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Fuzzy α-Baire spaces

S. Anjalmose, G. Thangaraj

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Abstract. In this paper the concepts of fuzzyα-Baire spaces are introduced and characterizations of fuzzy α-Baire spaces are studied. Several examples are given to illustrate the concepts introduced in this paper.

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Keywords: Fuzzy α-open set, Fuzzy α-nowhere dense set, Fuzzy α-first category, Fuzzy α-second category, Fuzzy α-Baire spaces, Fuzzy α-D-Baire space, Fuzzy α-D$'$-Baire space.

Corresponding Author: S. Anjalmose (ansalmose@gmail.com)

1. Introduction.

The fuzzy concept has invaded almost all branches of Mathematics ever since the introduction of fuzzy set by L. A. Zadeh [11]. The theory of fuzzy topological spaces was introduced and developed by C. L. Chang [2]. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. In recent years, fuzzy topology has been found to be very useful in solving many practical problems. In [3] El.Naschie showed that the notion of fuzzy topology might be relevant to Quantum Particle Physics and Quantum Gravity in connection with string theory and $\in^{\infty}$ theory. Tang [5] used a slightly changed version of Chang's fuzzy topological spaces to model spatial objects for GIS data bases and Structured Query Language (SQL) for GIS. In this paper we introduce the concepts of fuzzy α-Baire spaces, fuzzy α-D-Baire spaces and fuzzy α-D$'$-spaces. Also we discuss several characterizations of fuzzy α-Baire spaces and relations of fuzzy Baire, fuzzy α-Baire, fuzzy α-D-Baire and fuzzy α-D$'$-Baire spaces are discused. Several examples are given to illustrate the concepts introduced in this paper.

2. Preliminaries.

Now we introduce some basic notions and results that are used in the sequel. In this work by a fuzzy topological space we shall mean a non-empty set X together with a fuzzy topology T (in the sense of Chang) and denote it by (X, T). The interior, closure and the complement of a fuzzy set λ will be denoted by $int(\lambda), cl(\lambda)$ and $1 - \lambda$ respectively.

Definition 2.1 ([1]). Let (X, T) be any fuzzy topological space and λ be any fuzzy set in (X, T). we define $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T$ and $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T$. For any fuzzy set in a fuzzy topological space (X, T) it is easy to see that $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$.

Definition 2.2 ([4]). Let (X, T) be any fuzzy topological space and λ be any fuzzy set in (X, T). we define the fuzzy pre-interior and the fuzzy pre-closure of λ as follows:

(i) $\alpha\text{-}cl(\lambda) = \wedge\{\mu/\lambda \leq \mu,\ \ \mu$ is fuzzy α-closed set of $X\}$.
(ii) $\alpha\text{-}int(\lambda) = \vee\{\mu/\mu \leq \lambda,\ \ \mu$ is fuzzyα-open set of $X\}$.

Definition 2.3 ([4]). A fuzzy set λ in a fuzzy topological space X is called fuzzy α-open if $\lambda \leq int\ cl\ int(\lambda)$ and fuzzy α-closed if $cl\ int\ cl(\lambda) \leq \lambda$.

Definition 2.4 ([4]). A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy α-closed set if $\lambda = \alpha\text{-}cl(\lambda)$ and fuzzy α-open set if $\lambda = \alpha\text{-}int(\lambda)$.

Definition 2.5 ([10]). A fuzzy set λ in a fuzzy Topological space (X, T) is called fuzzy α-dense if there exists no fuzzy α-closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.6 ([8]). A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X,T) such that $\mu < cl(\lambda)$. That is, $int\ \ cl(\lambda) = 0$.

Definition 2.7 ([9]). A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy first category if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where λ_i's are fuzzy nowhere dense sets in (X,T). Any other fuzzy set in (X,T) is said to be of fuzzy second category.

Definition 2.8 ([9]). A fuzzy topological space (X, T) is called fuzzy first category space if $1 = \vee_{i=1}^{\infty}(\lambda_i)$ where (λ_i)'s are fuzzy nowhere dense sets in (X, T). A topological space which is not of fuzzy first category, is said to be of fuzzy second category space.

Definition 2.9 ([8]). Let λ be a fuzzy first category set in a fuzzy topological space (X, T). Then $1 - \lambda$ is called a fuzzy residual set in (X, T).

Definition 2.10 ([8]). Let (X, T) be a fuzzy topological space. Then (X, T) is called a fuzzy Baire space if $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where λ_i's are fuzzy nowhere dense sets in (X, T).

Definition 2.11 ([6]). A fuzzy topological space (X, T) is called a fuzzy D-Baire space if every fuzzy first category set in (X, T) is fuzzy nowhere dense set in (X, T). That is (X, T) is a fuzzy D-Baire space if $int\ cl(\lambda) = 0$ for each fuzzy first category set λ in (X, T).

Definition 2.12 ([7]). A fuzzy topological space (X,T) is fuzzy Baire space. Then (X,T) is called a fuzzy D$'$-Baire space if every fuzzy set with empty interior is fuzzy nowhere dense in (X,T).

3. Fuzzy α-nowhere dense sets.

Definition 3.1. Let (X,T) be a fuzzy topological space. A fuzzy set λ in (X,T) is called a fuzzy α-nowhere dense set if there exists no non-zero fuzzy α-open set μ in (X,T) such that $\mu < \alpha\text{-}cl(\lambda)$. That is, $\alpha\text{-}int\ \alpha\text{-}cl(\lambda) = 0$.

Definition 3.2. A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy α-first category if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where (λ_i)'s are fuzzy α-nowhere dense sets in (X,T). Any other fuzzy set in (X,T) is said to be of fuzzy α-second category.

Definition 3.3. Let λ be a fuzzy α-first category set in a fuzzy topological space (X,T). Then $1-\lambda$ is called a fuzzy α-residual set in (X,T).

Example 3.4. Let $X = \{a,b,c\}$. The fuzzy sets λ, μ and γ are defined on X as follows:

$\lambda : X \to [0,1]$ defined as $\lambda(a) = 0.9; \lambda(b) = 0.8; \lambda(c) = 0.6$,
$\mu : X \to [0,1]$ defined as $\mu(a) = 0.9; \mu(b) = 0.7; \mu(c) = 0.7$,
$\gamma : X \to [0,1]$ defined as $\gamma(a) = 0.8; \gamma(b) = 0.7; \gamma(c) = 0.6$.

Then $T = \{0, \lambda, \mu, \gamma, \lambda \vee \mu, \lambda \wedge \mu, 1\}$ is clearly a fuzzy topology on X.

Now consider the following fuzzy sets defined on X as followes:

$\eta : X \to [0,1]$ defined as $\eta(a) = 0.8; \eta(b) = 0.8; \eta(c) = 0.7$,
$\beta : X \to [0,1]$ defined as $\beta(a) = 0.9; \beta(b) = 0.9; \beta(c) = 0.8$,
$\delta : X \to [0,1]$ defined as $\delta(a) = 0.8; \delta(b) = 0.7; \delta(c) = 0.7$.

Then the fuzzy α-open sets in (X,T) are $\lambda, \mu, \gamma, \lambda \vee \mu, \lambda \wedge \mu, \alpha, \beta, \delta$. Thus

$\alpha\text{-}int\ \alpha\text{-}cl(1-\lambda) = 0, \alpha\text{-}int\ \alpha\text{-}cl(1-\mu) = 0, \alpha\text{-}int\ \alpha\text{-}cl(1-\gamma) = 0$,
$\alpha\text{-}int\ \alpha\text{-}cl[1-(\lambda \vee \mu)] = 0, \alpha\text{-}int\ \alpha\text{-}cl[1-(\lambda \wedge \mu)] = 0$.

So $1-\lambda, 1-\mu, 1-\gamma, 1-(\lambda \vee \mu), 1-(\lambda \wedge \mu)$ are fuzzy α-nowhere dense sets in (X,T). But η, β, δ are not fuzzy α-nowhere dense sets, since $\alpha\text{-}int\ \alpha\text{-}cl(\eta) = 1 \neq 0, \alpha\text{-}int\ \alpha\text{-}cl(\beta) = 1 \neq 0, \alpha\text{-}int\ \alpha\text{-}cl(\delta) = 1 \neq 0$. On one hand,

$(1-\lambda) \vee (1-\mu) \vee (1-\gamma) \vee [1-(\lambda \vee \mu)] \vee [1-(\lambda \wedge \mu)] = 1-\gamma$ is fuzzy α-first cetagory set and γ is a fuzzy α-residual set in (X,T).

Proposition 3.5. *If λ is a fuzzy α-closed set in (X,T) with $\alpha\text{-}int(\lambda) = 0$ then λ is a fuzzy α-nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy α-closed set in (X,T). Then $\alpha\text{-}cl(\lambda) = \lambda$. Now $\alpha\text{-}int\ \alpha\text{-}cl(\lambda) = \alpha\text{-}int(\lambda) = 0$. Thus λ is a fuzzy α-nowhere dense set in (X,T). □

Proposition 3.6. *If λ is fuzzy α-nowhere dense set in (X,T), then $\alpha\text{-}int(\lambda) = 0$.*

Proof. Let λ be a fuzzy α-nowhere dense set in (X,T). Now $\lambda \leq \alpha\text{-}cl(\lambda)$ implies that $\alpha\text{-}int(\lambda) \leq \alpha\text{-}int\ \alpha\text{-}cl(\lambda) = 0$. Then $\alpha\text{-}int(\lambda) = 0$. □

Remark 3.7. The converse of the above proposition need not be true, For consider the following example:

Example 3.8. Let $X = a, b$. The fuzzy sets λ, μ and γ are defined on X as follows:
$\lambda : X \to [0, 1]$ defined as $\lambda(a) = 0.9; \lambda(b) = 0.8$,
$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.9; \delta(b) = 0.7$.
$\gamma : X \to [0, 1]$ defined as $\gamma(a) = 0.8; \gamma(b) = 0.6$.
Then $T = \{0, \lambda, \mu, \gamma, 1\}$ is a fuzzy topology on X. Now consider the following fuzzy sets defined on X as followes:
$\eta : X \to [0, 1]$ defined as $\eta(a) = 0.8; \lambda(b) = 0.5$.
Thus the fuzzy α-open sets in (X, T) are λ, μ, γ. Now α-$int(\eta) = 0$ whereas α-int α-$cl(\eta) = 1 \neq 0$. Hence η is not a fuzzy α-nowhere dense set in(X, T).

Remark 3.9. The compliment of a fuzzy α-nowhere dense set need not be a fuzzy α-nowhere dense set. For consider the following example.

Example 3.10. Let $X = \{a, b\}$ and the fuzzy sets λ, μ be defined on X as follows:
$\lambda : X \to [0, 1]$ defined as $\lambda(a) = 0.9; \lambda(b) = 0.8$,
$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.9; \mu(b) = 0.7$.
Then $T = \{0, \lambda, \mu, 1\}$ is a fuzzy topology on X. The fuzzy α-open sets in (X, T) are λ, μ. Now $1 - \mu$ is a fuzzy α-nowhere dense set in (X, T) whereas μ is not a fuzzy α-nowhere dense set, since α-int α-$cl(\mu) = \alpha$-$int(1) = 1 \neq 0$.

Proposition 3.11. *If λ and μ are fuzzy α-nowhere dense sets in (X, T), then $(\lambda \wedge \mu)$ is a fuzzy α-nowhere dense set in (X, T).*

Proof. Let λ and μ are fuzzy α-nowhere dense sets in (X, T). Then
$$\alpha\text{-}int\ \alpha\text{-}cl(\lambda \wedge \mu) \leq \alpha\text{-}int\ \alpha\text{-}cl(\lambda) \wedge \alpha\text{-}int\ \alpha\text{-}cl(\mu) = 0 \wedge 0 = 0.$$
Since λ and μ are fuzzy α-nowhere dense sets in (X, T). Thus $\lambda \wedge \mu$ is fuzzy α-nowhere dense set in (X, T). □

Proposition 3.12. *If λ and μ are fuzzy α-nowhere dense sets, then $(\lambda \vee \mu)$ need not be a fuzzy α-nowhere dense set in (X, T). For consider the following example.*

Example 3.13. Let $X = a, b, c$. The fuzzy sets λ, μ and γ are defined on X as follows:
$\lambda : X \to [0, 1]$ defined as $\lambda(a) = 1; \lambda(b) = 0.2; \lambda(c) = 0.7.,$
$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.3; \mu(b) = 0.1; \mu(c) = 0.2$,
$\gamma : X \to [0, 1]$ defined as $\gamma(a) = 0.7; \gamma(b) = 0.4; \gamma(c) = 1$.
Then $T = \{0, \lambda, \mu, \gamma, \lambda \vee \gamma, \lambda \wedge \gamma, 1\}$ is clearly fuzzy topology on X. Thus the fuzzy α-open sets in (X, T) are $\lambda, \mu, \gamma, \lambda \vee \gamma, \lambda \wedge \gamma$. Now $1 - \lambda$ and $1 - \gamma$ are fuzzy α-nowhere dense sets in (X, T). But $[(1 - \lambda) \vee (1 - \gamma)] = 1 - (\lambda \wedge \gamma)$ is not a fuzzy α-nowhere dense set in (X, T). Since α-int α-$cl[1 - (\lambda \wedge \gamma)] \neq 0$, $(1 - \lambda) \vee (1 - \gamma)$ is not a fuzzy α-nowhere dense set in (X, T).

Proposition 3.14. *If λ is a fuzzy α-dense, fuzzy α-open set in (X, T) such that $\mu \leq (1 - \lambda)$ then μ is a fuzzy α-nowhere dense set in (X, T).*

Proof. Let λ be a fuzzy α-open set in (X, T) such that α-$cl(\lambda) = 1$. Now $\mu \leq (1 - \lambda)$ implies that α-$cl(\mu) \leq \alpha$-$cl(1 - \lambda) = 1 - \lambda$ [since $(1 - \lambda)$ is fuzzy α-closed in (X, T)]. Then we have α-int α-$cl(\mu) \leq \alpha$-$int(1 - \lambda) = 1 - \alpha$-$cl(\lambda) = 1 - 1 = 0$. Thus α-int α-$cl(\mu) = 0$. So μ is a fuzzy α-nowhere dense set in (X, T). □

Proposition 3.15. *If a non-zero fuzzy set λ in (X,T) is a fuzzy α-nowhere dense set then λ is fuzzy semi-closed in (X,T).*

Proof. Let λ be a fuzzy α-nowhere dense set in (X,T). Then $\alpha\text{-}int\ \alpha\text{-}cl(\lambda) = 0$. Thus $\alpha\text{-}int\ \alpha\text{-}cl(\lambda) \leq \lambda$. So λ is a fuzzy semi-closed set in (X,T). □

Remark 3.16. The converse of the above proposition need not be true. For consider the following example.

Example 3.17. Let $X = a, b, c$. The fuzzy sets λ and μ are defined on X as follows:
$\lambda : X \to [0,1]$ defined as $\lambda(a) = 0.3; \lambda(b) = 0.6; \lambda(c) = 0.5$,
$\mu : X \to [0,1]$ defined as $\mu(a) = 0.5; \mu(b) = 0.4; \mu(c) = 0.7$.
Then $T = \{0, \lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu, 1\}$ is a fuzzy topology on X. The fuzzy α-open sets in (X,T) are $\lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu$. Now $\lambda \wedge \mu$ is a fuzzy semi-closed set whereas $\lambda \wedge \mu$ is not a fuzzy α-nowhere dense set, since $\alpha\text{-}int\ \alpha\text{-}cl(\lambda \wedge \mu) = \lambda \neq 0$.

Proposition 3.18. *If λ is a fuzzy α-closed set in (X,T), then λ is a fuzzy α-nowhere dense set in (X,T), if and only if $\alpha\text{-}int(\lambda) = 0$.*

Proof. Let λ be a non-zero fuzzy α-closed set in (X,T) with $\alpha\text{-}int(\lambda) = 0$. Then by Proposition 3.5, λ is a fuzzy α-nowhere dense set in (X,T). Conversely let λ be a fuzzy α-nowhere dense set in (X,T). Then $\alpha\text{-}int\ \alpha\text{-}cl(\lambda) = 0$ which implies that $\alpha\text{-}int(\lambda) = 0$ [since λ is fuzzy α-closed, $\alpha\text{-}cl(\lambda) = \lambda$]. □

Proposition 3.19. *If (X,T) is open hereditarily irresolvable space, any non-zero fuzzy set λ with $\alpha\text{-}int(\lambda) = 0$ is a fuzzy α-nowhere dense set in (X,T).*

Proof. Let λ be a non-zero fuzzy set in a fuzzy open hereditarily irresolvable space (X,T) with $\alpha\text{-}int(\lambda) = 0$. Suppose that $\alpha\text{-}int\ \alpha\text{-}cl(\lambda) \neq 0$. Since (X,T) is fuzzy open hereditarily irresolvable, $\alpha\text{-}int(\lambda) \neq 0$ which is a contradiction to $\alpha\text{-}int(\lambda) = 0$. Then we must have $\alpha\text{-}int\ \alpha\text{-}cl(\lambda) = 0$. Thus λ is a fuzzy α-nowhere dense set in (X,T). □

Proposition 3.20. *If λ is a fuzzy α-nowhere dense set in (X,T), then $1 - \lambda$ is a fuzzy α-dense set in (X,T).*

Proof. Let λ be a fuzzy α-nowhere dense set in (X,T). Then, by Proposition 3.6, we have $\alpha\text{-}int(\lambda) = 0$. Now $\alpha\text{-}cl(1 - \lambda) = 1 - \alpha\text{-}int(\lambda) = 1 - 0 = 1$. Thus $1 - \lambda$ is a fuzzy α-dense set in (X,T). □

Proposition 3.21. *If λ is fuzzy α-dense and fuzzy α-open set in (X,T) then $1 - \lambda$ is fuzzy α-nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy α-open set in (X,T) such that $\alpha\text{-}cl(\lambda) = 1$. Now $int\ \alpha\text{-}cl(1 - \lambda) = 1 - \alpha\text{-}cl\ \alpha\text{-}int(\lambda) = 1 - \alpha\text{-}cl(\lambda) = 1 - 1 = 0$. Then $1 - \lambda$ is a fuzzy α-nowhere dense set in (X,T). □

Proposition 3.22. *If λ be a fuzzy α-nowhere dense set in (X,T), then $\alpha\text{-}cl(\lambda)$ is a fuzzy α-nowhere dense set in (X,T).*

Proof. Let $\alpha\text{-}cl(\lambda) = \mu$. Now $\alpha\text{-}int\ \alpha\text{-}cl(\mu) = \alpha\text{-}int\ \alpha\text{-}cl(cl(\lambda)) = \alpha\text{-}int\ \alpha\text{-}cl(\lambda) = 0$. Then $\mu = \alpha\text{-}cl(\lambda)$ is a fuzzy α-nowhere dense set in (X,T). □

Proposition 3.23. *If λ be a fuzzy α-nowhere dense set in (X, T), then $1 - \alpha\text{-}cl(\lambda)$ is a fuzzy α-dense set in (X, T).*

Proof. Let λ be a fuzzy α-nowhere dense set in (X, T), By Proposition 3.22, we have $\alpha\text{-}cl(\lambda)$ is a fuzzy α-nowhere dense set in (X, T). By Proposition 3.20, we have $1 - \alpha\text{-}cl(\lambda)$ is a fuzzy α-dense set in (X, T). □

Proposition 3.24. *Let λ be a fuzzy α-dense set in a fuzzy topological space (X, T). If μ is any fuzzy set in (X, T), then μ is fuzzy α-nowhere dense set in (X, T) if and only if $(\lambda \wedge \mu)$ is a fuzzy α-nowhere dense set in (X, T).*

Proof. Let μ be a fuzzy α-nowhere dense set in (X, T). Then

$$\begin{aligned} \alpha\text{-}int\ \alpha\text{-}cl(\lambda \wedge \mu) &= \alpha\text{-}int[\alpha\text{-}cl(\lambda) \wedge \alpha\text{-}cl(\mu)] \\ &= \alpha\text{-}int[1 \wedge \alpha\text{-}cl(\mu)] = \alpha\text{-}int\ \alpha\text{-}cl(\mu) \\ &= 0. \end{aligned}$$

Thus $\lambda \wedge \mu$ is a fuzzy α-nowhere dense set in (X, T).

Conversely, Let $\lambda \wedge \mu$ be a fuzzy α-nowhere dense set in (X, T). Then $\alpha\text{-}int\ \alpha\text{-}cl(\lambda \wedge \mu) = 0$ implies that $\alpha\text{-}int[\alpha\text{-}cl(\lambda) \wedge \alpha\text{-}cl(\mu)] = 0$. Thus $\alpha\text{-}int[1 \wedge \alpha\text{-}cl(\mu)] = 0$. So $\alpha\text{-}int\ \alpha\text{-}cl(\mu) = 0$ which means that μ is a fuzzy α-nowhere dense set in (X, T). □

Proposition 3.25. *If $\lambda \le \mu$ and μ is a fuzzy α-nowhere dense set in a fuzzy topological space (X, T), then λ is also a fuzzy α-nowhere dense set in (X, T).*

Proof. Now $\lambda \le \mu$ implies that $\alpha\text{-}int\ \alpha\text{-}cl(\lambda) \le \alpha\text{-}int\ \alpha\text{-}cl(\mu)$. Now μ is a fuzzy α-nowhere dense set implies that $\alpha\text{-}int\ \alpha\text{-}cl(\mu) = 0$. Then $\alpha\text{-}int\ \alpha\text{-}cl(\lambda) = 0$. Thus λ is a fuzzy α-nowhere dense set in (X, T). □

Proposition 3.26. *If $\lambda \le \mu$ and μ is a fuzzy α-first category set in a fuzzy topological space (X, T), then λ is also a fuzzy α-first category set in (X, T).*

Proof. Let μ_i be a fuzzy α-first category set in (X, T). Then $\mu = \vee_{i=1}^{\infty}(\mu_i)$, where (μ_i)'s are fuzzy α-nowhere dense sets in (X, T). Now $\lambda \wedge \mu$ gives $\lambda \wedge \mu = \lambda$ implies that $\lambda \wedge \mu \le \mu$. Then, by Proposition 3.25, $\lambda \wedge \mu$ is fuzzy α-nowhere dense sets in (X, T). That is $\lambda = \vee_{i=1}^{\infty}(\lambda \wedge \mu_i)$ and $(\lambda \wedge \mu_i)$'s are fuzzy α-nowhere dense sets in (X, T) gives λ is also a fuzzy α-first category set in (X, T). □

Proposition 3.27. *If $\lambda \le \mu$ and λ is a fuzzy α-residual set in a fuzzy topological space (X, T), then μ is also a fuzzy α-residual set in (X, T).*

Proof. Let λ be a fuzzy α-residual set in (X, T). Then $1 - \lambda$ is a fuzzy α-first category set in (X, T). Let $\eta = 1 - \lambda$ is a fuzzy α-first category set in (X, T). Now $\lambda \le \mu$ implies that $1 - \eta \le \mu$. Then $\eta \ge 1 - \mu$. Since η is a fuzzy α-first category set in (X, T), by proposition 3.26, $1 - \mu$ is a fuzzy α-first category set in (X, T) and hence μ is a fuzzy α-residual set in (X, T). □

4. Fuzzy α-Baire space

Definition 4.1. Let (X, T) be a fuzzy topological space. Then (X, T) is called a fuzzy α-Baire space if $\alpha\text{-}int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$ where (λ_i)'s are fuzzy α-nowhere dense sets in (X, T).

Example 4.2. Let $X = \{a, b, c\}$. The fuzzy sets λ, μ and γ are defined on X as follows:

$\lambda : X \to [0, 1]$ defined as $\lambda(a) = 0.9; \lambda(b) = 0.9; \lambda(c) = 0.8$,
$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.9; \mu(b) = 0.8; \mu(c) = 0.8$,
$\gamma : X \to [0, 1]$ defined as $\gamma(a) = 0.8; \gamma(b) = 0.8; \gamma(c) = 0.7$.

Then $T = \{0, \lambda, \mu, \gamma, 1\}$ is clearly a fuzzy topology on X. Thus the fuzzy α-open sets in (X, T) are λ, μ, γ. So

$\alpha\text{-}int\ \alpha\text{-}cl(1 - \lambda) = 0, \alpha\text{-}int\ \alpha\text{-}cl(1 - \mu) = 0$,
$\alpha\text{-}int\ \alpha\text{-}cl(1 - \gamma) = 0$.

So $1 - \lambda, 1 - \mu, 1 - \gamma$ are fuzzy α-nowhere dense sets in (X, T) implies that α-$int[(1 - \lambda) \vee (1 - \mu) \vee (1 - \gamma)] = 0$. Hence (X, T) is fuzzy α-Baire space.

Example 4.3. Let $X = a, b, c$. The fuzzy sets λ, μ and γ are defined on X as follows:

$\lambda : X \to [0, 1]$ defined as $\lambda(a) = 1; \lambda(b) = 0.2; \lambda(c) = 0.7$,
$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.3; \mu(b) = 0.1; \mu(c) = 0.2$,
$\gamma : X \to [0, 1]$ defined as $\gamma(a) = 0.7; \gamma(b) = 0.4; \gamma(c) = 1$.

Then $T = \{0, \lambda, \mu, \gamma, \lambda \vee \gamma, \lambda \wedge \gamma, 1\}$ is clearly fuzzy topology on X. Thus the fuzzy α-open sets in (X, T) are $\lambda, \mu, \gamma, \lambda \vee \gamma, \lambda \wedge \gamma$. Now $1-\lambda, 1-\gamma$ and $1-(\lambda \vee \gamma)$ are fuzzy α-nowhere dense sets in (X, T). But α-$int[(1-\lambda) \vee (1-\gamma) \vee (1-(\lambda \vee \gamma))] = 1-(\lambda \wedge \gamma) \neq 0$. So (X, T) is not of fuzzy α-Baire space.

Proposition 4.4. *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*

(1) *(X, T) is a fuzzy α-Baire space.*
(2) *α-$int(\lambda) = 0$ for every fuzzy α-first category set λ in (X, T).*
(3) *α-$cl(\mu) = 1$, for every fuzzy α-residual set μ in (X, T).*

Proof. (1) $\Rightarrow$ (2): Let λ be a fuzzy α-first category set in (X, T). Then $\lambda = (\vee_{i=1}^{\infty}(\lambda_i))$, where (λ_i)'s are fuzzy α-nowhere dense sets in (X, T). Now α-$int(\lambda) = \alpha$-$int(\vee_{i=1}^{\infty}(\lambda i)) = 0$ (since (X, T) is a fuzzy α-Baire space). Thus α-$int(\lambda) = 0$.

(2) $\Rightarrow$ (3): Let μ be a fuzzy α-residual set in (X, T). Then $1 - \mu$ is a fuzzy α-first category set in (X, T). By hypothesis, α-$int(1 - \mu) = 0$, which implies that $1 - \alpha$-$cl(\mu) = 0$. Thus α-$cl(\mu) = 1$.

(3)$\Rightarrow$ (1): Let λ be a fuzzy α-first category set in (X, T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where (λ_i)'s are fuzzy α-nowhere dense sets in (X, T). Now λ is a fuzzy α-first category set, $1 - \lambda$ is a fuzzy α-residual set in (X, T). By hypothesis, we have α-$cl(1 - \lambda) = 1$, which implies that $1 - \alpha$-$int(\lambda) = 1$. Thus α-$int(\lambda) = 0$. That is, α-$int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy α-nowhere dense sets in (X, T). So (X, T) is a fuzzy α-Baire space. □

Proposition 4.5. *If the fuzzy topological space (X, T) is a fuzzy α-Baire space, then (X, T) is a fuzzy α-second category space.*

Proof. Let (X, T) be a fuzzy α-Baire space. Then α-$int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy α-nowhere dense sets in (X, T). Then $\vee_{i=1}^{\infty}(\lambda_i) \neq 1_X$ [Other wise,$\vee_{i=1}^{\infty}(\lambda_i) = 1_X$ which implies that α-$int(\vee_{i=1}^{\infty}(\lambda_i)) = \alpha$-$int(1_X) = 1_X$, which implies that $0 = 1$, which is a contradiction]. Thus (X, T) is a fuzzy α-second category space. □

Remark 4.6. The converse of the above proposition need not be true. A fuzzy α-second category space need not be a fuzzy α-Baire space. For, consider the following example.

Example 4.7. Let $X = a, b, c$. The fuzzy sets λ, μ and γ are defined on X as follows:

$\lambda : X \to [0,1]$ defined as $\lambda(a) = 1; \lambda(b) = 0.2; \lambda(c) = 0.7$,

$\mu : X \to [0,1]$ defined as $\mu(a) = 0.3; \mu(b) = 0.1; \mu(c) = 0.2$,

$\gamma : X \to [0,1]$ defined as $\gamma(a) = 0.7; \gamma(b) = 0.4; \gamma(c) = 1$.

Then $T = \{0, \lambda, \mu, \gamma, \lambda \vee \gamma, \lambda \wedge \gamma, 1\}$ is clearly fuzzy topology on X. Thus the fuzzy α-open sets in (X,T) are $\lambda, \mu, \gamma, \lambda\vee\gamma, \lambda\wedge\gamma$. Now $1-\lambda, 1-\gamma$ and $1-(\lambda\vee\gamma)$ are fuzzy α-nowhere dense sets in (X,T). Now $[(1-\lambda)\vee(1-\gamma)\vee(1-(\lambda\vee\gamma))] \neq 1_X$, therefore (X,T) is a fuzzy α-second cetegory space. $\alpha\text{-}int[(1-\lambda)\vee(1-\gamma)\vee(1-(\lambda\vee\gamma))] = 1-(\lambda\wedge\gamma) \neq 0$. So (X,T) is not of fuzzy α-Baire space.

Proposition 4.8. *If the fuzzy topological space (X,T) is a fuzzy α-first category space, then (X,T) is not a fuzzy α-Baire space.*

Proof. Since (X,T) is a fuzzy α-first category space, then $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy α-nowhere dense sets in (X,T). Then $\alpha\text{-}int(\vee_{i=1}^{\infty}(\lambda_i)) = \alpha\text{-}int(1) \neq 0$, where (λ_i)'s are fuzzy α-nowhere dense sets in (X,T). Thus (X,T) is not a fuzzy α-Baire space. □

5. Fuzzy α-D-Baire space

Definition 5.1. A fuzzy topological space (X,T) is called a fuzzy α-D-Baire space if every fuzzy α-first category set in (X,T) is fuzzy α-nowhere dense set in (X,T). That is (X,T) is a fuzzy α-D-Baire space if $\alpha\text{-}int\ \alpha\text{-}cl(\lambda) = 0$ for each fuzzy α-first category set λ in (X,T).

Example 5.2. Let (X,T) be a fuzzy topological space. By Example 4.2, $1-\gamma$ is a fuzzy α-first category in (X,T). Now $\alpha\text{-}int\ \alpha\text{-}cl(1-\gamma) = 0$. Then the fuzzy αfirst category set $1-\gamma$ is a fuzzy α-nowhere dense set in (X,T). Thus (X,T) is a fuzzy α-D-Baire space.

Example 5.3. In Example 4.7, $1-(\lambda\wedge\gamma)$ is a fuzzy α-first category set in (X,T). Now $\alpha\text{-}int\ \alpha\text{-}cl[1-(\lambda\wedge\gamma)] \neq 0$. Then the fuzzy α-first category set $1-(\lambda\wedge\gamma)$ is not a fuzzy α-first category set. Thus (X,T) is not a fuzzy α-D-Baire space.

6. Fuzzy α-D′-Baire space

Definition 6.1. A fuzzy topological space (X,T) is fuzzy α-Baire space. Then (X,T) is called a fuzzy α-D′-Baire space if every fuzzy set with empty α-interior is fuzzy α-nowhere dense in (X,T).

Example 6.2. Let (X,T) be a fuzzy topological space. By Example 4.2, the fuzzy sets $1-\lambda, 1-\mu, 1-\gamma$ are empty α-interior in (X,T) and fuzzy α-nowhere dense sets in (X,T). Then (X,T) is a fuzzy α-D′-Baire space.

7. Fuzzy α-D′-Baire, fuzzy α-D-Baire, fuzzy α-Baire and fuzzy Baire spaces.

Proposition 7.1. *If the fuzzy topological space (X, T) is a fuzzy α-D′-Baire space, then (X, T) is a fuzzy α-D-Baire space.*

Example 7.2. Let (X, T) be a fuzzy topological space. By Example 4.2, (X, T) be a fuzzy α-Baire space. Then the empty αintererior fuzzy sets in (X, T) is $1-\lambda, 1-\mu, 1-\gamma$. Now $\alpha\text{-}int\ \alpha\text{-}cl(1-\lambda) = 0, \alpha\text{-}int\ \alpha\text{-}cl(1-\mu) = 0, \alpha\text{-}int\ \alpha\text{-}cl(1-\gamma) = 0$. Thus (X, T) is fuzzy α-Baire space and every fuzzy set with empty α-interior is fuzzy α-nowhere dense sets in (X, T). So (X, T) is a fuzzy α-D′-Baire space and fuzzy α-D-Baire space, since by Example 5.2 and Example 4.2.

Proposition 7.3. *If the fuzzy topological space (X, T) is a fuzzy α-D-Baire space, then (X, T) is a fuzzy α-Baire space.*

Example 7.4. Let (X, T) be a fuzzy topological space. By Example 5.2, (X, T) is a fuzzy α-D-Baire space. By Example 4.2, (X, T) is a fuzzy α-Baire space. Then (X, T) be a fuzzy α-D-Baire space. Thus (X, T) is fuzzy α-Baire space.

Proposition 7.5. *If the fuzzy topological space (X, T) is a fuzzy α-Baire space, then (X, T) is a fuzzy Baire space.*

Example 7.6. Let (X, T) be a fuzzy topological space. By Example 4.2, (X, T) be a fuzzy α-Baire space and the fuzzy open sets in (X, T) are λ, μ, γ and $int\ cl(1-\lambda) = 0, int\ cl(1-\mu) = 0, int\ cl(1-\gamma) = 0$. Now $int[(1-\lambda) \vee (1-\mu) \vee (1-\gamma)] = 0$, where the fuzzy sets $1-\lambda, 1-\mu, 1-\gamma$ are fuzzy nowhere dense sets in (X, T). Then (X, T) is a fuzzy Baire space. Thus (X, T) is fuzzy α-Baire space. So (X, T) is fuzzy Baire space.

Proposition 7.7. *If (X, T) is a fuzzy topological space, then (X, T) satisfied the following:*

(1) *(X, T) is s fuzzy Baire space.*
(2) *(X, T) is a fuzzy α-Baire space.*
(3) *(X, T) is a fuzzy α-D-Baire space.*
(4) *(X, T) is a fuzzy α-D′-Baire space.*

Consider the following examples:

Example 7.8. Let $X = \{a, b, c\}$. The fuzzy sets λ, μ and γ are defined on X as follows:

$\lambda : X \to [0, 1]$ defined as $\lambda(a) = 0.9; \lambda(b) = 0.9; \lambda(c) = 0.7$,
$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.9; \mu(b) = 0.8; \mu(c) = 0.7$,
$\gamma : X \to [0, 1]$ defined as $\gamma(a) = 0.8; \gamma(b) = 0.8; \gamma(c) = 0.7$.

Then $T = \{0, \lambda, \mu, \gamma, 1\}$ is clearly a fuzzy topology on X.

Now we prove Proposition 7.7.

Proof. (1) Clearly the fuzzy sets $1-\lambda, 1-\mu, 1-\gamma$ are fuzzy nowhere dense sets in (X, T). Then $int[(1-\lambda) \vee (1-\mu) \vee (1-\gamma)] = 0$. Thus (X, T) is fuzzy Baire space.

(2) Clearly the fuzzy α-open sets in (X, T) are λ, μ, γ. Then

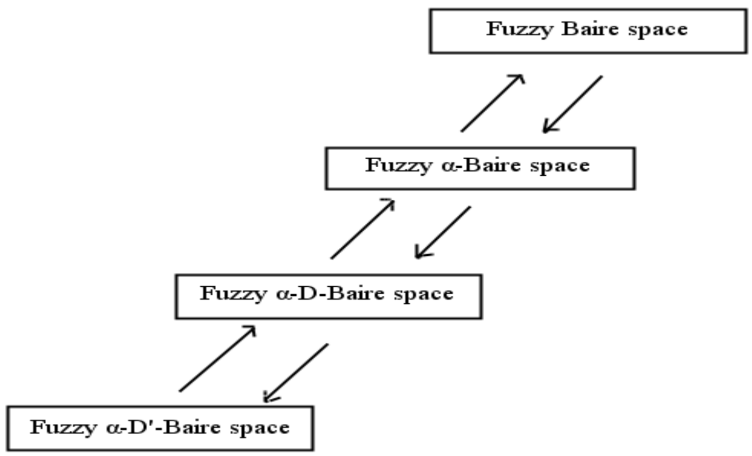
$$\alpha\text{-}int\ \alpha\text{-}cl(1-\lambda) = 0, \alpha\text{-}int\ \alpha\text{-}cl(1-\mu) = 0, \alpha\text{-}int\ \alpha\text{-}cl(1-\gamma) = 0.$$

Thus $1-\lambda, 1-\mu, 1-\gamma$ are fuzzy α-nowhere dense sets in (X,T) implies that α-$int[(1-\lambda) \vee (1-\mu) \vee (1-\gamma)] = 0$. So (X,T) is fuzzy α-Baire space.

(3) Clearly the fuzzy α-open sets in (X,T) are λ, μ, γ. Then $1-\gamma$ is a fuzzy α-first category in (X,T). Now α-int α-$cl(1-\gamma) = 0$. Thus the fuzzy αfirst category set $1-\gamma$ is a fuzzy α-nowhere dense set in (X,T). So (X,T) is a fuzzy α-D-Baire space.

(4) Clearly the fuzzy sets $1-\lambda, 1-\mu, 1-\gamma$ are empty α-interior in (X,T) and fuzzy α-nowhere dense sets in (X,T). Then (X,T) is a fuzzy α-D′-Baire space.

The following implications are true.



□

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S. Anjalmose (ansalmose@gmail.com)
Department of Mathematics, St. Joseph's College of Arts & Science (Autonomous), Manjakuppam, Cuddalore- 607 001, Tamil Nadu, India

G. Thangaraj (g.thangaraj@rediffmail.com)
Department of Mathematics, Thiruvalluvar University, Serkkadu, Vellore - 632 115, Tamil Nadu, India