\ &\leq \max\{\lambda(x * (x * x)), \lambda(x * x)\}\\ &= \max\{\lambda(x * 0), \lambda(0)\}\\ &= \max\{\lambda(x), \lambda(0)\} \leq s.\end{aligned}$$

So $0 \in \nu\ (\Psi\ ;\ i\ ,\ s\ )$.

Now letting $0 * (x * y)$ and $0 * y\ \in\ \nu\ (\Psi\ ;\ i,\ s)$, that means

$$\tilde{\mu}_A(0 * (x * y))\ \geq\ i \qquad \text{and} \qquad \tilde{\mu}_A(0 * y) \geq i.$$

Then $\tilde{\mu}_A(0 * x) \geq r\min\{\tilde{\mu}_A(0 * (x * y)), \tilde{\mu}_A(0 * y)\}\ =\ i$. Also

$$\lambda(0 * x) \leq \max\{\lambda(0 * (x * y), \lambda(0 * y)\} = s.$$

Thus $(0 * x) \in \nu\ (\ \Psi;\ i,\ s\ )$. Hence $\nu\ (\Psi\ ;\ i\ ,\ s\ )$is a BRK-ideal of $X$. This completes the proof. □

## 4. Image and inverse image of a cubic BRK-ideals

**Definition 4.1.** Let $f$ be a mapping from a set $X$ to a set $Y$. If $\Psi = \langle A, \lambda\rangle$ is a cubic subset of $X$, then the cubic subset $\Omega = \langle B, \eta\rangle$ of $Y$ is define by

$$\tilde{\mu}_A\ f^{-1}(y) = \tilde{\mu}_B(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \tilde{\mu}_A(x), & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases},$$

$$\lambda\ f^{-1}(y) = \eta(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \lambda(x), & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 1 & \text{otherwise} \end{cases},$$

for all $y \in Y$, is called the image of $\Psi = \langle A, \lambda\rangle$ under *f*.

Similarly, if $\Omega = \langle B, \eta\rangle$ is a cubic subset of $Y$, then the cubic subset $\Psi = \Omega \circ f$ in $X$ defined by $\tilde{\mu}_B(f(x)) = \tilde{\mu}_A(x)$and $\eta(f(x)) = \lambda(x)$for all $x\ \in X$, is said to be the inverse image of $\Omega$ under *f*.

**Theorem 4.2.** *An into homomorphic inverse image of a cubic BRK-ideal is also a Cubic BRK-ideal.*

*Proof.* Let $f : X \to X'$ be an into homomorphism of BRK-algebras, $\Omega = \langle B, \eta\rangle$ be a cubic BRK-ideal of $X'$ and $\Psi = \langle A, \lambda\rangle$ be the inverse image of $\Omega = \langle B, \eta\rangle$ under *f*. Then $\tilde{\mu}_B(f(x)) = \tilde{\mu}_A(x)$ and $\eta(f(x)) = \lambda(x)$ for all $x \in X$. Let $x \in X$. Then

$$\tilde{\mu}_A(0) = \tilde{\mu}_B(f(0)) \geq \tilde{\mu}_B(f(x)) = \tilde{\mu}_A(x)$$

and

$$\lambda(0) = \eta(f(0)) \le \eta(f(x)) = \lambda(x).$$

Let $x, y \in X$. Then

$$\begin{aligned}\tilde{\mu}_A(0 * x) &= \tilde{\mu}_B(f(0 * x))\\ &= \tilde{\mu}_B(f(0) *' f(x))\\ &\ge r\min\{\tilde{\mu}_B(f(0) *' (f(x) *' f(y))), \tilde{\mu}_B(f(0) *' f(y))\}\\ &= r\min\{\tilde{\mu}_B(f(0 * (x * y)), \tilde{\mu}_B(f(0 * y))\}\\ &= r\min\{\tilde{\mu}_A(0 * (x * y)), \tilde{\mu}_A(0 * y)\}\end{aligned}$$

and

$$\begin{aligned}\lambda(0 * x) &= \eta(f(0 * x))\\ &= \eta(f(0) *' f(x))\\ &= \tilde{\mu}_B(f(0) *' f(x))\\ &\le \max\{\ \eta\ (f(0) *' (f(x) *' f(y))),\ \eta\ (f(0) *' f(y))\}\\ &= \max\{\ \eta\ (f(0 * (x * y)), \eta\ (f(0 * y))\}\\ &= \max\{\ \lambda\ (0 * (x * y)), \lambda\ (0 * y)\}.\end{aligned}$$

This completes the proof. □

**Definition 4.3** (sup and inf properties)**.** A cubic subset $\Psi = \langle A, \lambda\rangle$ of $X$ has "sup" and "inf " properties if for any subset $K$ of $X$, there exist$m, n \in K$ such that $\tilde{\mu}_A(m) = \sup\limits_{m\in K} \tilde{\mu}_A(m)$ and $\lambda(n) = \inf\limits_{n\in K} \lambda(n)$.

**Theorem 4.4.** *An onto homomorphic image of a cubic BRK-ideal is also a cubic BRK-ideal.*

*Proof.* Let $f : X \to X'$ be an onto homomorphism of BRK-algebras $(X;\ *,\ 0)$ and $(X';\ *,\ 0')$, $\Psi = \langle A, \lambda\rangle$ be a cubic BRK-ideal of $X$ with sup and inf properties and $\Omega = \langle B, \eta\rangle$ is the image of $\Psi = \langle A, \lambda\rangle$ under *f*. By Definition 4.1, we get

$$\tilde{\mu}_B(y') = \tilde{\mu}_A\ f^{-1}(y') = \sup_{x\in f^{-1}(y)} \tilde{\mu}_A(x)$$

and

$$\eta(y') = \lambda\ f^{-1}(y') = \inf_{x\in f^{-1}(y)} \lambda(x)$$

for all $y' \in Y$, sup $\phi = 0$ and inf $\phi = 1$. Since $\Psi = \langle A, \lambda\rangle$ is a cubic BRK-ideal of $X$, we have $\tilde{\mu}_A(0) \ge \tilde{\mu}_A(x)$ and$\lambda(0) \le \lambda(x)$for all $x \in X$ . Note that $0 \in f^{-1}(0')$. Thus, for all $x \in X$,

$$\tilde{\mu}_B(0') = \tilde{\mu}_A f^{-1}(0') = \sup_{t\in f^{-1}(0')} \tilde{\mu}_A(t) = \tilde{\mu}_A(0) \ge \tilde{\mu}_A(x).$$

This implies that $\tilde{\mu}_B(0') \ge \sup\limits_{t\in f^{-1}(x')} \tilde{\mu}_A(t) = \tilde{\mu}_B(x')$ for any $x' \in X'$.

On the other hand, for all $x \in X$,

$$\eta(0') = \lambda f^{-1}(0') = \inf_{t\in f^{-1}(0')} \lambda(t) = \lambda(0) \leq \lambda(x).$$

This implies that $\eta(0') \leq \inf_{t\in f^{-1}(x')} \lambda(t) = \eta(x')$ for any $x' \in X'$. For any $x', y', z' \in X'$, let$x_0 \in f^{-1}(x')$ , $y_0 \in f^{-1}(y')$ , and $0_0 \in f^{-1}(0')$ be such that

$$\tilde{\mu}_A(0_0 * x_0) = \sup_{t\in f^{-1}(0'*x')} \tilde{\mu}_A(t),$$

$$\tilde{\mu}_A(0_0 * y_0) = \sup_{t\in f^{-1}(0'*y')} \tilde{\mu}_A(t),$$

and

$$\begin{aligned}\tilde{\mu}_A(0_0 * (x_0 * y_0)) &= \tilde{\mu}_B\{f(0_0 * (x_0 * y_0))\}\\ &= \tilde{\mu}_B(0' * (x' * y'))\\ &= \sup_{(0_0*(x_0*y_0))\in f^{-1}(0'*(x'*y'))} \tilde{\mu}_A(0_0 * (x_0 * y_0))\\ &= \sup_{t\in f^{-1}(0'*(x'*y'))} \tilde{\mu}_A(t).\end{aligned}$$

Thus

$$\begin{aligned}\tilde{\mu}_B(0' * x') &= \sup_{t\in f^{-1}(0'*x')} \tilde{\mu}_A(t)\\ &= \tilde{\mu}_A(0_0 * x_0)\\ &\geq r \min\{ \tilde{\mu}_A(0_0 * (x_0 * y_0)), \tilde{\mu}_A(0_0 * y_0)\}\\ &= r \min\{ \sup_{t\in(0'*(x'*y')} \tilde{\mu}_A(t) \ , \ \sup_{t\in(0'*y')} \tilde{\mu}_A(t)\}\\ &= r \min\{\tilde{\mu}_B(0' * (x' * y')), \tilde{\mu}_B(0' * y')\}.\end{aligned}$$

Also

$$\lambda(0_0 * x_0) = \inf_{t\in f^{-1}(0'*x')} \lambda(t) \ ,$$

$$\lambda(0_0 * y_0) = \inf_{t\in f^{-1}(0'*y')} \lambda(t),$$

and

$$\begin{aligned}\lambda(0_0 * (x_0 * y_0)) &= \eta\{f(0_0 * (x_0 * y_0))\}\\ &= \eta(0' * (x' * y'))\\ &= \inf_{(0_0*(x_0*y_0))\in f^{-1}(0'*(x'*y'))} \lambda(0_0 * (x_0 * y_0))\\ &= \inf_{t\in f^{-1}(0'*(x'*y'))} \lambda(t).\end{aligned}$$

So

$$\begin{aligned}\eta(0' * x')) &= \inf_{t\in f^{-1}(0' * x')} \lambda(t)\\ &= \lambda(0_0 * x_0)\\ &\leq \max\{\lambda(0_0 * (x_0 * y_0)), \lambda(0_0 * y_0)\}\\ &= \ max\{\inf_{t\in(0' *(x' * y')} \lambda(t)\ ,\ \inf_{t\in(0' * y')} \lambda(t)\}\\ &= \max\{\ \eta\ (0' * (x' * y')),\ \eta\ (0' * y')\}.\end{aligned}$$

Hence $\Omega = \langle B, \eta\rangle$ is a cubic BRK-ideal of $X'$. This completes the proof. □

## 5. Conclusions

In this paper, we have introduced the concept of cubic BRK-ideal of BRK-algebra and studied their properties. In our future work, we introduce the concept of Intuitionistic Fuzzy Homomorphism of BRK-algebra with Interval valued Membership and Non Membership Functions and intuitionistic L-fuzzy BRK- ideals of BRK-algebra. I hope this work would serve as a foundation for further studies on the structure of BRK-algebras.

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