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Contact relation on De Morgan algebra

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Abstract. In the paper, the notion of a contact relation on de Morgan algebra is defined, the algebraic structure of all contact relations and the connection to the set of all close, reflexive and symmetric relations on all maximal filters are investigated.

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1. Introduction

Contact relations have been studied on two different contexts: the proximity relations and the theory of pointless geometry(topology) since the early 1920's. Recently, it has become a powerful tool in several areas of artificial intelligence, such as qualitative spatial reasoning and ontology building (See [1, 10, 11, 12, 18]). In [5], we generalized the notion of a contact relation in fuzzy setting, and defined a way below relation in continuous lattice.

On the other hand, the notion of a de Morgan algebra of topology was introduced in [9], which is a generalization of algebra of topology [17], and unifies the classical point-set topology [16], I-topology [2].

In this paper, we want to investigate the theory of contact relation on a de Morgan algebra. First, Section 2 surveys an overview of contact relation, De Morgan algebra of topology. Then, Section 3 recalls the notions of a filter, a maximal filter on De Morgan algebra. Section 4 generalizes the notion of a contact relation on De Morgan algebra. Section 5 establishes the order preserving correspondence between the set of all contact relations on (L, Q), and the set of all close, reflexive, symmetric relations on $\mathrm{Max}(L)$. Section 6 focuses on the algebraic structure of all contact relations.

2. **Preliminaries**

Let us recall some main notions for each area, i.e., contact relations [10, 11, 12], De Morgan topological algebra[7, 8, 9].

2.1. **Contact relations.** We assume familiarity with the notions of Boolean algebra and lattice [14]. Suppose $(B, +, \cdot, *, 0, 1)$ is a Boolean algebra , $R \subseteq B \times B$ is called a binary relation on B, and all relations are denoted by $\mathrm{Rel}(B)$.

Definition 2.1. Suppose $C \in \mathrm{Rel}(B)$, and consider the following properties : For all $x, y, z \in B$,
(C_0) $0(-C)x$, i.e., for $x \neq 0$, 0 and x are not C-related.
(C_1) xCx, for $x \neq 0$.
(C_2) $xCy \Rightarrow yCx$.
(C_3) xCy and $y \leq z \Rightarrow xCz$.
(C_4) $xC(y+z) \Rightarrow xCy$ or xCz.
C is called a contact relation, and (B, C) is called a Boolean contact algebra, if C satisfies $C_0 - C_4$.

For further information on contact relations, Boolean contact algebras (See [1, 10, 11, 12, 18]).

2.2. **De Morgan topological algebra.** De Morgan algebra of topology was initiated by Deng in [9]. Suppose $(L, \leq, ')$ is a completely distributivity complete de Morgan algebra([7, 8]), for $a, b \in L$, the complete way-below relation $\prec$ was defined.

Definition 2.2. $a \prec b$, if for every $A \subseteq L$, $\bigvee A = b$, then there exists $c \in A$, such that $a \leq c$.

Remark 2.3. The notion of a way below relation is important in the study of the theory of continuous lattices, and generalized continuous lattices, please refer to [6, 14, 15, 19].

Definition 2.4. Let $\triangle(L) = \{\bigwedge_{a \prec b} b \mid a \neq 0, \bigvee_{a \not\leq c} c < 1\}$, $p \in \triangle(L)$ is called a quasi-atom, in short q-atom.

For $p \in \triangle(L)$, $p^{\sim} = \bigwedge\{q \mid q \in \triangle(L), q \not\prec p^{'}\}$. It is easy to prove $p^{\sim} \in \triangle(L)$ and if $p^{\sim\,\sim} = p$, then $p^{\sim}$ is also a q-atom.
$Q \subseteq \triangle(L)$ is called a p-base for L, if
(i) $p \in Q \Rightarrow p^{\sim} \in Q$,
(ii) for every $a \in L$, $a = \bigvee_{p \prec a} p$.
Then (L, Q) is said to be a de Morgan algebra with p-base Q.

Definition 2.5. $T \subseteq L$ is called a topology on (L, Q), if
(i) T is a $\bigwedge$-set for Q, i.e., for $a, b \in T$, $p \prec a$, and $p \prec b$ implies $p \prec a \bigwedge b$,
(ii) T is closed under finite infimum and arbitrary suprema.

Suppose (L, Q, T) is a de Morgan algebra of topology, $p \in Q$, $a \in T$, a is an open neighborhood of p, if $p \prec a$.
Suppose (L_1, Q_1, T_1), (L_2, Q_2, T_2) are two de Morgan topological algebras, the product topological algebra $(L(Q_1 \otimes Q_2), \Phi(Q_1 \otimes Q_2), T_1 \otimes T_2)$ was defined, where $a \in L_1, b \in L_2, a \otimes b = \{(p, q)_{-} \mid p \in Q_1, p \prec a; q \in Q_2, q \prec b\}$.

Example 2.6. Suppose X is a classical point set, $L = 2^X$ the power set of X, the set of all q-atoms $\triangle(L) = X$. (L, Q, T) is a point-set topology [16].

Example 2.7. When L is a Boolean lattice, $a \vee a^{'} = 1$, $p = p^{\sim}$, the notion of a q-atom coincides with the notion of a atom in a Boolean lattice. (L, Q, T) is a topology on a Boolean lattice [17].

Example 2.8. Suppose X is a universe set, $A : X \to [0, 1]$ is a fuzzy set, then $L = [0, 1]^X$ of all fuzzy sets, $A \vee A^{'} \neq 1_X, A \in L$. Where $1_X : X \to [0, 1]$, for every $x \in X$, $1_X(x) = 1$.

For every $a \in X$, $r \in [0, 1]$, we defined a fuzzy point

$$x_a^r = \begin{cases} r & \text{if} \quad x = a \\ 0 & \text{if} \quad x \neq a \end{cases} \quad \forall x \in X.$$

Let $p = x_a^r$, then $p^{\sim} = x_a^{1-r}$. The notion of a q-atom coincides with the notion of a fuzzy point. (L, Q, T) is a I-topology [2].

For further information on de Morgan algebra of topology (See [3, 4, 7, 8, 9]).

3. Filters on (L, Q)

In the section, we introduce the notion of a filter on (L, Q), a topology on the set Max(L) of all maximal filters, and the collection of all reflexive, symmetric, close relations on Max(L).

First, we introduce the notion of a filter on (L, Q).

Definition 3.1. Suppose (L, Q) is a de Morgan algebra, $\eta \subseteq L$ is called a filter if
(i) $a \in \eta \Rightarrow a^{'} \notin \eta$,
(ii) $a, b \in \eta \Rightarrow a \wedge b \in \eta$,
(iii) $a \leq b, a \in \eta \Rightarrow b \in \eta$.

We present two examples of filters on (L, Q).

Example 3.2. $1 \in L$, clearly $\{1\}$ is a filter on (L, Q).

Example 3.3. Suppose $p \in Q$, then $\eta_p = \{a \mid p \prec a, a \in L\}$ is a filter on (L, Q).

Clearly η_p satisfies Definition 3.1 (2) and (3). For $a \in \eta_p$, we have $p \prec a$; if $a^{'} \in \eta_p$, so $p \prec a^{'}$, thus $p \prec a \wedge a^{'} = 0$, a contradiction, Definition 3.1 (1) holds.

When X is a universal set, $L = 2^X$, then $\eta_p = \{A \mid A \subseteq X, p \in A\}$ for every $p \in Q = X$.

Suppose η, ν are two filters on (L, Q), we define $\eta \subseteq \nu$, if for every $a \in \eta$, we have $a \in \nu$. A filter η is called a maximal filter, if for any filter ν satisfying : $\eta \subseteq \nu$, we have $\eta = \nu$. Obviously, a filter η is a maximal filter if and only if for every $a \in L$, we have $a \in \eta$, or $a^{'} \in \eta$ holds and only one holds. We use the symbol Max(L) to denote all maximal filters on (L, Q). Then we obtain, par

Lemma 3.4. *Suppose $a \in L$, $a \neq 0$, there exists a maximal filter μ_a, such that $a \in \mu_a$.*

Second, the notion of a topology is defined on Max(L). For $\eta_i \in$Max(L), $\eta_1 \vee \eta_2$, $\eta_1 \wedge \eta_2$, $\bigvee_{i \in I} \eta_i$ are defined :

$$\eta_1 \vee \eta_2 = \{a \vee b \mid a \in \eta_1, b \in \eta_2\},$$

$$\eta_1 \wedge \eta_2 = \{a \wedge b \mid a \in \eta_1, b \in \eta_2\},$$
$$\bigvee_{i \in I} \eta_i = \{\bigvee_{i \in I} a_i \mid a_i \in \eta_i\}.$$

Definition 3.5. $T \subseteq \mathrm{Max}(L)$ is called a topology on $\mathrm{Max}(L)$, if it is closed with respect to the infinite join and finite meet, i.e., if $\eta_i \in \mathrm{Max}(L)$, then $\bigvee_{i \in I} \eta_i \in \mathrm{Max}(L)$, and $\bigwedge_{i=1}^{m} \eta_i \in \mathrm{Max}(L)$.

We consider the product topology $T \otimes T$ on $\mathrm{Max}(L) \times \mathrm{Max}(L)$. Let $h(a) = \{\eta \mid a \in \eta\}, h(b) = \{\mu \mid b \in \mu\}$, then $\{h(a) \times h(b) \mid a, b \in L\} = \{\eta \times \mu \mid \eta \in h(a), \mu \in h(b), a, b \in L\}$ is a base for the product topology $T \otimes T$.

Finally, we introduce the notion of a relation on (L, Q) with three properties : reflexivity, symmetric, close [13].

$\mathrm{Max}(L) \times \mathrm{Max}(L) = \{(\eta, \nu) \mid \eta, \nu \in \mathrm{Max}(L)\}$. $R \subseteq \mathrm{Max}(L) \times \mathrm{Max}(L)$ is called a relation on $\mathrm{Max}(L)$. $\mathrm{Rel}(\mathrm{Max}(L))$ denotes all relations on $\mathrm{Max}(L)$. It is reflexive if $R(\eta, \eta) = 1$ for every $\eta \in \mathrm{Max}(L)$. It is symmetric if $R(\eta, \nu) = R(\nu, \eta)$ for any $\eta, \nu \in \mathrm{Max}(L)$.

A relation R is called closed if it is a close set in the product topology $T \otimes T$ on $\mathrm{Max}(L) \times \mathrm{Max}(L)$. All reflexive, symmetric, closed relations will be denoted by $\mathrm{Rel}^{rsc}\mathrm{Max}(L)$

Suppose $R_1, R_2 \in \mathrm{Rel}(\mathrm{Max}(L))$, we define $R_1 \wedge R_2$, $R_1 \vee R_2$ as follows :

$(R_1 \wedge R_2)(\eta_1, \eta_2) = R_1(\eta_1, \eta_2) \wedge R_2(\eta_1, \eta_2)$,

$(R_1 \vee R_2)(\eta_1, \eta_2) = R_1(\eta_1, \eta_2) \vee R_2(\eta_1, \eta_2)$ for any $\eta_1, \eta_2 \in \mathrm{Max}(L)$.

Example 3.6. For $\eta_1, \eta_2 \in \mathrm{Max}(L)$, $1_{Max(L)}(\eta_1, \eta_2) = 1$.
Clearly, $1_{Max(L)}$ is the largest element in $\mathrm{Rel}(\mathrm{Max}(L))$ and $\mathrm{Rel}^{rsc}\mathrm{Max}(L)$.

Example 3.7. For $\eta_1, \eta_2 \in \mathrm{Max}(L)$, $0_{Max(L)}(\eta_1, \eta_2) = 0$.
$0_{Max(L)}$ is the smallest element in $\mathrm{Rel}(\mathrm{Max}(L))$, but $0_{Max(L)} \notin \mathrm{Rel}^{rsc}\mathrm{Max}(L)$.

Example 3.8. For $\nu_1, \nu_2 \in \mathrm{Max}(L)$, we define a relation on $\mathrm{Max}(L)$,

$$1^{rsc}_{Max(L)}(\nu_1, \nu_2) \begin{cases} 1 & \text{if} \quad \nu_1 = \nu_2, \\ 0 & \text{otherwise.} \end{cases}$$

$1^{rsc}_{Max(L)}$ is the smallest element in $\mathrm{Rel}^{rsc}\mathrm{Max}(L)$.

4. Contact relations on De Morgan algebra

In the section, we introduce the notion of a contact relation on de Morgan algebra (L, Q).

Definition 4.1. Suppose (L, Q) is a de Morgan algebra, a relation θ on (L, Q) is called a contact relation, if it satisfies the following:

(i) $\theta(0, a) = 0$ for every $a \in L, a \neq 0$.
(ii) $\theta(p, p^{\sim}) = 1$.
(iii) $\theta(a, b) = \theta(b, a)$.
(iv) $b \leq c \Rightarrow \theta(a, b) \leq \theta(a, c)$.
(v) $\theta(a, b \vee c) \leq \theta(a, b) \vee \theta(a, c)$.

When L is a Boolean algebra, the above definition coincides with Definition 2.1.

All contact relations on (L, Q) will be denoted by CR(L). Suppose $\theta_1, \theta_2 \in$ CR(L), $\theta_1 \subseteq \theta_2$ is defined as: for any $a, b \in L$, $\theta_1(a, b) \leq \theta_2(a, b)$.

In [3], the notion of proximity relation δ was introduced on a de Morgan algebra (L, Q). δ is called a proximity on (L, Q), if δ satisfies the following conditions:

(i) $a\delta b \Rightarrow b\delta a$.

(ii) $a\delta(b \vee c) \Rightarrow a\delta b$, or $a\delta c$.

(iii) $p\delta p^{\sim}$.

(iv) $0\bar{\delta}1$.

(v) if $a\bar{\delta}b$, then there exist $c, d \in L$, such that $a\bar{\delta}c$, $b\bar{\delta}d$, and $c \vee d = 1$.

Clearly, the proximity relation δ is a contact relation on (L, Q).

Example 4.2. For $a, b \in L$, we define a contact relation on (L, Q),

$$1_L(a, b) = \begin{cases} 1 & \text{if} \quad a \neq 0 \text{ and } \quad b \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly 1_L is the largest element in CR(L).

Example 4.3. For $a, b \in L$, let

$$O_L(a, b) = \begin{cases} 1 & \text{if} \quad a \wedge b \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then O_L is a contact relation on (L, Q).

Proposition 4.4. (1) *For* $a, b \in L$, $a \not\leq b^{'} \Rightarrow \theta(a, b) = 1$.

(2) $p \prec a \Rightarrow \theta(p^{\sim}, a) = 1$.

Proof. (1) For $a, b \in L$, suppose $a \not\leq b^{'}$. Then there exists $p \in Q$ such that $p \prec a$ *and* $p \not\leq b^{'}$. Thus $p \prec a$, $p^{\sim} \prec b$. So by Definition 4.1 (2), $\theta(p, p^{\sim}) = 1$. Hence, by Definition 4.1 (4), we obtain $\theta(a, b) = 1$.

(2) $p \prec a \Rightarrow p^{\sim} \not\leq a^{'}$, by (1), the result holds. ☐

By the above proposition, we obtain that O_L is the smallest element on (L, Q). It is clear that for any contact relation $\theta \in$CR(L), and $a, b \in L$,

(1) $a \wedge b \neq 0 \Rightarrow a \not\leq b^{'} \Rightarrow \theta(a, b) = 1 \Rightarrow O_L(a, b) \leq \theta(a, b)$.

(2) $a \wedge b = 0 \Rightarrow O_L(a, b) = 0 \leq \theta(a, b)$.

Then $O_L \subseteq \theta$ holds.

5. The correspondence between Rel^{rsc}(Max(L))) and CR(L)

In the section, we investigate the correspondence between Rel^{rsc}(Max(L))) and CR(L). On the one hand, for $R \in$Rel^{rsc}(Max(L))), and $a, b \in L$, we define

$$\theta_R(a, b) = \bigvee_{\eta, \nu \in Max(L), a \in \eta, b \in \nu} R(\eta, \nu)$$

Then we have the following result.

Lemma 5.1. *Suppose R is a close, reflexive and symmetric relation on Max(L). Then q : Rel^{rsc}(Max(L))) $\rightarrow$ CR(L), $q(R) = \theta_R \in$CR(L) is an injective mapping.*

Proof. First we have to show that θ_R is a contact relation, that is, to verify θ_R satisfies the conditions of Definition 4.1 :

(i) $\theta_R(0, a) = 0$ for every $a \neq 0$.

(ii) For $p \in Q$, we have $p \prec p$, thus $p^{\sim} \nleq p^{'}$, which leads to $p \wedge p^{\sim} \neq 0$. Then there exists $q \in Q$ such that $q \prec p \wedge p^{\sim}$. By Lemma 3.1, we obtain a maximal filter μ_q, $q \in \mu_q$. Certainly $p,\ p^{\sim} \in \mu_q$. Thus

$$\theta_R(p, p^{\sim}) = \bigvee_{\eta,\nu \in Max(L), p \in \eta, p^{\sim} \in \nu} R(\eta, \nu) \geq R(\mu_q, \mu_q) = 1.$$

(iii) $\theta(a, b) = \theta(b, a)$ is obviously.

(iv) Since $b \leq c$, $\{\nu \mid b \in \nu\} \subseteq \{\nu \mid c \in \nu\}$. Then

$$\theta(a, b) = \bigvee_{\eta,\nu \in Max(L), a \in \eta, b \in \nu} R(\eta, \nu) \leq \bigvee_{\eta,\nu \in Max(L), a \in \eta, c \in \nu} R(\eta, \nu) = \theta(a, c).$$

(v) Since ν is a maximal filter, $b \vee c \in \nu$ implies that $b \in \nu$ or $c \in \nu$ holds. Then $\{\nu \mid b \vee c \in \nu\} \subseteq \{\nu \mid b \in \nu\} \cup \{\nu \mid c \in \nu\}$. Thus

$$\begin{aligned}\theta(a, b \vee c) &= \bigvee_{\eta,\nu \in Max(L), a \in \eta, b \vee c \in \nu} R(\eta, \nu) \\ &\leq \bigvee_{\eta,\nu \in Max(L), a \in \eta, b \in \nu} R(\eta, \nu) \vee \bigvee_{\eta,\nu \in Max(L), a \in \eta, c \in \nu} R(\eta, \nu) \\ &= \theta(a, b) \vee \theta(a, c).\end{aligned}$$

By the above proof, we know that θ_R is a contact relation on (L, Q). But the key of the proof is to prove q is an injective mapping, which is equivalent to show for any two close relations $R_1, R_2 \in$Rel^{rsc}(Max(L))), $R_1 \neq R_2$ yields $\theta_{R_1} \neq \theta_{R_2}$ holds.

Suppose $R_1, R_2 \in$Rel^{rsc}(Max(L))) and $R_1 \neq R_2$. Then there exist $\eta_0, \mu_0 \in$Max(L) such that $(\eta_0, \mu_0) \in R_1, (\eta_0, \mu_0) \notin R_2$. Since R_2 is closed, there are two elements $a, b \in L$, such that $a \in \eta_0, b \in \mu_0$, $(h(a) \times h(b)) \cap R_2 = \varnothing$.

If $\theta_{R_1} = \theta_{R_2}$, then we have

$$\theta_{R_2}(a, b) = \theta_{R_1}(a, b) = \bigvee_{\eta,\nu \in Max(L), a \in \eta, b \in \nu} R_1(\eta, \nu) \geq R_1(\eta_0, \mu_0) = 1.$$

Thus we obtain

$$1 = \theta_{R_2}(a, b) = \bigvee_{\eta,\nu \in Max(L), a \in \eta, b \in \nu} R_2(\eta, \nu).$$

So there exist η_1, μ_1 satisfying $a \in \eta_1, b \in \mu_1,\ R_2(\eta_1, \mu_1) = 1$. A contradiction to $(h(a) \times h(b)) \cap R_2 = \varnothing$. Hence $\theta_{R_1} \neq \theta_{R_2}$ holds, which implies that q is an injective mapping. □

Remark 5.2. Suppose X is a universal set, $L = 2^X$ is the power set of X, then $q(R) = \{\nu_1 \times \nu_2 \mid R(\nu_1, \nu_2) = 1\}$, and $\theta_R(A, B) = 1 \Longleftrightarrow$ there exist $\nu_1, \nu_2 \in$Max(2^X), such that $A \in \nu_1, B \in \nu_2$, and $R(\nu_1, \nu_2) = 1$, i.e., $\nu_1 \times \nu_2 \subseteq q(R)$.

On the other hand, for each contact relation $\theta \in$CR(L), we consider to define the corresponding $R_\theta \in$ Rel^{rsc}(Max(L))).

For a maximal filter η, a element in L is defined, i.e., $\bigwedge \eta = \bigwedge_{a \in \eta} a$. Since $L - \{0\}$ is not a filter on (L, Q), for a maximal filter η, we have $\bigwedge \eta \neq 0$.

Suppose $\theta \in$CR(L), in other words, θ is a contact relation on Max(L), for any maximal filters $\eta_1, \eta_2 \in$ Max(L), let

$$R_\theta(\eta_1, \eta_2) = \theta(\bigwedge \eta_1, \bigwedge \eta_2).$$

Then we have the following lemma.

Lemma 5.3. *Suppose $\theta \in CR(L)$, then $p: CR(L) \to Rel^{rsc}(Max(L))$, for $\theta \in CR(L)$, $p(\theta) = CL(R_\theta) \in Rel^{rsc}(Max(L))$ is an injective mapping. Where $CL(R_\theta)$ is the closure of R_θ in the product topology $T \otimes T$.*

Proof. The closeness, symmetry and reflexivity of R_θ follow directly from the above definition. By the above proof and Lemma 5.1, we know that p is also an injective mapping. This complete the proof. ☐

Remark 5.4. Suppose X is a universal set, $L = 2^X$, then $p(\theta) = \{A \times B \mid \theta(A, B) = 1\}$, and $R(\nu_1, \nu_2) = 1 \Longleftrightarrow \forall A \in \nu_1, B \in \nu_2, \theta(A, B) = 1$ holds.

In the end of the section, by Lemmas 5.1 and 5.2, we obtain the order preserving correspondence, which is called the representation theorem in [9].

Proposition 5.5. *Suppose (L, Q) is a de Morgan algebra, then there exists a bijective order preserving correspondence between the set of all contact relations on (L, Q) and the set of all close, reflexive, symmetric relations on $Max(L)$.*

Proof. By Lemmas 5.1 and 5.2, we obtain q and p are injective mappings. The remainder is to prove $p(q(R)) = R$ or $q(p(\theta)) = \theta$. We have to prove the first equation.

$$\begin{aligned} R(\nu_1, \nu_2) = 1 & \\ & \Rightarrow \forall a \in \nu_1, b \in \nu_2, \theta_R(a, b) = 1 \\ & \Rightarrow \theta_R(\bigwedge \nu_1, \bigwedge \nu_2) = 1 \\ & \Rightarrow p(\theta_R)(\nu_1, \nu_2) = \theta_R(\bigwedge \nu_1, \bigwedge \nu_2) = 1 \\ & \Rightarrow p(q(R))(\nu_1, \nu_2) = R(\nu_1, \nu_2) = 1, \\ R(\nu_1, \nu_2) = 0 & \\ & \Rightarrow \nu_1 \times \nu_2 = \{a \otimes b \mid a \in \nu_1, b \in \nu_2\} \cap R = \varnothing \\ & \Rightarrow \theta_R(a, b) = 0 \\ & \Rightarrow \theta_R(\bigwedge \nu_1, \bigwedge \nu_2) = 0 \\ & \Rightarrow p(\theta_R)(\nu_1, \nu_2) = \theta_R(\bigwedge \nu_1, \bigwedge \nu_2) = 0 \\ & \Rightarrow p(q(R))(\nu_1, \nu_2) = R(\nu_1, \nu_2) = 0. \end{aligned}$$

By the above proof, we obtain $p(q(R)) = R$. So there exists a bijective order preserving correspondence between the set of contact relations on (L, Q) and the set of all close, reflexive, symmetric relations on Max(L). ☐

6. The structure of contact relations

In the section, we consider two problem, the one is discuss the structure of Rel^{rsc}(Max(L)), the two is obtain the corresponding results about the CR(L).

First, we focus on the structure of $\mathrm{Rel}^{rsc}(\mathrm{Max}(L))$.

Let $\mathrm{Rel}^{rsc}(\mathrm{Max}(L))$ be the set of all close, reflexive and symmetric relations on $\mathrm{Max}(L)$. For each $R \in \mathrm{Rel}^{rsc}(\mathrm{Max}(L))$, R is a close set in the product topology $T \otimes T$ on $\mathrm{Max}(L)\times\mathrm{Max}(L)$. From the topological point of view, we know that $\mathrm{Rel}^{rsc}(\mathrm{Max}(L))$ is closed with finite join, and infinity meet.

In other words, suppose $\{R_i \mid i \in I\} \subseteq \mathrm{Rel}^{rsc}(\mathrm{Max}(L))$, we have $\bigvee_{j=1}^{m} R_j \in \mathrm{Rel}^{rsc}(\mathrm{Max}(L))$ and $\bigwedge_{i\in I} R_i \in \mathrm{Rel}^{rsc}(\mathrm{Max}(L))$. Furthermore, $\mathrm{Rel}^{rsc}(\mathrm{Max}(L))$ is also closed with the operator defined as follows : $\sum_{i\in I} R_i = CL(\bigvee_{i\in I} R_i)$, that is to say, $\sum_{i\in I} R_i \in \mathrm{Rel}^{rsc}(\mathrm{Max}(L))$.

Clearly, on $\mathrm{Rel}^{rsc}(\mathrm{Max}(L))$, the largest element is $1_{Max(L)}$; the smallest element is $1^{rsc}_{Max(L)}$.

Second, with the help of Proposition 5.3, we obtain the corresponding results on $\mathrm{CR}(L)$.

Let $\mathrm{CR}(L)$ be the set of all contact relations on (L, Q). It is closed with the two operators : $\sum$ and Π, i.e., for $\{\theta_i \mid i \in I\} \subseteq \mathrm{CR}(L)$,

$$\sum_{i\in I} \theta_i = q(CL(\bigvee_{i\in I} p(\theta_i))), \qquad \Pi_{i\in I}\theta_i = q(\bigwedge_{i\in I} p(\theta_i)).$$

Then we obtain.

Proposition 6.1. *Suppose $CR(L)$ is the collection of all contact relations on (L, Q). Then it forms a complete lattice with respect to $\sum$ and Π. On which the largest element is $1_{CR(L)}$; the smallest element is $O_{CR(L)}$.*

Note.

(1) $q(1^{rsc}_{Max(L)}) = O_{CR(L)}$.

For any $a, b \in L$,

(i) $O_{CR(L)}(a, b) = 1$

$\Rightarrow a \wedge b \neq 0$

$\Rightarrow$ there exists $\mu_{a\wedge b}$, $a \wedge b \in \mu_{a\wedge b}$ (Lemma 3.1)

$\Rightarrow 1^{rsc}_{Max(L)}(\mu_{a\wedge b}, \mu_{a\wedge b}) = 1$ and $a \in \mu_{a\wedge b}, b \in \mu_{a\wedge b}$

$$\Rightarrow q(1^{rsc}_{Max(L)})(a, b) = \bigvee_{\nu_1,\nu_2\in Max(L)} 1^{rsc}_{Max(L)}(\nu_1, \nu_2) \geq 1^{rsc}_{Max(L)}(\mu_{a\wedge b}, \mu_{a\wedge b}) = 1.$$

(ii) $q(1^{rsc}_{Max(L)})(a, b) = 0$

$$\Rightarrow \bigvee_{\nu_1,\nu_2\in Max(L)} 1^{rsc}_{Max(L)}(\nu_1, \nu_2) = 0$$

$\Rightarrow a \wedge b = 0$. If not, $a \wedge b \neq 0$, by (i), we obtain $1^{rsc}_{Max(L)}(\mu_{a\wedge b}, \mu_{a\wedge b}) = 1$, a contradiction.

$\Rightarrow O_{CR(L)}(a, b) = 0$.

(2) $p(O_{CR(L)}) = 1^{rsc}_{Max(L)}$.

For any $\nu_1, \nu_2 \in$Max(L),

(i) $1^{rsc}_{Max(L)}(\nu_1, \nu_2) = 1$

$$\Rightarrow \nu_1 = \nu_2$$

$$\Rightarrow \bigwedge \nu_1 = \bigwedge \nu_2,$$

$$\Rightarrow p(O_{CR(L)})(\nu_1, \nu_2) = CL(O_{CR(L)})(\nu_1, \nu_2) \geq O_{CR(L)}(\bigwedge \nu_1, \bigwedge \nu_2) = 1.$$

(ii) $p(O_{CR(L)})(\nu_1, \nu_2) = 0$

$$\Rightarrow \bigwedge \nu_1 \neq \bigwedge \nu_2$$

$$\Rightarrow \nu_1 \neq \nu_2$$

$$\Rightarrow 1^{rsc}_{Max(L)}(\nu_1, \nu_2) = 0.$$

7. Conclusion

In the paper, we introduced the notion of a contact relation on de Morgan algebra (L, Q), proved all contact relations on (L, Q) form a complete lattice with the two operators $\sum$ and Π, investigated the correspondence between the contact relations on (L, Q) and the close, reflexive, symmetric relations on Max(L).

Since De Morgan algebra with $p-$base is only generalization of atomic Boolean algebra, our work only shows the results of [12] may be generalized to a point-structure. How to construct these results in a general pointless framework is need to further study.

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