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A contribution to the algebraic structure of fuzzy numbers

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Abstract. In this paper, we deal with (t_1, t_2)- type fuzzy numbers. In this sense, by constructing a new algebraic structure, we give a contribution to the fuzzy set theory. In this way, we talk about the normed and linear spaces of (t_1, t_2)- type fuzzy numbers. Additionally, the generalized difference sequence spaces of (t_1, t_2)- type fuzzy numbers $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ consisting of all sequences $\tilde{u} = (u^k_{(t_1,t_2)})$ such that $\mathscr{B}(r,s)\tilde{u}$ is in the spaces $\ell_\infty(\mathscr{F})$, $c(\mathscr{F})$ and $c_0(\mathscr{F})$, have been constructed, respectively. Moreover, we prove that $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ are linearly isomorphic to the spaces $\ell_\infty(\mathscr{F})$, $c(\mathscr{F})$ and $c_0(\mathscr{F})$, respectively. Furthermore, some theorems are given on the $\alpha(r)$-, $\beta(r)$- and $\gamma(r)$- real duals of the spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$. Finally, some classes of matrix transformations from the space $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $\mu(\mathscr{F})$ to $\mu(\mathscr{F})$ and $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ are characterized, respectively, where $\mu(\mathscr{F})$ is any sequence space..

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1. Preliminaries, background and notation

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [20]. After his innovation, mathematical structures were altered with fuzzy numbers. Matloka [11] introduced the class of bounded and convergent sequences of fuzzy numbers with respect to the Hausdorff metric. In [13], Nanda has studied the spaces of bounded and convergent sequences of fuzzy numbers and shown that these spaces are complete metric spaces with the metric $\widehat{d}(\mathrm{u},\mathrm{v}) = \sup_k \max\{|\mathrm{u}_k^-(\alpha) - \mathrm{v}_k^-(\alpha)|, |\mathrm{u}_k^+(\alpha) - \mathrm{v}_k^+(\alpha)|\}$, $(\alpha \in [0,1])$.

By using the metric $\widehat{d}$ so many spaces of fuzzy sequences have been built and published in famous maths journals.

The idea to construct a new sequence space of real or complex numbers using by matrix domain of a particular limitation method has been employed by many authors, for example you can see: Altay, Başar and Mursaleen [1], Başar and Altay [2, 4], Ng and Lee [14] and Wang [18]. They introduced the sequence spaces $(\ell_p)_{R_t} = e_p^r$, $(\ell_\infty)_{R_t} = r_\infty^t$, $c_{R_t} = r_c^t$ and $(c_0)_{R^t} = r_0^t$ in [2], $(\ell_p)_\Delta = bv_p$ in [4], $(\ell_p)_{\mathscr{C}^*} = X_p$ in [14], $(\ell_p)_{N^q}$ in [18]; where E^r, R^t, Δ, $\mathscr{C}^*$ and N_q denote the Euler, means order r, Riesz means with respect to the sequence $t = (t_k)$, backward difference matrix, Cesàro means of order one and Nörlund means with respect to the sequence $q = (q_k)$, respectively, with $1 \leq p \leq \infty$.

Some important problems on sequence spaces of fuzzy numbers can be ordered as follows:1- Construct a sequence space of fuzzy numbers and compute α-, β- and γ- dual, 2- Find some isomorphic spaces of it, 3- Give some theorems about matrix transformations on sequence space of fuzzy numbers, 4- Study some inclusion problems and other properties.

It will not be right to regard this article as a copy of classical summability theory because both a big generalization and definitions of fuzzy zero are presented in this article. Therefore, the readers are advised to take these into consideration while reading the article, and using the definition of fuzzy zero that we have solved some equations in the form $x + u_{(t_1,t_2)} = v_{(t_1,t_2)}$, where $u_{(t_1,t_2)}$ and $v_{(t_1,t_2)}$ are fuzzy number.

In the present paper, the light of these properties mentioned above, following [1, 4, 10, 14, 15, 18], we will define matrix domain of sequence spaces of triangular fuzzy numbers and introduce the sequence spaces of fuzzy numbers $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}, [c(F_t)]_{\mathscr{B}(r,s)}$ and $[c_0(F_t)]_{\mathscr{B}(r,s)}$. Besides, we redefine fuzzy identity elements according to addition and multiplication for constructing an algebraic structure on the (t_1, t_2)- type sets of fuzzy numbers.

The rest of our paper is organized, as follows:

In Section 2, some basic definitions and theorems related with the fuzzy numbers are given. Also, the sets $\ell_\infty(\mathscr{F})$, $c(\mathscr{F})$, $c_0(\mathscr{F})$ and $\ell_p(\mathscr{F})$ consisting of the bounded, convergent, null and absolutely p - summable sequences of fuzzy numbers, respectively, are defined. In Section 3, we have introduced generalized difference sequence space of (t_1, t_2)- type fuzzy numbers and proved some inclusion relations on these sequence spaces. It is also established in Section 3 that the sequence spaces of (t_1, t_2)- type fuzzy numbers showed by $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}, [c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ are linearly isomorphic to the spaces $\ell_\infty(\mathscr{F}), c(\mathscr{F})$ and $c_0(\mathscr{F})$, respectively. Finally, in Section 3, it is proved that the spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ are complete with the norm defined by $||\tilde{u}|| = \sup_k \max\{\mathscr{B}(r,s)|u^k - v^k - t_1|, \mathscr{B}(r,s)|u^k - v^k|, \mathscr{B}(r,s)|u^k - v^k + t_2|\}$. Section 4 is devoted to the calculation of the $\beta(r)$- and $\gamma(r)$- duals of the spaces $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$. In the final Section 5, some classes of matrix transformations from the space $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $\mu(\mathscr{F})$ to $\mu(\mathscr{F})$ and $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ characterized, respectively, where $\mu(\mathscr{F})$ is a any sequence space.

In this section, we recall some of the basic definitions and notions in the theory of fuzzy numbers. In the following, we provide the essential results for the development of our theory. Let suppose that $\mathbb{N}$, $\mathbb{R}$ and $E_i = \{a = [a^-, a^+] : a^- \leq x \leq a^+, \ \ a^- \ \ and \ \ a^+ \in \mathbb{R}\}$ be the set of all positive integers, all real numbers and all bounded and closed intervals on the real line $\mathbb{R}$, respectively. For $a, b \in E_i$ define

$$d(a,b) = \max\{|a^- - b^-|, |a^+ - b^+|\}. \tag{1.1}$$

It can easily be seen that d defines a metric on E_i and the pair (E_i, d) is a complete metric space [12]. We denote the set of all real and complex valued sequences by w which is a linear space with addition and scalar multiplication of sequences. Each linear subspace of w is called a sequence space. Let λ and μ be two sequence spaces and $\mathscr{A} = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we can say that $\mathscr{A}$ defines a matrix mapping from λ to μ, and we denote it by writing $\mathscr{A} \in (\lambda : \mu)$, if for every sequence $x = (x_k)$ is in λ and the sequence $\mathscr{A}x = \{(\mathscr{A}x)_n\}$, the $\mathscr{A}$- transform of x, is in μ, where k runs from 0 to ∞. The domain $\lambda_{\mathscr{A}}$ of an infinite matrix $\mathscr{A}$ in a sequence space λ is defined by

$$\lambda_{\mathscr{A}} = \{x = (x_k) \in w : \mathscr{A}x \in \lambda\} \tag{1.2}$$

which is a sequence space. If we take $\lambda = c$, then $c_{\mathscr{A}}$ is called convergence domain of $\mathscr{A}$. We write the limit of $\mathscr{A}x$ as $\mathscr{A} - \lim_n x_n = \lim_n \sum_{k=0}^{\infty} a_{nk}x_k$, and the $\mathscr{A}$ is called regular if $\lim_{\mathscr{A}} x = \lim x$ for every $x \in c$. Also, a matrix $\mathscr{A} = (a_{nk})$ is called triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$.

Let X be a nonempty set. According to Zadeh a fuzzy subset of X is a nonempty subset $\{(x, \mathrm{u}(x)) : x \in X\}$ of $X \times [0,1]$ for some function $\mathrm{u} : X \to [0,1]$. Consider a function called as membership function, $\mathrm{u} : \mathbb{R} \to [0,1]$ as a nonempty subset of $\mathbb{R}$ and denote the family of all such functions or fuzzy sets by E. Let us suppose that the function u satisfies the following properties:

(1) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $\mathrm{u}(x_0) = 1$,

(2) u is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\mu \in [0,1]$, $\mathrm{u}[\mu x + (1-\mu)y] \geq \min\{\mathrm{u}(x), \mathrm{u}(y)\}$,

(3) u is upper semi-continuous,

(4) The closure of $\{x \in \mathbb{R} : \mathrm{u}(x) > 0\}$, denoted by u^0, is compact[20].

Then the function u is called a fuzzy number.

The properties (1)-(4) imply that for each $\alpha \in [0,1]$, the α- cut set of the fuzzy number u defined by $\mathrm{u}(\alpha) = \{x \in \mathbb{R} : \mathrm{u}(x) \geq \alpha\}$ is in E_i. That is the equality $\mathrm{u}(\alpha) = [\mathrm{u}^-(\alpha), \mathrm{u}^+(\alpha)]$ holds for each $\alpha \in [0,1]$. We denote the set of all fuzzy numbers by F and the set of all triangular fuzzy numbers by F_t and write ℓ_∞, c, c_0 and ℓ_p for the classical sequence spaces of all bounded, convergent, null and absolutely p - summable sequences, respectively, where $1 \leq p < \infty$. Also by bs and cs, we denote the spaces of all bounded and convergent series, respectively. For brevity in notation, through all the text, we shall write $\sum_n$, $\sup_n$, and $\lim_n$ instead of $\sum_{n=0}^{\infty}$, $\sup_{n \in \mathbb{N}}$, and $\lim_{n \to \infty}$.

Sometimes, the representation of fuzzy numbers with α- cut sets induce errors according to algebraic operations. For example, if u is any fuzzy number then $\mathrm{u} - \mathrm{u} = [\mathrm{u}^-(\alpha), \mathrm{u}^+(\alpha)] - [\mathrm{u}^-(\alpha), \mathrm{u}^+(\alpha)] = [\mathrm{u}^-(\alpha) - \mathrm{u}^+(\alpha), \mathrm{u}^+(\alpha) - \mathrm{u}^-(\alpha)]$ is not equal to zero in the classical mean.

Furthermore, we know that shape similarity of the membership functions does not reflect the conception of itself, but it will be used for examining the context of the membership functions. Whether a particular shape is suitable or not can be determined only in the context of a particular application. However, many applications are not overly sensitive to variations in the shape. In such cases, it is convenient to use a simple shape, such as the triangular shape of membership function.

Let us consider any triangular fuzzy number u = $u_{(t_1,t_2)}$, as in the following.

If the function

$$u_{(t_1,t_2)}(x) = \begin{cases} (x-(u-t_1))t_1^{-1}, & x \in [u-t_1, u] \\ ((u+t_2)-x)t_2^{-1}, & x \in [u, u+t_2] \\ 0, & \text{otherwise} \end{cases}$$

is the membership function of the triangular fuzzy number $u_{(t_1,t_2)}$ then $u_{(t_1,t_2)}$ can be represented with the notation

$$u_{(t_1,t_2)} = (u-t_1, u, u+t_2) \tag{1.3}$$

where $t_1, t_2, u \in \mathbb{R}$ and $t_1 \leq t_2$. If $t_1 = t_2$, then the fuzzy number u is called symmetric fuzzy number and if $t_1 = t_2 = 0$ then u be a real number. In generally, the fuzzy number $u_{(t_1,t_2)}$ is called (t_1, t_2)- type fuzzy number through all the text.

The set of all (t_1, t_2)- type fuzzy numbers is defined as in the following:

$$\mathscr{F} = \{(u-t_1, u, u+t_2) : t_1, t_2, u \in \mathbb{R},\ t_1 \leq t_2\}. \tag{1.4}$$

The notations $u-t_1, u, u+t_2$ are called first, middle and end points of triangular fuzzy number u, respectively. In addition this, the notation u means that the height of the fuzzy number u is 1 at the point u. Any element of the set $\mathscr{F}$ will be denoted with $u_{(t_1,t_2)}$. If $t_1 = \frac{1}{3}$, $t_2 = \frac{1}{2}$ and $u = 2$ then $u_{(\frac{1}{3},\frac{1}{2})}$ is represented as, $u_{(\frac{1}{3},\frac{1}{2})} = (2-\frac{1}{3}, 2, 2+\frac{1}{2})$. For every $t_1, t_2 \in \mathbb{R}$, the sets $\mathscr{F}$ are different from each other and every element in the form $u_{(t_1,t_2)}$ of the these sets belongs to $\mathscr{F}$. Another mean of the (1.4) is that for every $t_1, t_2 \in \mathbb{R}$, there is not an unique set of fuzzy numbers, conversely there are infinite sets of fuzzy numbers and these sets are different from each other according to structure of their elements. So, we can use the most appropriate one of these infinite sets for our problem. In this study, we will take $t_1, t_2 \in [0, 1)$.

2. Algebraic structure of the set $\mathscr{F}$

Let $u_{(t_1,t_2)}, v_{(t_1,t_2)} \in \mathscr{F}$ and $\lambda \in \mathbb{R}$. Let us define addition and scalar multiplication on $\mathscr{F}$ as follows:

(2.1)

$$\begin{aligned} u_{(t_1,t_2)} + v_{(t_1,t_2)} &= (u-t_1, u, u+t_2) + (v-t_1, v, v+t_2) \\ &= (u+v-t_1, u+v, u+v+t_2) = (w-t_1, w, w+t_2) = w_{(t_1,t_2)}, \end{aligned}$$

$$\lambda u_{(t_1,t_2)} = (\lambda u - t_1, \lambda u, \lambda u + t_2) = (r-t_1, r, r+t_2) = r_{(t_1,t_2)}. \tag{2.2}$$

Let $u_{(t_1,t_2)} = (u-t_1, u, u+t_2) \in \mathscr{F}$. Then

$$(u-t_1, u, u+t_2) + (0-t_1, 0, 0+t_2) = (u+0-t_1, u+0, u+0+t_2\}) = (u-t_1, u, u+t_2).$$

It means that $\theta_{(t_1,t_2)} = (0-t_1, 0, 0+t_2)$ is identity element of $\mathscr{F}$ according to operation which is given in (2.1). Therefore, we say that, the inverse of the fuzzy number $u_{(t_1,t_2)}$ is equal to $-u_{(t_1,t_2)} = (-u - t_1, -u, -u + t_2)$, according to addition and the $-u_{(t_1,t_2)}$ determines a fuzzy number. With this idea, we can solve equations in the form $x_{(t_1,t_2)} + u_{(t_1,t_2)} = \theta_{(t_1,t_2)}$ which may be given by inexact data. The fuzzy zeros of the sets $\mathscr{F}$ are represented as follows:

$$\theta = \theta_{(t_1,t_2)} = (0 - t_1, 0, 0 + t_2), t_1, t_2 \in \mathbb{R}, t_1 \leq t_2. \tag{2.3}$$

It is clear that fuzzy zero is different for each element of the set $\mathscr{F}$. In this way, we can talk about the normed and linear spaces of fuzzy numbers. This idea will fill a large gap in the literature.

Theorem 2.1. *All sets in the form $\mathscr{F}$ are linear spaces according to algebraic operations (2.1) and (2.2), where $t_1, t_2 \in \mathbb{R}$ and $t_1 \leq t_2$.*

The second important matter is the topology on the set $\mathscr{F}$. Similar to Şengönül [15] we can construct a topology on $\mathscr{F}$ by using the metric $\overline{d} : \mathscr{F} \times \mathscr{F} \to \mathbb{R}$ defined as follows

$$\overline{d}(u_{(t_1,t_2)}, v_{(t_1,t_2)}) := \max\{|u - v - t_1|, |u - v|, |u - v + t_2|\}. \tag{2.4}$$

We can easily show that the set $\mathscr{F}$ is a complete metric space with the metric $\overline{d}$.

Clearly, the representation (1.3) for u(x) is unique. We know that, generally in the practical applications, the spread of fuzziness is not very large. So the value of $\max\{|u - v - t_1|, |u - v|, |u - v + t_2|\}$ should be small to the greatest extent possible. In the case of applications, this smallness is considered more suitable. For example, the "*approximately 5*" can be taken as $5_{(t_1,t_2)} = (-4 - t_1, 5, 15 + t_2)$ but in the applications, generally, "*approximately 5*" is taken as $5_{(t_1,t_2)} = (5 - t_1, 5, 5 + t_2), (0 \leq t_1 \leq t_2 < 1)$. This choice is more accurate than $5_{(t_1,t_2)} = (-4 - t_1, 5, 15 + t_2)$.

Furthermore, it must be emphasized here that a fuzzy number is determined according to specific processes and this fuzzy number may not in the same sense in the another specific systems. For instance, let A and B be two different specific system. If u is "*approximately 5*" for the system A then "*approximately 5*" may not in the same sense for another system B. So, algebraic properties of the systems A and B are different. This can be explained as follows:

Let us suppose that the spread of left and right fuzziness of every number u be equal to t in the A. Then fuzzy zero is equal to $(-2t, 0, 2t), (0 \leq t < 1)$ for the system A and this fuzzy zero is unique for A.

The function $f : \mathbb{N} \to \mathscr{F}, k \to f(k) = u^k_{(t_1,t_2)}$ is called a sequence of (t_1, t_2)- type fuzzy numbers and is represented by $\tilde{u} = (u^k_{(t_1,t_2)})$.

Let us denote the set of all sequences of (t_1, t_2)- type fuzzy numbers by $w(\mathscr{F})$, that is

$$w(\mathscr{F}) := \{(u^k_{(t_1,t_2)}) : u : \mathbb{N} \to F_t, u(k) = u^k_{(t_1,t_2)} = (u^k - t_1, u^k, u^k + t_2)\}$$

where $u^k - t_1 \leq u^k \leq u^k + t_2$, $t_1, t_2 \in \mathbb{R}$ and $u^k_{(t_1,t_2)} \in \mathscr{F}$ for all $k \in \mathbb{N}$. The elements $u^k - t_1$, u^k and $u^k + t_2$ are called first, middle and end points of general term of a sequences of fuzzy numbers, respectively. If degree of membership at u^k is equal to 1 then $\tilde{u}$ is a (t_1, t_2)- type fuzzy number, if it is not equal to 1 then $(u^k_{(t_1,t_2)})$ is a sequence of the fuzzy sets.

Each subspace of $w(F)$ is called a sequence space of fuzzy numbers. We define the classical sets $\ell_\infty(\mathscr{F})$, $c(\mathscr{F})$, $c_0(\mathscr{F})$ and $\ell_p(\mathscr{F})$ consisting of the bounded, convergent, null and absolutely p - summable sequences of fuzzy numbers, respectively, it means that

$$\ell_\infty(\mathscr{F}) = \left\{ \tilde{u} = (u^k_{(t_1,t_2)}) \in w(F) : \sup_k \overline{d}(u^k_{(t_1,t_2)}, \theta) < \infty \right\},$$

$$c(\mathscr{F}) = \left\{ \tilde{u} = (u^k_{(t_1,t_2)}) \in w(F) : \lim_k \overline{d}(u^k_{(t_1,t_2)}, u^0_{(t_1,t_2)}) = 0, \quad u^0_{(t_1,t_2)} \in \mathscr{F} \right\},$$

$$c_0(\mathscr{F}) = \left\{ \tilde{u} = (u^k_{(t_1,t_2)}) \in w(F) : \lim_k \overline{d}(u^k_{(t_1,t_2)}, \theta) = 0 \right\},$$

$$\ell_p(\mathscr{F}) = \left\{ \tilde{u} = (u^k_{(t_1,t_2)}) \in w(F) : \sup_k \sum_k \overline{d}(u^k_{(t_1,t_2)}, \theta)^p < \infty, 1 \le p < \infty \right\}.$$

We should emphasize here that the sequence spaces of fuzzy numbers $\ell_\infty(\mathscr{F})$, $c(\mathscr{F})$, $c_0(\mathscr{F})$ and $\ell_p(\mathscr{F})$ can be reduced to the classical sequence spaces of real numbers ℓ_∞, c, c_0 and ℓ_p, respectively in the special case $(u^k_{(t_1,t_2)}) = (u^k_{(1,1)})$, where $u^k_1 \in \mathbb{R}$ and $u^k_{(t_1,t_2)} \in \mathscr{F}$. So, the properties and results related to the sequence spaces of $\ell_\infty(\mathscr{F}), c(\mathscr{F}), c_0(\mathscr{F})$ and $\ell_p(\mathscr{F})$ are more general and more useful than the corresponding implications of the spaces ℓ_∞, c, c_0 and ℓ_p, respectively.

Definition 2.2. Let $\lambda(\mathscr{F}) \subset w(F)$, θ is identity element according to addition and algebraic operations on $\lambda(\mathscr{F})$ in the sense of (2.1) and (2.2). The function $||.|| : \lambda(\mathscr{F}) \to \mathbb{R}$ is called norm on the set $\lambda(\mathscr{F})$ if it has the following properties:
(i) $||u_{(t_1,t_2)}|| = 0 \Leftrightarrow u_{(t_1,t_2)} = \theta$.
(ii) $||\alpha u_{(t_1,t_2)}|| = |\alpha| ||u_{(t_1,t_2)}||$, $\alpha \in \mathbb{R}$.
(iii) $||u_{(t_1,t_2)} + v_{(t_1,t_2)}|| \le ||u_{(t_1,t_2)}|| + ||v_{(t_1,t_2)}||$.

If the function $||.|| : \lambda(\mathscr{F}) \to \mathbb{R}$ satisfy (i)-(iii) then $\lambda(\mathscr{F})$ is called normed sequence space of the (t_1,t_2)- type fuzzy numbers. If $\lambda(\mathscr{F})$ is complete with respect to the norm $||.||$ then $\lambda(\mathscr{F})$ is called complete normed sequence space of the (t_1,t_2)-type fuzzy numbers.

Lemma 2.3. *The sets $c(\mathscr{F}), c_0(\mathscr{F})$ and $\ell_\infty(\mathscr{F})$ are complete normed sequence spaces with the norm defined by as follows:*

$$||\tilde{u}|| = \sup_k \max\{|u^k - v^k - t_1|, |u^k - v^k|, |u^k - v^k + t_2|\}$$

where $\tilde{u}$ is in the any sets of $\{c(\mathscr{F}), c_0(\mathscr{F}), \ell_\infty(\mathscr{F})\}$.

Proof. We take into account only the space $\ell_\infty(\mathscr{F})$. Others can be proved similarly. It is clear that $|| \cdot ||$ is a norm on $\ell_\infty(\mathscr{F})$. For showing the completeness of $\ell_\infty(\mathscr{F})$, we accept the fact that $(u^k_{n(t_1,t_2)}) = (u^k_n - t_1, u^k_n, u^k_n + t_2)$ is a Cauchy sequence in $\ell_\infty(\mathscr{F})$ for each n. Then, we have

$$||u^k_{n(t_1,t_2)} - u^k_{m(t_1,t_2)}|| = \sup_k \max\{|u^k_n - u^k_m - t_1|, |u^k_n - u^k_m|, |u^k_n - u^k_m + t_2|\} < \varepsilon.$$

From here, we can obtain

$$|u^k_n - u^k_m - t_1| < \varepsilon, |u^k_n - u^k_m| < \varepsilon, |u^k_n - u^k_m + t_2| < \varepsilon.$$

From hence, we have that $(u_n^k - u_m^k - t_1)$, $(u_n^k - u_m^k)$ and $(u_n^k - u_m^k + t_2)$ are Cauchy sequences of real numbers. Since $\mathbb{R}$ is complete, the sequences $(u_n^k - u_m^k - t_1)$, $(u_n^k - u_m^k)$ and $(u_n^k - u_m^k + t_2)$ are convergent in in $\mathbb{R}$, for all $n \in \mathbb{N}$. Let us assume that $\lim_n u^k_{n(t_1,t_2)} = u^k_{0(t_1,t_2)}$ for all $t_1, t_2 \in \mathbb{R}$ and $k \in \mathbb{N}$. Because of the fact that

$$\sup_k \max\{|u_n^k - u_m^k - t_1|, |u_n^k - u_m^k|, |u_n^k - u_m^k + t_2|\} < \varepsilon$$

for all $n, m \geq k$, we obtain the following equality

$$\begin{aligned} &\lim_m \sup_k \max\{|u_n^k - u_m^k - t_1|, |u_n^k - u_m^k|, |u_n^k - u_m^k + t_2|\} \\ &= \sup_k \max\{|u_n^k - u_0^k - t_1|, |u_n^k - u_0^k|, |u_n^k - u_0^k + t_2|\} < \varepsilon. \end{aligned}$$

Thus, we can write for $t_1, t_2 \in \mathbb{R}$ that $u^k_{n(t_1,t_2)} \to u^k_{0(t_1,t_2)}$ as $n \to \infty$. On the other side, since the belows hold

$$\begin{aligned} ||u^k_{0(t_1,t_2)}|| &\leq \sup_k \max\{|u_0^k - u_n^k - t_1|, |u_0^k - u_n^k|, |u_0^k - u_n^k + t_2|\} \\ &+ \sup_k \max\{|u_n^k - t_1|, |u_n^k|, |u_n^k + t_2|\} \\ &= ||u^k_{0(t_1,t_2)} - u^k_{n(t_1,t_2)}|| + ||u^k_{n(t_1,t_2)}|| \end{aligned}$$

we conclude that $(u^k_{0(t_1,t_2)}) \in \ell_\infty(\mathscr{F})$. □

Now, let $\lambda(\mathscr{F})$ and $\mu(\mathscr{F})$ be two spaces of triangular fuzzy valued sequences and $\mathscr{A} = (a_{nk})$ be an infinite matrix of positive real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that $\mathscr{A}$ defines a *real - matrix mapping* from $\lambda(\mathscr{F})$ to $\mu(\mathscr{F})$ and we denote it by writing $\mathscr{A} : \lambda(\mathscr{F}) \to \mu(\mathscr{F})$, if for every sequence $\tilde{u} = (u^k_{(t_1,t_2)}) \in \lambda(\mathscr{F})$ the sequence $\left\{(\mathscr{A} u_{(t_1,t_2)})^n\right\}$, is in $\mu(\mathscr{F})$ where

$$(2.5)\quad (\mathscr{A} u_{(t_1,t_2)})^n = \sum_k a_{nk} u^k_{(t_1,t_2)} = \left(\sum_k a_{nk} u^k - t_1, \sum_k a_{nk} u^k, \sum_k a_{nk} u^k + t_2\right)$$

and the series $\sum_k a_{nk} u^k - t_1, \sum_k a_{nk} u^k, \sum_k a_{nk} u^k + t_2$ are convergent for all $n \in \mathbb{N}$. By $(\lambda(\mathscr{F}) : \mu(\mathscr{F}))$, we denote the class of matrices $\mathscr{A}$ such that $\mathscr{A} : \lambda(\mathscr{F}) \to \mu(\mathscr{F})$. Thus, $\mathscr{A} \in (\lambda(\mathscr{F}) : \mu(\mathscr{F}))$ if and only if the series on the right side of (2.5) are convergent for each $n \in \mathbb{N}$ and every $\tilde{u} \in \lambda(\mathscr{F})$, and we have $\mathscr{A}\tilde{u} = \left\{(\mathscr{A} u_{(t_1,t_2)})^n\right\}_{n\in\mathbb{N}} \in \mu(\mathscr{F})$ for all $\tilde{u} \in \lambda(\mathscr{F})$.

Let $\lambda(\mathscr{F})$ be a sequence space of (t_1, t_2)- type fuzzy numbers. Then, the set $[\lambda(\mathscr{F})]_{\mathscr{A}}$ of sequences of (t_1, t_2)- type fuzzy numbers defined as follows, is called the domain of an infinite matrix $\mathscr{A}$ in $\lambda(\mathscr{F})$,

$$(2.6)\qquad [\lambda(\mathscr{F})]_{\mathscr{A}} = \left\{(u^k_{(t_1,t_2)}) \in w(F_t) : \mathscr{A}\tilde{u} \in \lambda(\mathscr{F})\right\}.$$

Now, we give the definition of generalized difference matrix used in the later part of this work.

Let r, s be non-zero real numbers and define the generalized difference matrix $\mathscr{B}(r,s) = (b_{nk}(r,s))$ by

$$b_{nk}(r,s) = \begin{cases} r & , \quad k = n \\ s & , \quad k = n-1 \\ 0 & , \quad 0 \leq k < n-1 \quad or \quad k > n \end{cases} \quad ; \ (k, n \in \mathbb{N}) . \tag{2.7}$$

It is easy to calculate that the inverse $\mathscr{B}^{-1}(r,s) = (a_{nk}(r,s))$ of the generalized difference matrix $\mathscr{B}(r,s)$ is given by

$$a_{nk}(r,s) = \begin{cases} \frac{1}{r}(\frac{-s}{r})^{n-k} & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n \end{cases} \ ; \ (k, n \in \mathbb{N}) . \tag{2.8}$$

3. The generalized difference sequence spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$

In this section, we wish to introduce the $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ spaces, as the set of all sequences such that $\mathscr{B}(r,s)$- transforms of them are in the spaces $\ell_\infty(\mathscr{F})$, $c(\mathscr{F})$ and $c_0(\mathscr{F})$, respectively, that is

$$[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)} = \left\{\tilde{u} = (u^k_{(t_1,t_2)}) \in w(F) : \mathscr{B}(r,s)u^k_{(t_1,t_2)} \in \ell_\infty(\mathscr{F})\right\}, \tag{3.1}$$

$$[c(\mathscr{F})]_{\mathscr{B}(r,s)} = \left\{\tilde{u} = (u^k_{(t_1,t_2)}) \in w(F) : \mathscr{B}(r,s)u^k_{(t_1,t_2)} \in c(\mathscr{F})\right\} \tag{3.2}$$

and

$$[c_0(\mathscr{F})]_{\mathscr{B}(r,s)} = \left\{\tilde{u} = (u^k_{(t_1,t_2)}) \in w(F) : \mathscr{B}(r,s)u^k_{(t_1,t_2)} \in c_0(\mathscr{F})\right\}. \tag{3.3}$$

We should emphasize here that the sequence spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ of (t_1, t_2)- type fuzzy numbers can be reduced to the sets $\ell_\infty(\mathscr{F}), c(\mathscr{F})$ and $c_0(\mathscr{F})$, respectively in the case $r = 1, s = 0$ in the structure of generalized difference matrix. So, the properties and results related to the sequence spaces of $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ are more general and more extensive than the corresponding consequences of the spaces $\ell_\infty(\mathscr{F}), c(\mathscr{F})$ and $c_0(\mathscr{F})$, respectively.

Let us define the sequence of fuzzy numbers $\tilde{v} = (v^k_{(t_1,t_2)})$ which will be frequently used, as the $\mathscr{B}(r,s)$- transform of a sequence of fuzzy numbers $\tilde{u} = (u^k_{(t_1,t_2)})$, i.e.,

$$v^k_{(t_1,t_2)} = su^{k-1}_{(t_1,t_2)} + ru^k_{(t_1,t_2)} \tag{3.4}$$

where $u^{-1} = 0$, $r, s \in \mathbb{R} - \{0\}$.

Now, we may begin with the following theorem which is essential in the text:

Theorem 3.1. *The sequence spaces* $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ *and* $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ *are linearly isomorphic to the spaces* $\ell_\infty(\mathscr{F})$, $c(\mathscr{F})$ *and* $c_0(\mathscr{F})$ *respectively, i.e.,* $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)} \cong \ell_\infty(\mathscr{F})$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)} \cong c(\mathscr{F})$ *and* $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)} \cong c_0(\mathscr{F})$.

Proof. Since the others can be similarly proved, we consider only the case $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)} \cong \ell_\infty(\mathscr{F})$. To prove this, we should show the existence of a linear bijection between the spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $\ell_\infty(\mathscr{F})$. Consider the transformation T defined with the notation of (3.4), from $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$ to $\ell_\infty(\mathscr{F})$ by $\tilde{u} \mapsto \tilde{v} = T\tilde{u} =$

$su^{k-1}_{(t_1,t_2)} + ru^k_{(t_1,t_2)}$. The equality $T(\tilde{u}+\tilde{w}) = T\tilde{u} + T\tilde{w}$ where $\tilde{u}, \tilde{w} \in [\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$ are clear. Let us suppose that $\alpha \in \mathbb{R}$ then

$$T(\alpha\tilde{u}) \quad = T(\alpha u^k_{(t_1,t_2)}) = s(\alpha u^{k-1}_{(t_1,t_2)}) + r(\alpha u^k_{(t_1,t_2)}) = \alpha[su^{k-1}_{(t_1,t_2)} + ru^k_{(t_1,t_2)}] = \alpha T\tilde{u}.$$

It means that, T has the property homogeneity. Thus, T is linear. Let us take any $\tilde{v} \in \ell_\infty(\mathscr{F})$ and represent the sequence $\tilde{u} = (u^k_{(t_1,t_2)})$ using the $\mathscr{B}^{-1}(r,s)$ as follows

$$\tilde{u} = (u^k_{(t_1,t_2)}) = (\mathfrak{B}^k v^k_{(t_1,t_2)})$$

where $\mathfrak{B}^k = \frac{1}{r}\sum_{j=0}^{k}(\frac{-s}{r})^j$, $\mathfrak{B}^{k-1} = \frac{1}{r}\sum_{j=0}^{k-1}(\frac{-s}{r})^j$. Then, we have

$$\begin{aligned} ||\tilde{u}||_{[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}} &= \sup_k \overline{d}(\mathscr{B}(r,s)u^k_{(t_1,t_2)}, \theta) \\ &= \sup_k \overline{d}(su^{k-1}_{(t_1,t_2)} + ru^k_{(t_1,t_2)}, \theta) \\ &= \sup_k \overline{d}(s[\mathfrak{B}^{k-1}v^{k-1-j}_{(t_1,t_2)}] + r[\mathfrak{B}^k v^{k-j}_{(t_1,t_2)}], \theta) \\ &= \sup_k \overline{d}(v^k_{(t_1,t_2)}, \theta) = ||\tilde{v}||_{\ell_\infty(\mathscr{F})}. \end{aligned}$$

That is, T is norm preserving. Consequently, the spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $\ell_\infty(\mathscr{F})$ are linearly isomorphic. It is clear here that if the spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $\ell_\infty(\mathscr{F})$ are respectively replaced by the spaces $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $c(\mathscr{F})$, $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $c_0(\mathscr{F})$, then we obtain the fact that $[c(\mathscr{F})]_{B(r,s)} \cong c(\mathscr{F})$ and $[c_0(\mathscr{F})]_{B(r,s)} \cong c_0(\mathscr{F})$. This step completes the proof. □

Theorem 3.2. *The sets* $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ *and* $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$ *are complete normed sequence space of the* (t_1,t_2)*- type fuzzy numbers with the norm defined by*

$$\sup_k \max\{\mathscr{B}(r,s)|u^k - v^k - t_1|, \mathscr{B}(r,s)|u^k - v^k|, \mathscr{B}(r,s)|u^k - v^k + t_2|\}. \tag{3.5}$$

Proof. It was seen that in Theorem 3.1, the sequence spaces of (t_1,t_2)- type fuzzy numbers $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ are linearly isomorphic to the spaces $\ell_\infty(\mathscr{F})$, $c(\mathscr{F})$ and $c_0(\mathscr{F})$, respectively. Additionally, since the matrix $\mathscr{B}(r,s)$ is normal (see, [19]) and $\ell_\infty(\mathscr{F})$, $c(\mathscr{F})$ and $c_0(\mathscr{F})$ are complete normed sequence spaces, it is clear that the sequence spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ are complete normed spaces with the norm defined in (3.5). □

Theorem 3.3. *Let* $(u^k_{(t_1,t_2)})$ *be a sequence of* (t_1,t_2)*- type fuzzy numbers. If* $\overline{d}(u^k_{(t_1,t_2)}, u^0_{(t_1,t_2)}) \to 0$ *as* $n \to \infty$ *then* $\overline{d}(su^{k-1}_{(t_1,t_2)} + ru^k_{(t_1,t_2)}, u^0_{(t_1,t_2)}) \to 0$ *as* $n \to \infty$. *It means that the matrix* $\mathscr{B}(r,s)$ *is regular.*

Proof. Let $(u^k_{(t_1,t_2)})$ be a sequence of (t_1,t_2)- type fuzzy numbers and $\overline{d}(u^k_{(t_1,t_2)}, u^0_{(t_1,t_2)})$ $\to 0$ as $n \to \infty$. Then for a given $\epsilon > 0$ there exits a positive integer n_0 such that

$$\overline{d}(u^k_{(t_1,t_2)}, u^0_{(t_1,t_2)}) < \frac{\epsilon}{2M} \quad \text{for each} \quad k \geq n_0.$$

Therefore, it can be written as in the following, $\overline{d}(\mathscr{B}(r,s)u^k_{(t_1,t_2)}, u^0_{(t_1,t_2)}) = \overline{d}(su^{k-1}_{(t_1,t_2)} + ru^k_{(t_1,t_2)}, u^0_{(t_1,t_2)}) \leq s\overline{d}(u^{k-1}_{(t_1,t_2)}, u^0_{(t_1,t_2)}) + r\overline{d}(u^k_{(t_1,t_2)}, u^0_{(t_1,t_2)})$. If we take $M = \max\{s, r\}$

then we see that $\overline{d}(\mathscr{B}(r,s)u^k_{(t_1,t_2)}, u^0) < \epsilon$. That is

$$\lim_k (su^{k-1}_{(t_1,t_2)} + ru^k_{(t_1,t_2)}) = u^0_{(t_1,t_2)}$$

holds. This completes the proof. □

Theorem 3.4. *The inclusions* $c(\mathscr{F}) \subset [c(\mathscr{F})]_{\mathscr{B}(r,s)}$ *and* $c_0(\mathscr{F}) \subset [c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ *strictly hold.*

Proof. To prove the validity of the inclusion $c_0(\mathscr{F}) \subset [c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$, let us take any $\tilde{v} \in c_0(\mathscr{F})$. Then, bearing in mind the regularity of the method $\mathscr{B}(r,s)$ (see, Theorem 3.3), we immediately observe that $\mathscr{B}(r,s)\tilde{v} \in c_0(\mathscr{F})$ which means that $\tilde{v} \in [c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$. Hence, the inclusion $c_0(\mathscr{F}) \subset [c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ holds. Furthermore, let us consider the sequence $\tilde{u} = (u^k_{(t_1,t_2)})$ defined by $u^k_{(t_1,t_2)} = \frac{1}{r}(\frac{-s}{r})^{n-k}(\frac{3}{2} - \frac{(k+2)}{2k} - t_1, \frac{3}{2}, \frac{3}{2} + \frac{3k-2}{2k} + t_2)$ for all $k \in \mathbb{N}$. Then, we have $(\mathscr{B}(r,s)u^k_{(t_1,t_2)}) = ((0 - t_1, \frac{3}{2}, 2 + t_2), ..., \theta, \theta, ...)$ which implies that $\tilde{u}$ is in $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ but $\tilde{u} \notin c_0(\mathscr{F})$. This shows that the inclusion is strict. One can see by analogy that the strict inclusion $c(\mathscr{F}) \subset [c(\mathscr{F})]_{\mathscr{B}(r,s)}$ also holds. This completes the proof. □

Let $\lambda(F), \mu(F) \subset w(F)$ and $\mathscr{A} = (a_{nk})$ be an infinite matrix of fuzzy numbers and consider following expressions:

$$\sup_n \sum_k \overline{d}(a_{nk}, \theta) < \infty, \tag{3.6}$$

$$\sup_n \sum_k [\overline{d}(a_{nk}, \theta)]^q < \infty, \tag{3.7}$$

$$\overline{d}(a_{nk}, \alpha^k) = 0, \quad where \quad \alpha^k \in w(F), \tag{3.8}$$

$$\sum_k \overline{d}(a_{nk}, \theta) = 0, \tag{3.9}$$

$$\lim_n \sum_k a_{nk} = 1_{(t_1,t_2)}, \tag{3.10}$$

$$\lim_n a_{nk} = \theta, \quad k \in \mathbb{N}. \tag{3.11}$$

In [17], some matrix classes are characterized by Talo and Başar which is given in the following lemma:

Lemma 3.5. [17] *The following statements hold:*

(1) $\mathscr{A} \in (\ell_\infty(F) : \ell_\infty(F))$, $\mathscr{A} \in (c(F) : \ell_\infty(F))$, $\mathscr{A} \in (c_0(F) : \ell_\infty(F))$ *if and only if (3.6) holds.*

(2) $\mathscr{A} \in (\ell_\infty(F) : c_0(F))$ *if and only if (3.9) holds.*

(3) $\mathscr{A} \in (c_0(F) : c(F))$ *if and only if (3.6) and (3.8) hold.*

(4) $\mathscr{A} \in (c_0(F) : c_0(F))$ *if and only if (3.6) and (3.8) hold with* $\alpha^k = \theta$ *for all* $k \in \mathbb{N}$.

(5) $\mathscr{A} \in (\ell_p(F) : c(F))$ *if and only if (3.7) and (3.8) hold.*

(6) $\mathscr{A} \in (\ell_p(F) : c_0(F))$ *if and only if (3.7) and (3.8) hold with* $\alpha^k = \theta$ *for all* $k \in \mathbb{N}$.

(7) $\mathscr{A} \in (c(F) : c(F), p)$ *if and only if (3.6), (3.10) and (3.8) hold with* $\alpha^k = \theta$ *for all* $k \in \mathbb{N}$.

Analogously to Talo and Başar, we can prove following propositions:

Proposition 3.6. *$A \in (c(\mathscr{F}) : c(\mathscr{F}))$ if and only if (3.6) and (3.8) also hold with $\alpha^k = \theta$ for all $k \in \mathbb{N}$.*

Proposition 3.7. *$A \in (c_0(\mathscr{F}) : \ell_1(\mathscr{F}))$ if and only if $\sup_G \sum_k \overline{d}\left(\sum_{n\in G} a_{nk}, \theta\right) < \infty$, where G is the finite subset of $\mathbb{N}$.*

4. Real duals of the spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$

In this section, we state and prove the theorems determining the $\alpha(r)$-, $\beta(r)$- and $\gamma(r)$- real duals of the spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$. For the sequence spaces $\lambda(\mathscr{F})$ and $\mu(\mathscr{F})$, define the set $S(\lambda(\mathscr{F}), \mu(\mathscr{F}))$ by

$$\left\{\hat{a} = (a^k) \in w(\mathbb{R}) : (a^k x^k_{(t_1,t_2)}) \in \mu(\mathscr{F}) \text{ for all } \tilde{x} \in \lambda(\mathscr{F})\right\} \tag{4.1}$$

where $w(\mathbb{R})$ denotes all real valued sequences space. With the notation of (4.1), the $\alpha(r)$-, $\beta(r)$- and $\gamma(r)$- duals of a sequence space $\lambda(\mathscr{F})$, which are respectively denoted by $\lambda^{\alpha(r)}(\mathscr{F})$, $\lambda^{\beta(r)}(\mathscr{F})$ and $\lambda^{\gamma(r)}(\mathscr{F})$ are defined by $\lambda^{\alpha(r)}(\mathscr{F}) = S(\lambda(\mathscr{F}), \ell_1(\mathscr{F}))$, $\lambda^{\beta(r)}(\mathscr{F}) = S(\lambda(\mathscr{F}), cs(\mathscr{F}))$ and $\lambda^{\gamma(r)}(\mathscr{F}) = S(\lambda(\mathscr{F}), bs(\mathscr{F}))$. We will use a technique, in the proof of the Theorems 4.1 and 4.4, which is used in [1], [4] and [16].

Theorem 4.1. *The $\gamma(r)$- dual of the spaces $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ is the set*

$$D_1 = \left\{\hat{a} \in w(\mathbb{R}) : \sup_n \sum_{k=0}^{n} \overline{d}\left(\frac{1}{r}\sum_{j=k}^{n}\left(\frac{-s}{r}\right)^{j-k} a^j, \theta\right) < \infty\right\}. \tag{4.2}$$

Proof. Since the proof is similar for the rest of the spaces, we determine only $\gamma(r)$-dual of the set $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$. Let $\hat{a} \in w(\mathbb{R})$ and define the matrix $\mathscr{Z} = (z_{nk})$ via the sequence $\hat{a} = (a^i)$ by

$$z_{nk} = \begin{cases} \frac{1}{r}\sum_{j=k}^{n}(\frac{-s}{r})^{j-k} a^j & , \quad (0 \le k \le n) \\ 0 & , \quad (k > n) \end{cases} ; \ (k, j, n \in \mathbb{N}). \tag{4.3}$$

Bearing in mind the relation (3.4) we immediately derive that

$$\sum_{k=0}^{n} a^k x^k_{(t_1,t_2)} = \sum_{k=0}^{n}\left(\sum_{j=k}^{n}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k} a^j\right) y^k_{(t_1,t_2)} = (\mathscr{Z} y_{(t_1,t_2)})^n. \tag{4.4}$$

From (4.4), we realize that $\hat{a}\tilde{x} = (a^k x^k_{(t_1,t_2)}) \in bs(\mathscr{F})$ whenever $\tilde{x} \in [c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$ if and only if $\mathscr{Z} y^k_{(t_1,t_2)} \in \ell_\infty(\mathscr{F})$ whenever $y^k_{(t_1,t_2)} \in c_0(\mathscr{F})$. Then, by considering the Part (1) of Lemma 3.5, we have

$$\sup_n \sum_{k=0}^{n} \overline{d}\left(\sum_{j=k}^{n}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k} a^j, \theta\right) < \infty$$

which yields the consequence that $[c_0(\mathscr{F})]^{\gamma(r)}_{\mathscr{B}(r,s)} = D_1$. □

Theorem 4.2. $[c_0(\mathscr{F})]^{\alpha(r)}_{\mathscr{B}(r,s)}$ *is the set* D_2*, where*

$$D_2 = \left\{ \hat{a} \in w(\mathbb{R}) : \sup_G \sum_k \overline{d} \left(\sum_{n\in G} \sum_{j=k}^{n} \frac{1}{r} \left(\frac{-s}{r} \right)^{j-k} a^j, \theta \right) < \infty \right\}.$$

Proof. This is clear from Proposition 3.7. □

Lemma 4.3. ([3], Theorem 3.1) *Let* $\mathscr{C} = (c_{nk})$ *be defined via a sequence* $\hat{a} = (a^k) \in w$ *and the inverse matrix* $\mathscr{V} = (v_{nk})$ *of the triangle matrix* $\mathscr{U} = (u_{nk})$ *by*

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} a^j v_{jk}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$*. Then*

$$\{\lambda_{\mathscr{U}}\}^{\gamma} = \{\hat{a} = (a^k) \in w : \mathscr{C} \in (\lambda : \ell_\infty)\}$$

and

$$\{\lambda_{\mathscr{U}}\}^{\beta} = \{\hat{a} = (a^k) \in w : \mathscr{C} \in (\lambda : c)\}.$$

Combining Lemma 3.5 and Lemma 4.3, we have following theorems.

Theorem 4.4. *Define the sets* D_3, D_4 *and* D_5 *by*

$$D_3 = \left\{ \hat{a} \in w(\mathbb{R}) : \overline{d} \left(\frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a^j, \alpha^k \right) = 0, \alpha^k \in w(F) \right\},$$

$$D_4 = \left\{ \hat{a} \in w(\mathbb{R}) : \overline{d} \left(\frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a^j, \theta \right) \quad \textit{exists} \right\}$$

and

$$D_5 = \left\{ \hat{a} \in w(\mathbb{R}) : \lim_n \sum_{k=0}^{n} \left(\frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a^j \right) = 1_{(t_1,t_2)} \right\}.$$

Then, $[c_0(F_t)]^{\beta(r)}_{\mathscr{B}(r,s)} = D_1 \cap D_3$*. And* $[c(F_t)]^{\beta(r)}_{\mathscr{B}(r,s)} = D_1 \cap D_4 \cap D_5$

Proof. It is clear from the Lemma 3.5 and Lemma 4.3. □

5. Matrix Transformations

First of all, Lorentz introduced the concept of dual summability methods for the limitation which depends on a Stieltjes integral and passed to the discontinuous matrix methods by means of a suitable step function in [8]. Later, many authors, such as Başar [5], Başar - Çolak[6], Kuttner [7], Lorentz - Zeller [9] and Şengönül - Başar [16] worked on the dual summability methods.

Let us suppose that the set $[\lambda(\mathscr{F})]_{\mathscr{B}(r,s)}$ be any of the sets $[c_0(\mathscr{F})]_{\mathscr{B}(r,s)}$, $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $[\ell_\infty(\mathscr{F})]_{\mathscr{B}(r,s)}$. In this section, we characterize the matrix mappings from $[\lambda(\mathscr{F})]_{\mathscr{B}(r,s)}$ into any given sequence space of triangular fuzzy numbers via the concept of the dual summability methods and vice versa, so we call it as the generalized difference dual summability methods. Let us suppose that the sequences

$\tilde{u} = (u^i_{(t_1,t_2)})$ and $\tilde{v} = (v^i_{(t_1,t_2)})$ are connected with (3.4) and let the $\mathscr{A}$- transform of the sequence $\tilde{u} = (u^i_{(t_1,t_2)})$ be $\tilde{z} = (z^i_{(t_1,t_2)})$ and the $\mathscr{B}$- transform of the sequence $\tilde{v} = (v^i_{(t_1,t_2)})$ be $\tilde{t} = (t^i_{(t_1,t_2)})$, i.e.,

$$z^i_{(t_1,t_2)} = (\mathscr{A}u_{(t_1,t_2)})^i = \sum_i a_{ni}u^i_{(t_1,t_2)}, \quad (i \in \mathbb{N}) \tag{5.1}$$

and

$$t^i_{(t_1,t_2)} = (\mathscr{B}v_{(t_1,t_2)})^i = \sum_i b_{ni}v^i_{(t_1,t_2)}, \quad (i \in \mathbb{N}). \tag{5.2}$$

It is clear here that the method $\mathscr{B}$ is applied to the $\mathscr{B}(r,s)$- transform $\tilde{v} = (v^i_{(t_1,t_2)}) = (\mathscr{B}(r,s)u_{(t_1,t_2)})^i$ of the sequence $\tilde{u} = (u^i_{(t_1,t_2)})$ while the method $\mathscr{A}$ is directly applied to the terms of the sequence $\tilde{u} = (u^i_{(t_1,t_2)})$. So, the methods $\mathscr{A}$ and $\mathscr{B}$ are essentially different (see, [5]).

Let us assume the existence of the matrix product $\mathscr{B}\mathscr{B}(r,s)$ which is a much weaker assumption than the conditions on the matrix $\mathscr{B}$ belonging to any matrix class, in general. If $z^i_{(t_1,t_2)}$ becomes $t^i_{(t_1,t_2)}$(*or* $t^i_{(t_1,t_2)}$becomes $z^i_{(t_1,t_2)}$), under the application of the formal summation by parts, then the methods $\mathscr{A}$ and $\mathscr{B}$ in (5.1), (5.2) are called generalized difference dual type matrices. This leads us to the fact that $\mathscr{B}\mathscr{B}(r,s)$ exists and is equal to $\mathscr{A}$ and $\mathscr{A}\tilde{u} = (\mathscr{B}\mathscr{B}(r,s))u^k_{(t_1,t_2)} = \mathscr{B}(\mathscr{B}(r,s)u^k_{(t_1,t_2)}) = \mathscr{B}\tilde{v}$ formally holds. This statement is equivalent to the following relation between the elements of the matrices $\mathscr{A} = (a_{ni})$ and $\mathscr{B} = (b_{ni})$

$$b_{ni} = \sum_{i=1}^{n} \frac{1}{r}\left(\frac{-s}{r}\right)^{n-i} a_{ni} \quad \text{or} \quad a_{ni} = rb_{ni} + sb_{n,i+1} \tag{5.3}$$

for all $n, i \in \mathbb{N}$. Now, we may give the following theorem concerning to the generalized difference dual matrices.

Theorem 5.1. *Let $\mathscr{A} = (a_{ni})$ and $\mathscr{B} = (b_{ni})$ be the generalized difference dual type matrices, $\mu(\mathscr{F})$ be any given sequence space and $(a_{ni})_{k\in\mathbb{N}} \in \ell_1(\mathscr{F})$. Then, $\mathscr{A} \in ([c(\mathscr{F})]_{\mathscr{B}(r,s)} : \mu(\mathscr{F}))$ if and only if $\mathscr{B} \in (c(\mathscr{F}) : \mu(\mathscr{F}))$.*

Proof. Suppose that $\mathscr{A} = (a_{ni})$ and $\mathscr{B} = (b_{ni})$ be generalized difference dual type matrices which means that (5.3) holds. Additionally, let $\mu(\mathscr{F})$ be any given sequence space and take account that the spaces $[c(\mathscr{F})]_{\mathscr{B}(r,s)}$ and $c(\mathscr{F})$ are linearly isomorphic. Let $\mathscr{A} \in ([c(\mathscr{F})]_{\mathscr{B}(r,s)} : \mu(\mathscr{F}))$ and take any $\tilde{y} \in c(\mathscr{F})$. Then, $\mathscr{B}\mathscr{B}(r,s)$ is equal to $\mathscr{A}$ and $(a_{ni})_{i\in\mathbb{N}} \in D_1 \cap D_4 \cap D_5 = [c(\mathscr{F})]^{\beta(r)}_{\mathscr{B}(r,s)}$ which yields that $(b_{ni})_{k\in\mathbb{N}} \in \ell_1(\mathscr{F})$ for each $n \in \mathbb{N}$. Hence $\mathscr{B}\tilde{y}$ exists for each $\tilde{y} \in c(\mathscr{F})$ and we have the following equation

$$\sum_i b_{ni}y^i_{(t_1,t_2)} = \sum_i a_{ni}x^i_{(t_1,t_2)} \quad (n \in \mathbb{N}). \tag{5.4}$$

Subsequently, it is clear from (5.3) that $\mathscr{B}\tilde{y} = \mathscr{A}\tilde{x}$ which leads us to the consequence that $\mathscr{B} \in (c(\mathscr{F}) : \mu(\mathscr{F}))$.

Conversely, suppose that $\mathscr{B} \in (c(\mathscr{F}) : \mu(\mathscr{F}))$ and take any $\tilde{x} \in [c(\mathscr{F})]_{\mathscr{B}(r,s)}$. Then, $\mathscr{A}\tilde{x}$ exists. Therefore, we obtain from the following equality as $n \to \infty$ that $\mathscr{A}\tilde{x} = \mathscr{B}\tilde{y}$ and this shows that $\mathscr{A} \in ([c(\mathscr{F})]_{\mathscr{B}(r,s)} : \mu(\mathscr{F}))$ and

$$\sum_{i=0}^{n} a_{ni} x^i_{(t_1,t_2)} = \sum_{i=0}^{n} \left\{ \sum_{i=0}^{n} \frac{1}{r} \left(\frac{-s}{r} \right)^{n-i} a_{ni} \right\} y^i_{(t_1,t_2)} \ ; \quad (n \in \mathbb{N}).$$

This completes the proof. □

Theorem 5.2. *Suppose that the elements of the infinite matrices* $\mathscr{D} = (d_{ni})$ *and* $\mathscr{E} = (e_{ni})$ *are connected with the relation*

$$e_{ni} = sd_{n-1,i} + rd_{ni}, \quad (n, i \in \mathbb{N}) \tag{5.5}$$

and $\mu(\mathscr{F})$ *be any given sequence space. Then,* $\mathscr{D} \in (\mu(\mathscr{F}) : [c(\mathscr{F})]_{\mathscr{B}(r,s)})$ *if and if only* $\mathscr{E} \in (\mu(\mathscr{F}) : c(\mathscr{F}))$.

Proof. Let $\tilde{x} = (x^i_{(t_1,t_2)}) \in \mu(\mathscr{F})$ and consider the following equality

$$\begin{aligned} \{\mathscr{B}(r,s)(\mathscr{D}x_{(t_1,t_2)})\}^n &= s(\mathscr{D}x^{n-1}_{(t_1,t_2)}) + r(\mathscr{D}x^n_{(t_1,t_2)}) \\ &= s\sum_i d_{n-1,i} x^i_{(t_1,t_2)} + r\sum_i d_{ni} x^i_{(t_1,t_2)} \\ &= \sum_i (sd_{n-1,i} + rd_{ni}) x^i_{(t_1,t_2)} = (\mathscr{E}x_{(t_1,t_2)})^n, \ (n, i \in \mathbb{N}) \end{aligned}$$

which yields that $\mathscr{D}\tilde{x} \in [c(\mathscr{F})]_{\mathscr{B}(r,s)}$ if and only if $\mathscr{E}\tilde{x} \in c(\mathscr{F})$. This step completes the proof. □

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