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Reticulation of 0 - distributive lattices by fuzzy prime spectrum

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Abstract. A congruence relation $\cong$ on a 0-distributive lattice L is defined by using fuzzy prime spectrum such that the quotient lattice $L/\cong$ is a distributive lattice and the fuzzy prime spectrum of L and $L/\cong$ are homeomorphic. It is proved that the lattices of fuzzy filters of L and $L/\cong$ are isomorphic.

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1. Introduction

The notion of a 0-distributive lattice is due to Varlet [19], which generalizes a distributive lattice and a pseudo-complemented lattice. A lattice L with 0 is said to be a 0-distributive lattice if for all $x, y, z \in L$, $x \wedge y = 0 = x \wedge z$ implies $x \wedge (y \vee z) = 0$. This concept has been widely studied by many researchers (see [1, 2, 12, 19]). It can be seen that a large part of the theory of filters in distributive lattices can be extended to 0-distributive lattices. Balsubramani [1] has studied the space of prime filters of a bounded 0-distributive lattice together with the hull-kernel topology.

The reticulation was first introduced by Simmonns [17] for commutative rings and Belluce [3] made this construction for MV-algebras. This concept was extended later on to non-commutative rings [4], BL-algebras [10] and Residuated lattices [11]. Recently Pawar [13] has furnished the reticulation of a 0-distributive lattice by following the method of construction used by Dan [6] for Heyting algebras. The reticulation of an algebra A is a pair $(L(A), \lambda)$ consisting of a bounded distributive lattice $L(A)$ and a surjection $\lambda : A \to L(A)$ such that λ induces a homeomorphism between the prime spectrum of $L(A)$ and that of A. The existence of the reticulation permits to transfer many properties between A and $L(A)$.

Following the introduction of fuzzy sets by Zadeh [20] many researchers have applied this concept to lattices and founded the theory of fuzzy ideals and fuzzy filters in a lattice[5, 9, 18]. We have studied various properties of spectrum of L-fuzzy prime ideals of a distributive lattice in [14]. 0-distributive lattice being a generalization of a distributive lattice we have introduced and studied spectrum of fuzzy prime filters of a 0-distributive lattice in [15]. Motivated by the results in [15] and [13] we construct the reticulation of a 0-distributive lattice by using fuzzy prime spectrum on the lines of [13]. In this paper we define the congruence relation $\cong$ on a 0-distributive lattice L by using basic closed sets of the fuzzy prime spectrum of L. Then we will show that $L/\cong$ is a bounded distributive lattice and the lattice of all fuzzy filters of L and $L/\cong$ are isomorphic.

2. Preliminaries

For basic concepts in lattice theory and fuzzy set theory the reader is referred to [7] and [8] respectively. For topological concepts the reader is referred to [16]. For notions and notation in fuzzy lattices we follow [5] and [15]. Below we give some definitions and results regarding fuzzy lattices that we need to develop the text of this paper.

Let $L = \langle L, \wedge, \vee \rangle$ denotes a lattice. A fuzzy subset of L is a map of L into $\langle [0,1], \wedge, \vee \rangle$, where $\alpha \wedge \beta = \min(\alpha, \beta)$ and $\alpha \vee \beta = \max(\alpha, \beta)$ for all $\alpha, \beta \in [0,1]$. Let μ be a fuzzy subset of L. For $\alpha \in [0,1]$, the set $\mu_\alpha = \{x \in L : \mu(x) \geq \alpha\}$ is called α - cut (or α - level set) of μ. A fuzzy subset μ of L is proper if it is a non constant function. A fuzzy subset μ of L is said to be a fuzzy sublattice of L if for all $x, y \in L$, $\mu(x \wedge y) \wedge \mu(x \vee y) \geq \mu(x) \wedge \mu(y)$. A monotonic (antimonotonic) fuzzy sublattice is a fuzzy filter (ideal) of L. Here μ is monotonic (antimonotonic) means $\mu(x) \leq \mu(y)$ ($\mu(x) \geq \mu(y)$) whenever $x \leq y$ in L. A proper fuzzy filter (ideal) μ of L is called a fuzzy prime filter (ideal) if $\mu(x \vee y) \leq \mu(x) \vee \mu(y)$ ($\mu(x \wedge y) \leq \mu(x) \vee \mu(y)$) holds for all $x, y \in L$.

Let Σ denote the set of fuzzy prime filters of a 0-distributive lattice L. We assume that for each $\mu \in \Sigma$, $\mu(1) = 1$. For a fuzzy subset σ of L define $V(\sigma) = \{\mu \in \Sigma \mid \sigma \subseteq \mu\}$. Let $a \in L$. We will denote $V\left(\chi_{\{a\}}\right)$ by simply $V(a)$. Thus $V(a) = \{\mu \in \Sigma : \mu(a) = 1\}$. The family $\mathfrak{B} = \{X(a) \mid a \in L\}$, where $X(a) = X\left(\chi_{\{a\}}\right) = \Sigma \setminus V(a)$, constitutes a base for the open sets of some topology on Σ. Let τ denote the topology with the base $\mathfrak{B}$ on Σ. The topological space $\langle \Sigma, \tau \rangle$ is called fuzzy prime spectrum of L and is denoted by $Fspec(L)$.

Proposition 2.1 ([5]). *A fuzzy subset μ of L is a fuzzy filter (ideal) of L if and only if $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ ($\mu(x \vee y) = \mu(x) \wedge \mu(y)$) holds for all $x, y \in L$.*

Proposition 2.2 ([5]). *A fuzzy subset μ of L is a fuzzy prime filter (ideal) of L if and only if*

$$\mu(x \wedge y) = \mu(x) \wedge \mu(y) \text{ and } \mu(x \vee y) = \mu(x) \vee \mu(y), \text{ for all } x,\ y \in L$$
$$(\mu(x \wedge y) = \mu(x) \vee \mu(y) \text{ and } \mu(x \vee y) = \mu(x) \wedge \mu(y), \text{ for all } x,\ y \in L().$$

Proposition 2.3 ([5]). *Let L and L' be two lattices and $f : L \to L'$ an onto homomorphism. Then*

(1) *If μ is a fuzzy sublattice (ideal, filter) of L then $f(\mu)$ is a fuzzy sublattice (ideal, filter) of L' where $f(\mu)$ is defined as*

$$f(\mu)(y) = sup\{\mu(x) : f(x) = y, x \in L\}, \text{ for all } y \in L';$$

(2) *If ν is a fuzzy sublattice (ideal, prime ideal, filter, prime filter) of L' then $f^{-1}(\nu)$ is a fuzzy sublattice (ideal, prime ideal, filter, prime filter) of L where $f^{-1}(\nu)$ is defined as*

$$f^{-1}(\nu)(x) = \nu(f(x)), \text{ for all } x \in L.$$

Proposition 2.4 ([9] Dual of Corollary 2.16)**.** *A proper subset F of L is a prime filter of L if and only if the characteristic function of F is a fuzzy prime filter of L.*

Proposition 2.5 ([9])**.** *A non-constant fuzzy filter μ of L is a fuzzy prime filter of L if and only if for all $\alpha \in [0, 1]$, if μ_α is a proper filter of L, then μ_α is a prime filter of L.*

Proposition 2.6 ([15])**.** *Let θ and σ be fuzzy subsets of a 0-distributive lattice L. Then*

(1) *If $\theta \subseteq \sigma$, then $V(\sigma) \subseteq V(\theta)$.*
(2) *$V(\sigma) \cup V(\theta) \subseteq V(\sigma \cap \theta)$.*
(3) *$V(\theta) = V(\langle\theta\rangle)$, where $\langle\theta\rangle$ is the fuzzy filter of L generated by θ.*
(4) *$V(0) = \varnothing$ and $V(1) = \Sigma$.*

Proposition 2.7 ([15])**.** *If $\{\theta_i \mid i \in \Lambda\}$ (Λ is any indexing set) is a family of fuzzy subsets of L, then $V(\bigcup\{\theta_i \mid i \in \Lambda\}) = \bigcap\{V(\theta_i) \mid i \in \Lambda\}$. Moreover, $V(a) \cup V(b) = V(a \vee b), \forall\, a, b \in L$.*

Proposition 2.8 ([7])**.** *Let L and L' be two lattices. Let $f : L \to L'$ be a bijection. If f and f^{-1} are both isotone maps (i.e. order preserving maps), then f is a lattice homomorphism (and hence an isomorphism).*

For any bounded lattice L, $\mathcal{F}(L)$ denotes the set of all fuzzy filters of L which is a complete bounded lattice (see [9]). Now onwards L will denote a bounded 0-distributive lattice.

3. Reticulation of a 0-distributive lattice

Define a relation $\cong$ on L by, for all $a, b \in L$,

$$a \cong b \text{ if and only if } V(a) = V(b).$$

Hence $a \cong b \Leftrightarrow (\mu(a) = 1 \Leftrightarrow \mu(b) = 1, \forall \mu \in \Sigma)$.

It can be easily seen that $\cong$ is an equivalence relation on L. Moreover we have,

Theorem 3.1. *The relation $\cong$ is a congruence relation on L.*

Proof. Suppose that $a \cong b$ and $c \cong d$. First, we prove that $a \wedge c \cong b \wedge d$. Let $\mu \in V(a \wedge c)$. Then $\mu(a \wedge c) = 1$. Since μ is a fuzzy filter, we get $\mu(a) \wedge \mu(c) = 1$, i.e., $\mu(a) = 1$ and $\mu(c) = 1$. By assumption, $\mu(b) = 1$ and $\mu(d) = 1$ and thus $\mu(b \wedge d) = \mu(b) \wedge \mu(d) = 1$ and consequently, $\mu \in V(b \wedge d)$. So $V(a \wedge c) \subseteq V(b \wedge d)$.

Similarly, we can prove that $V(b \wedge d) \subseteq V(a \wedge c)$. Hence $a \wedge c \cong b \wedge d$.

Now we prove that $a \vee c \cong b \vee d$. Let $\mu \in V(a \vee c)$. Then $\mu(a \vee c) = 1$. As μ is a fuzzy prime filter, we get $\mu(a) \vee \mu(c) = 1$, i.e., $\mu(a) = 1$ or $\mu(c) = 1$. By assumption, $\mu(b) = 1$ or $\mu(d) = 1$. Thus $\mu(b \vee d) = \mu(b) \vee \mu(d) = 1$. So $\mu \in V(b \vee d)$ proving that

$V(a \vee c) \subseteq V(b \vee d)$. Similarly we can prove that $V(b \vee d) \subseteq V(a \vee c)$. Hence $a \vee c \cong b \vee d$. This proves that the relation $\cong$ is a congruence relation on L. □

For any $a \in L$, the equivalence class of a with respect to the congruence relation $\cong$ is denoted by $\bar{a}$ that is $\bar{a} = \{x \in L : x \cong a\}$. The set of all such distinct equivalence classes is denoted by $L/\cong$. We define $\sqcap$ and $\sqcup$ on $L/\cong$ as

$$\bar{a} \sqcap \bar{b} = \overline{a \wedge b} \text{ and } \bar{a} \sqcup \bar{b} = \overline{a \vee b}.$$

By Theorem 3.1, these operations are well defined on $L/\cong$ and we have,

Theorem 3.2. *The algebra $\langle L/\cong, \sqcap, \sqcup, \bar{0}, \bar{1}\rangle$ is a bounded lattice.*

Proof. It is easy to prove that the operations $\sqcap$ and $\sqcup$ satisfies the idempotent, absorption, associative and commutative properties in $L/\cong$. For any $\bar{x} \in L/\cong$, $\bar{x} \sqcap \bar{0} = \overline{x \wedge 0} = \bar{0}$. Then $\bar{0} \leq \bar{x}$ in $L/\cong$ for all $\bar{x} \in L/\cong$.

Similarly, $\bar{x} \leq \bar{1}$ hold for all $\bar{x} \in L/\cong$. Thus the algebra $\langle L/\cong, \sqcap, \sqcup, \bar{0}, \bar{1}\rangle$ is a bounded lattice. □

Lemma 3.3. *For $a, b \in L$*

(1) *$\bar{a} \leq \bar{b}$ in $L/\cong$ if and only if $V(a) \subseteq V(b)$.*
(2) *If $a \leq b$ in L then $\bar{a} \leq \bar{b}$ in $L/\cong$.*

Proof. (1) By definition of $\sqcup$, we have $\bar{a} \leq \bar{b}$ in $L/\cong$ if and only if $\bar{a} \sqcup \bar{b} = \bar{b}$, i.e., $\overline{a \vee b} = \bar{b}$. Now by Proposition 2.7,

$$\begin{aligned}\overline{a \vee b} = \bar{b} &\Leftrightarrow a \vee b \cong b\\ &\Leftrightarrow V(a \vee b) = V(b)\\ &\Leftrightarrow V(a) \cup V(b) = V(b)\\ &\Leftrightarrow V(a) \subseteq V(b).\end{aligned}$$

Thus $\bar{a} \leq \bar{b}$ in $L/\cong$ if and only if $V(a) \subseteq V(b)$.

(2) If $a \leq b$ in L, then $a \wedge b = a$. Thus $\overline{a \wedge b} = \bar{a}$, that is, $\bar{a} \sqcap \bar{b} = \bar{a}$. This yields $\bar{a} \leq \bar{b}$ in $L/\cong$. □

Theorem 3.4. *The lattice $\langle L/\cong, \sqcap, \sqcup, \bar{0}, \bar{1}\rangle$ is distributive.*

Proof. Let $a, b, c \in L$. Then $\bar{a} \sqcap (\bar{b} \sqcup \bar{c}) = \bar{a} \sqcap (\overline{b \vee c}) = \overline{a \wedge (b \vee c)}$.

Similarly, $(\bar{a} \sqcap \bar{b}) \sqcup (\bar{a} \sqcap \bar{c}) = \overline{(a \wedge b) \vee (a \wedge c)}$.

Let $x \in \overline{a \wedge (b \vee c)}$. Then $x \cong a \wedge (b \vee c)$. Thus, for any fuzzy prime filter μ, we have

$$\begin{aligned}\mu(x) = 1 &\Leftrightarrow \mu(a \wedge (b \vee c)) = 1\\ &\Leftrightarrow \mu(a) \wedge \mu(b \vee c) = 1 \quad \text{(By Proposition 2.1)}\\ &\Leftrightarrow \mu(a) = 1 \text{ and } \mu(b) \vee \mu(c) = 1 \quad \text{(By Proposition 2.2)}\\ &\Leftrightarrow \mu(a) = 1 \text{ and } (\mu(b) = 1 \text{ or } \mu(c) = 1)\\ &\Leftrightarrow \mu(a) \wedge \mu(b) = 1 \text{ or } \mu(a) \wedge \mu(c) = 1\\ &\Leftrightarrow \mu(a \wedge b) \vee \mu(a \wedge c) = 1\\ &\Leftrightarrow \mu((a \wedge b) \vee (a \wedge c)) = 1.\end{aligned}$$

So $x \cong (a \wedge b) \vee (a \wedge c)$ resulting into $x \in \overline{(a \wedge b) \vee (a \wedge c)}$. This gives

$$\overline{a \wedge (b \vee c)} \subseteq \overline{(a \wedge b) \vee (a \wedge c)}.$$

Hence $\overline{a} \sqcap (\overline{b} \sqcup \overline{c}) \subseteq (\overline{a} \sqcap \overline{b}) \sqcup (\overline{a} \sqcap \overline{c})$.

Similarly, we can prove $(\overline{a} \sqcap \overline{b}) \sqcup (\overline{a} \sqcap \overline{c}) \subseteq \overline{a} \sqcap (\overline{b} \sqcup \overline{c})$. Combining both the inclusions, we get, $\overline{a} \sqcap (\overline{b} \sqcup \overline{c}) = (\overline{a} \sqcap \overline{b}) \sqcup (\overline{a} \sqcap \overline{c})$. Therefore the lattice $L/\cong$ is a distributive lattice. □

Definition 3.5. Let $\lambda : L \to L/\cong$ be the canonical surjection defined by, $\lambda(a) = \bar{a}$.

Now we prove that $(L/\cong, \lambda)$ is a reticulation of L. Before that we furnish some necessary results.

Lemma 3.6. *Let L_1 and L_2 be 0-distributive lattices and $f : L_1 \to L_2$ be a homomorphism. Then $V(a) = V(b)$ imply $V(f(a)) = V(f(b))$, for all $a, b \in L_1$.*

Proof. Let $\mu \in V(f(a))$. Then $\mu \in Fspec(L_2)$ and $\mu(f(a)) = 1$. By Proposition 2.3, $f^{-1}(\mu) \in Fspec(L_1)$ and $f^{-1}(\mu)(a) = \mu(f(a)) = 1$. Thus $f^{-1}(\mu) \in V(a) = V(b)$. So $\mu(f(b)) = f^{-1}(\mu)(b) = 1$. Hence $\mu \in V(f(b))$ proving that $V(f(a)) \subseteq V(f(b))$.

Similarly, we can show that $V(f(b)) \subseteq V(f(a))$. Therefore the lemma follows. □

Let L_1 and L_2 be 0-distributive lattices and $f : L_1 \to L_2$ be a {0,1} - homomorphism i.e., $f(0) = 0$, $f(1) = 1$. Define $f^* : L_1/\cong \to L_2/\cong$ by $f^*(\bar{a}) = \overline{f(a)}$, for all $\bar{a} \in L_1/\cong$. Then we have,

Theorem 3.7. *f^* is a {0,1} - homomorphism.*

Proof. Let $\bar{a} = \bar{b}$. Then $a \cong b$. Thus $V(a) = V(b)$. By Lemma 3.6, we get $V(f(a)) = V(f(b))$. So $f(a) \cong f(b)$ imply $\overline{f(a)} = \overline{f(b)}$, that is, $f^*(\bar{a}) = f^*(\bar{b})$. This proves f^* is well defined.

Now, as f is a homomorphism, we have

$$f^*(\bar{a} \sqcap \bar{b}) = f^*(\overline{a \wedge b}) = \overline{f(a \wedge b)} = \overline{f(a) \wedge f(b)} = \overline{f(a)} \sqcap \overline{f(b)} = f^*(\bar{a}) \sqcap f^*(\bar{b}).$$

Similarly, we can show that $f^*(\bar{a} \sqcup \bar{b}) = f^*(\bar{a}) \sqcup f^*(\bar{b})$. Also $f^*(\bar{0}) = \overline{f(0)} = \bar{0}$ and $f^*(\bar{1}) = \overline{f(1)} = \bar{1}$. This completes the proof. □

Lemma 3.8. *Let μ be a fuzzy prime filter of L and $a, b \in L$ such that $\overline{a} = \overline{b}$, then $\mu(a) = \mu(b)$.*

Proof. Since $\overline{a} = \overline{b}$, $a \cong b$. Then $V(a) = V(b)$. Suppose $\mu(a) \neq \mu(b)$. Then either $\mu(a) < \mu(b)$ or $\mu(b) < \mu(a)$. Without loss of generality, we can assume $\mu(a) < \mu(b)$. Take $\alpha = \mu(b)$ and $F = \mu_\alpha = \{x \in L : \mu(x) \geq \alpha\}$. Then F is a prime filter of L (by Proposition 2.5) such that $b \in F$ and $a \notin F$. By Proposition 2.4, χ_F is a fuzzy prime filter of L such that $\chi_F(b) = 1$ but $\chi_F(a) \neq 1$. Thus there exists $\chi_F \in V(b)$ but $\chi_F \notin V(a)$. This contradicts to the fact that $V(a) = V(b)$. So we must have $\mu(a) = \mu(b)$. □

Theorem 3.9. *Let μ be a fuzzy filter of L, then $\lambda(\mu)$ is fuzzy filter of $L/\cong$.*

Proof. In view of Proposition 2.3 and as λ is an surjective homomorphism, $\lambda(\mu)$ is a fuzzy filter of $L/\cong$. □

Theorem 3.10. *Let ν be a fuzzy filter of $L/\cong$, then $\lambda^{-1}(\nu)$ is fuzzy filter of L and $\lambda(\lambda^{-1}(\nu)) = \nu$.*

Proof. By Proposition 2.3, $\lambda^{-1}(\nu)$ is a fuzzy filter of L. For any $\overline{x} \in L/\cong$,
$$\begin{aligned}&\lambda(\lambda^{-1}(\nu))(\overline{x})\\ &= sup\left\{\lambda^{-1}(\nu)(y) : \lambda(y) = \overline{x}\right\}\\ &= sup\left\{\nu(\lambda(y)) : \overline{y} = \overline{x}\right\}\\ &= sup\left\{\nu(\overline{y}) : \overline{y} = \overline{x}\right\} = \nu(\overline{x}).\end{aligned}$$
This proves that $\lambda(\lambda^{-1}(\nu)) = \nu$. □

Theorem 3.11. *Let μ and ν be a fuzzy filters of L then $\mu \leq \nu$ if and only if $\lambda(\mu) \leq \lambda(\nu)$.*

Proof. First suppose $\mu \leq \nu$. Then for any $\overline{x} \in L/\cong$,
$$\lambda(\mu)(\overline{x}) = \sup\{\mu(y) : \lambda(y) = \overline{x}\} \leq \sup\{\nu(y) : \lambda(y) = \overline{x}\} = \lambda(\nu)(\overline{x}).$$
Thus $\lambda(\mu) \leq \lambda(\nu)$.

Conversely, suppose $\lambda(\mu) \leq \lambda(\nu)$. On the contrary assume that $\mu > \nu$. Then for any $a \in L$, $\sup\{\mu(z) : \lambda(z) = \overline{a}\} > \sup\{\nu(z) : \lambda(z) = \overline{a}\}$. But it means $\lambda(\mu)(\overline{a}) > \lambda(\nu)(\overline{a})$, which contradicts to our hypothesis. So we must have $\mu \leq \nu$. □

Corollary 3.12. *Let μ and ν be a fuzzy filters of L, then $\mu = \nu$ if and only if $\lambda(\mu) = \lambda(\nu)$.*

This enables us to prove the following theorem for the lattices of all fuzzy filters of L and $L/\cong$.

Theorem 3.13. *The mapping $\psi : \mathcal{F}(L) \to \mathcal{F}(L/\cong)$ induced by λ and defined by*
$$\psi(\mu) = \lambda(\mu), \quad \forall\ \mu \in \mathcal{F}(L),$$
is an isomorphism which preserves fuzzy prime filters and $\lambda^{-1}(\lambda(\mu)) = \mu$.

Proof. By Corollary 3.12, the map ψ is well defined and injective. By Theorem 3.10, ψ is surjective. By Proposition 2.8 and Theorem 3.11, ψ is a homomorphism. Then ψ is an isomorphism.

Let μ be a fuzzy prime filter of L. So μ is a non constant, in particular $\mu(0) \neq \mu(1)$. By Lemma 3.3, it is clear that $\lambda(\mu)(\overline{0}) \neq \lambda(\mu)(\overline{1})$. Thus, by Theorem 3.9, $\lambda(\mu)$ is a proper fuzzy filter of $L/\cong$.

To prove it is prime, let $\overline{a},\ \overline{b} \in L/\cong$. If $z \in L$ such that $\overline{z} = \overline{a \vee b}$, then $z \cong a \vee b$, that is, $V(z) = V(a \vee b)$. Thus for any fuzzy prime filter θ of L,
$$\begin{aligned}\theta(z) = 1 &\Leftrightarrow \theta(a \vee b) = 1\\ &\Leftrightarrow \theta(a) \vee \theta(b) = 1\\ &\Leftrightarrow \theta(a) = 1 \text{ or } \theta(b) = 1.\end{aligned}$$
So $V(z) = V(a)$ or $V(z) = V(b)$ and consequently, $\overline{z} = \overline{a}$ or $\overline{z} = \overline{b}$. Hence,
$$\begin{aligned}\lambda(\mu)(\overline{a} \sqcup \overline{b}) &= \lambda(\mu)(\overline{a \vee b})\\ &= \mu(a \vee b) \quad \text{(by Lemma 3.8)}\\ &= \mu(a) \vee \mu(b)\\ &= \lambda(\mu)(\overline{a}) \vee \lambda(\mu)(\overline{b}) \quad \text{(by Lemma 3.8)}\end{aligned}$$

This proves $\lambda(\mu)$ is a fuzzy prime filter of $L/\cong$.

Now let $x \in L$. Then, by Lemma 3.8,

$$\begin{aligned}\lambda^{-1}(\lambda(\mu))(x) &= \lambda(\mu)(\lambda(x)) = \lambda(\mu)(\overline{x}) \\ &= sup\{\mu(z) : \lambda(z) = \overline{z} = \overline{x}\} = \mu(x).\end{aligned}$$

Hence $\lambda^{-1}(\lambda(\mu)) = \mu$. □

Theorem 3.14. *Let ν be a fuzzy prime filter of $L/\cong$, then $\lambda^{-1}(\nu)$ is fuzzy prime filter of L.*

Proof. Let ν be a fuzzy prime filter of $L/\cong$. By Theorem 3.10, $\lambda^{-1}(\nu)$ is fuzzy filter of L. Also ν is non constant so that $\nu(\overline{0}) \neq \nu(\overline{1})$. Then $\lambda^{-1}(\nu)(0) = \nu(\lambda(0)) = \nu(\overline{0})$ and $\lambda^{-1}(\nu)(1) = \nu(\lambda(1)) = \nu(\overline{1})$ gives $\lambda^{-1}(\nu)(\overline{0}) \neq \lambda^{-1}(\nu)(\overline{1})$, proving that $\lambda^{-1}(\nu)$ is non constant and hence proper fuzzy filter of L. Now using the fact that ν is a fuzzy prime filter and using the Proposition 2.2, we get,

$$\begin{aligned}\lambda^{-1}(\nu)(a \vee b) &= \nu(\lambda(a \vee b)) = \nu(\overline{a \vee b}) \\ &= \nu(\overline{a} \sqcup \overline{b}) = \nu(\overline{a}) \vee \nu(\overline{b}) \\ &= \nu(\lambda(a)) \vee \nu(\lambda(b)) = \lambda^{-1}(\nu)(a) \vee \lambda^{-1}(\nu)(b).\end{aligned}$$

This proves $\lambda^{-1}(\nu)$ is a fuzzy prime filter of L. □

Theorem 3.15. *The two topological spaces $Fspec(L)$ and $Fspec(L/\cong)$ are homeomorphic.*

Proof. Define a map $g : Fspec(L) \to Fspec(L/\cong)$ by

$$g(\mu) = \lambda(\mu), \quad \text{for all } \mu \in Fspec(L).$$

Then g is a restriction of the map ψ in Theorem 3.13 to $Fspec(L)$. By Theorem 3.13, g is well defined and injective. Also for any $\nu \in Fspec(L/\cong)$, we have, by Theorem 3.14,

$$\lambda^{-1}(\nu) \in Fspec(L)$$

and by Theorem 3.10,

$$g(\lambda^{-1}(\nu)) = \lambda(\lambda^{-1}(\nu)) = \nu.$$

Thus g is surjective. So g is a bijection.
We have $V(\overline{a})$ is a basic closed set in $Fspec(L/\cong)$ and

$$\begin{aligned}g^{-1}(V(\overline{a})) &= \{\mu \in Fspec(L) : g(\mu) \in V(\overline{a})\} \\ &= \{\mu \in Fspec(L) : \lambda(\mu) \in V(\overline{a})\} \\ &= \{\mu \in Fspec(L) : \lambda(\mu)(\overline{a}) = 1\} \\ &= \{\mu \in Fspec(L) : \mu(a) = 1\}.\end{aligned}$$

Here $\lambda(\mu)(\overline{a}) = \sup\{\mu(x) : \lambda(x) = \overline{x} = \overline{a}\} = \mu(a)$ (by Lemma 3.8). Hence we get, $g^{-1}(V(\overline{a})) = V(a)$, which is a basic closed set in $Fspec(L)$. Therefore g is continuous.
Also

$$\begin{aligned}g(V(a)) &= \{g(\mu) : \mu \in V(a)\} \\ &= \{\lambda(\mu) : \mu \in V(a)\} \\ &= \{\nu = \lambda(\mu) \in Fspec(L/\cong) : \mu(a) = 1\} \\ &= \{\nu \in Fspec(L/\cong) : \nu = \lambda(\mu),\ \mu(a) = 1\}.\end{aligned}$$

Again if $\nu = \lambda(\mu)$ and $\mu(a) = 1$, then we have $\nu(\overline{a}) = \lambda(\mu)(\overline{a}) = \mu(a) = 1$. This gives, $g(V(a)) = \{\nu \in Fspec(L/\cong) : \nu(\overline{a}) = 1\} = V(\overline{a})$ which is a basic closed set in $Fspec(L/\cong)$. This proves g is a closed map.

Since g is bijective, continuous and closed map, g is a homeomorphism. □

Combining all the results, we get,

Theorem 3.16. *The pair $(L/\cong, \lambda)$ forms a reticulation of a bounded 0-distributive lattice L.*

4. Conclusions

Algebraic reticulation is obtained for various algebras viz commutative rings, MV-algebras, BL-algebras and Residuated lattices. Here a generalization of the concept of algebraic reticulation for a 0 - distributive lattice using fuzzy theory is obtained successfully. This accentuates applicability of fuzzy theory in algebra.

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References

[1] P. Balasubramani, Stone topology of the set of prime filters of a 0-distributive lattice, Indian J. Pure Appl. Math. 35 (2) (2004) 149–158.
[2] P. Balasubramani and P. V. Venkatanarsimhan, Characterizations of the 0-distributive lattice, Indian J. Pure and Appl. Math. 32 (3) (2001) 315–324.
[3] L. P. Belluce, Semisimple algebras of Infinite valued logic and Bold fuzzy set theory, Can. J. Math. 38 (6) (1986) 1356–1379.
[4] L. P. Belluce, Spectral spaces and Non-commutative rings, Comm. Algebra 19 (1991) 1855–1865.
[5] Bo Yuan and Wangming Wu, Fuzzy ideals on a distributive lattice, Fuzzy Sets and Systems 35 (1990) 231–240.
[6] C. T. Dan, Reticulation in Heyting algebra, Annals of University of Craiova, Math. Comp. Sci. Ser. 30 (2) (2003) 66–70.
[7] G. Grätzer, Lattice Theory - First concepts and Distributive Lattices, Freeman and Company, San Francisco 1971.
[8] J. Klir George and Bo Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall of India Pvt. Ltd., New Delhi 1997.
[9] B. B. N. Koguep, C. Nkuimi C. and C. Lele , On Fuzzy prime ideals of lattice, SJPAM 3 (2008) 1–11.
[10] L. Leustean L., The Prime and Maximal spectra and the Reticulation of BL-algebras, Central European Journal of Mathematics 1 (3) (2003) 382–397.
[11] C. Muresan, The reticulation of a residuated lattices, Bull. Math. Soc. Sci. Math. Roumania, Tome 51 (99) No.1 (2008) 47–65.
[12] Y. S. Pawar, 0-1 distributive lattices, Indian J. Pure Appl. Math. 24 (1993) 173–179.
[13] Y. S. Pawar, Reticulation of a 0 distributive lattice, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica 54 (1) (2015) 121–128.
[14] Y. S. Pawar and S. S Khopade, Spectrum of L - fuzzy prime ideals of a distributive lattice, Fuzzy Systems and Mathematics 27 (1) (2013) 12–19.
[15] Y. S. Pawar and S. S Khopade, Spectrum of fuzzy prime filters of a 0 - distributive lattice, Malaya Journal of Matematik 3 (4) (2015) 591–597.
[16] W. J. Perwin, Foundations of General topology, Academic Press, 1904.
[17] H. Simmons, Reticulated Rings, J. Algebra 66 (1980) 169–192.

[18] Swamy U. M. and Raju D. Viswanadha, Fuzzy ideals and congruence of lattices, Fuzzy Sets and Systems 95 (1998) 249–253.

[19] J. Varlet, A generalization of the notion of pseudo-complementedness, Bull. Soc. Liege. 37 (1968) 149–158.

[20] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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