Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 12, No. 1, (July 2016), pp. 93–98
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

ON fuzzy $T_{\ddot{g}}$-spaces

M. Jeyaraman, S. Vijayalakshmi, R. Muthuraj

Received 16 06 2015; Revised 05 08 2015; Accepted 06 12 2015

Abstract. The aim of this paper is to introduce fuzzy $T_{\ddot{g}}$-spaces, fuzzy $gT_{\ddot{g}}$-spaces. Moreover, we obtain certain new characterizations for the fuzzy $T_{\ddot{g}}$-spaces, fuzzy $gT_{\ddot{g}}$-spaces.

2010 AMS Classification: 2010 Mathematics Subject Classification: 54C10, 54C08, 54C05

Keywords: Fuzzy $T_{\ddot{g}}$-spaces, Fuzzy $gT_{\ddot{g}}$-spaces, Fuzzy T_{ω}-spaces, Fuzzy αT_b-spaces.

Corresponding Author: S. Vijayalakshmi (vimalvishali@gmail.com.)

1. Introduction

In 1970, Levine [7] introduced the notion of generalized closed sets in topological spaces as a generalization of closed sets. Since then, many concepts related to generalized closed sets were defined and investigated. Recently, Balasubramanian and Sundaram [2] introduced the concepts of generalized fuzzy closed sets and fuzzy $T_{1/2}$-spaces.

Quite Recently, Jeyaraman et al.[6] have introduced the concept of fuzzy $\ddot{g}$-closed sets and studied its basic fundamental properties in fuzzy topological spaces. In this paper, we introduce the notions called fuzzy $T_{\ddot{g}}$-spaces, fuzzy $gT_{\ddot{g}}$-spaces and obtain their properties and characterizations.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) (or X and Y) represent fuzzy topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a fuzzy subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

Definition 2.1 ([12, 14]). If X is a set, then any function $A : X \to [0, 1]$ (from X to the closed unit interval [0, 1] is called a fuzzy set in X.

Definition 2.2 ([12]). (i) The complement of a fuzzy set A, denoted by A^c, is defined by

$$A^c(x) = 1 - A(x), \forall\, x \in X.$$

(ii) Union of two fuzzy sets A and B, denoted by A ∨ B, is defined by

$$(A \cup B)(x) = \max\{A(x),\, B(x)\}, \forall\, x \in X.$$

(iii) Intersection of two fuzzy sets A and B, denoted by A ∧ B, is defined by

$$(A \cap B)(x) = \min\{A(x),\, B(x)\}, \forall\, x \in X.$$

Definition 2.3 ([14]). Let $f : X \to Y$ be a function from a set X into a set Y. Let A be a fuzzy subset in X and B be a fuzzy subset in Y. Then the Zadeh's functions $f(A)$ and $f^{-1}(B)$ are defined by : for each y ∈ Y and for each x ∈ X,

(i) $f(A)$ is a fuzzy subset of Y, where

$$f(A) = \begin{cases} sup\ A(z), & \text{if } f^{-1}(y) \neq \varnothing, \\ z \in f^{-1}(y) & \\ 0, & \text{otherwise.} \end{cases}$$

(ii) $f^{-1}(B)$ is a fuzzy subset of X, where

$$f^{-1}(B)(x) = B(f(x)).$$

Definition 2.4 ([5, 12]). Let X be a set and τ be a family of fuzzy sets in X. Then τ is called a fuzzy topology if τ satisfies the following conditions :

(i) 0, 1 ∈ τ,

(ii) If A $_{i\epsilon I} \in \tau$ then $\vee_{i\epsilon I} A_\imath \in \tau$,

(iii) If A, B ∈ τ then A ∧ B ∈ τ.

The pair (X, τ) is called a fuzzy topological space (or fts). The elements of τ are called fuzzy open sets. Complements of fuzzy open sets are called fuzzy closed sets.

Definition 2.5 ([12]). Let A be a fuzzy set in a fts (X, τ). Then

(i) The closure of A, denoted by cl(A) is defined by

$$\text{cl(A)} = \wedge\,\{\text{F: A} \leq \text{F and F is fuzzy closed}\}.$$

(ii) The interior of A, denoted by int(A) is defined by

$$\text{int(A)} = \vee\,\{\text{G: G} \leq \text{A and G is fuzzy open}\}.$$

Definition 2.6. A fuzzy subset A of a space (X, τ) is called :

(i) fuzzy semi-open set [1] if A ≤ cl(int(A)),

(ii) fuzzy α-open set [4] if A ≤ int(cl(int(A))).

The complements of the above mentioned fuzzy open sets are called their respective fuzzy closed sets.

The fuzzy semi-closure [13] (resp. fuzzy α-closure [9]) of a fuzzy subset A of X, denoted by scl(A) (resp. αcl(A)), is defined to be the intersection of all fuzzy semi-closed (resp. fuzzy α-closed) sets of (X, τ) containing A. It is known that scl(A) (resp. αcl(A)) is a fuzzy semi-closed (resp. fuzzy α-closed) set.

Definition 2.7. A fuzzy subset A of a space (X, τ) is called :

(i) a fuzzy generalized closed (briefly fg-closed) set [2] if $cl(A) \leq U$, whenever $A \leq U$ and U is fuzzy open in (X, τ).

The complement of fuzzy g-closed set is called fuzzy g-open set.

(ii) a fuzzy semi-generalized closed (briefly fsg-closed) set [3] if $scl(A) \leq U$, whenever $A \leq U$ and U is fuzzy semi-open in (X, τ).

The complement of fsg-closed set is called fsg-open set.

(iii) a fuzzy generalized semi-closed (briefly fgs-closed) set [8] if $scl(A) \leq U$, whenever $A \leq U$ and U is fuzzy open in (X, τ).

The complement of fgs-closed set is called fgs-open set.

(iv) a fuzzy α-generalized closed (briefly fαg-closed) set [10] if αcl(A) $\leq$ U, whenever A $\leq$ U and U is fuzzy open in (X, τ).

The complement of fαg-closed set is called fαg-open set.

(v) a fuzzy $\ddot{g}$-closed set [6] if cl(A) $\leq$ U, whenever A $\leq$ U and U is fgs-open in (X, τ).

The complement of fuzzy $\ddot{g}$-closed set is called fuzzy $\ddot{g}$-open set.

(vi) a fuzzy ω-closed set [11] if cl(A)$\leq$U, whenever A$\leq$U and U is fuzzy semi-open in (X, τ).

Definition 2.8 ([2])**.** A fuzzy topological space (X, τ) is called a fuzzy $T_{1/2}$-space if every fuzzy g-closed set in it is fuzzy closed.

Remark 2.9 ([6])**.** For a fuzzy topological space (X, τ), the following hold :

(1) Every fuzzy closed set is fuzzy $\ddot{g}$-closed but not conversely.
(2) Every fuzzy $\ddot{g}$-closed set is fuzzy ω-closed but not conversely.
(3) Every fuzzy $\ddot{g}$-closed set is fuzzy g-closed but not conversely.
(4) Every fuzzy $\ddot{g}$-closed set is fuzzy αg-closed but not conversely.
(5) Every fuzzy $\ddot{g}$-closed set is fuzzy gs-closed but not conversely.

Theorem 2.10 ([6])**.** *If A and B are fuzzy $\ddot{g}$-closed sets in (X, τ), then $A \vee B$ is fuzzy $\ddot{g}$-closed sets in (X, τ).*

3. Properties of fuzzy $T_{\ddot{g}}$-space

We introduce the definition of fuzzy $T_{\ddot{g}}$-Space and study the relationships of such sets.

Definition 3.1. A fuzzy topological spaces (X, τ) is called a fuzzy $T_{\ddot{g}}$-Space if every fuzzy $\ddot{g}$-closed set in it is fuzzy closed.

Example 3.2. Let X = {a, b} and $\alpha : X \to [0, 1]$ with $\tau = \{0_X,\ \alpha,\ 1_X\}$ where α is a fuzzy set in X defined by α(a)=0.5, α(b)=0.5. Then $F\ddot{G}$C(X)= $\{(\frac{a}{u},\ \frac{b}{v})$ $|u = 0, v = 0,\ u = 0.5, v = 0.5$ and $u = 1, v = 1\}$. Thus (X, τ) is a fuzzy topological space. Clearly (X, τ) is fuzzy $T_{\ddot{g}}$-Space.

Example 3.3. Let X = {a, b} and $\alpha : X \to [0, 1]$ with $\tau = \{0_X, \alpha, 1_X\}$ where α is a fuzzy set in X defined by α(a)=1, α(b)=0. Then $F\ddot{G}$C(X)=$\{(\frac{a}{u},\ \frac{b}{v})\ |u = 0, v = 0$ $u \in [0, 1], v = 1$ and $u = 1, v = 1\}$. Thus (X, τ) is a fuzzy topological space. Clearly (X, τ) is not fuzzy $T_{\ddot{g}}$-Space.

Proposition 3.4. *Every fuzzy $T_{1/2}$-space is fuzzy $T_{\ddot{g}}$-Space but not conversely.*

Proof. Let A be any fuzzy $\ddot{g}$-closed set of (X, τ). Every fuzzy $\ddot{g}$ -closed set is fg-closed. Since (X, τ) is a fuzzy $T_{1/2}$ space, A is fuzzy closed. Then (X, τ) is a fuzzy $T_{\ddot{g}}$-Space. □

Example 3.5. Let X = {a, b} and $\alpha : X \to [0,1]$ with $\tau = \{0_X,\ \alpha,\ 1_X\}$ where α is a fuzzy set in X defined by α(a)=α(b)=0.5. Then $F\ddot{G}$C(X)= $\{(\frac{a}{u},\ \frac{b}{v})\ |u = 0, v = 0$ and $u = 0.5, v = 0.5,\ u = 1, v = 1\}$ and FGC(X)= $\{(\frac{a}{u},\ \frac{b}{v})\ |u = 0, v = 0,$ $u \in [0, 0.5], v \in [0, 0.5]$ and $u = 1, v = 1\}$. Thus (X, τ) is a fuzzy topological space. Clearly (X, τ) is fuzzy $T_{\ddot{g}}$-Space but not fuzzy $T_{1/2}$-space.

Definition 3.6. A fuzzy topological spaces (X, τ) is called a fuzzy T_ω-Space if every fuzzy ω-closed set in it is fuzzy closed.

Proposition 3.7. *Every fuzzy T_ω-space is fuzzy $T_{\ddot{g}}$-Space but not conversely*

Proof. Let A be any fuzzy $\ddot{g}$-closed set of (X, τ). Every fuzzy $\ddot{g}$-closed set is fuzzy ω-closed. Since (X, τ) be a fuzzy T_ω-space, A is fuzzy closed. Then (X, τ) be a fuzzy $T_{\ddot{g}}$-Space. □

Example 3.8. Let X = {a, b} and $\alpha : X \to [0,1]$ with $\tau = \{0_X,\ \alpha,\ 1_X\}$ where α is a fuzzy set in X defined by α(a)=α(b)=0.5. Then $F\ddot{G}$C(X)= $\{(\frac{a}{u},\ \frac{b}{v})\ |u = 0, v = 0$ and $u = 0.5, v = 0.5,\ u = 1, v = 1\}$ and ωC(X)= $\{(\frac{a}{u},\ \frac{b}{v})\ |u = 0, v = 0,\ u \in [0, 1], v \in [0, 1]$ and $u = 1, v = 1\}$. Thus (X, τ) is a fuzzy topological space. Clearly (X, τ) be a fuzzy $T_{\ddot{g}}$-Space but not a fuzzy T_ω-space.

Definition 3.9. A fuzzy topological spaces (X, τ) is called a fuzzy αT_b-Space if every fuzzy αg-closed set in it is fuzzy closed.

Proposition 3.10. *Every fuzzy αT_b-Space is fuzzy $T_{\ddot{g}}$-Space but not conversely*

Proof. Let A be any fuzzy $\ddot{g}$-closed set of (X, τ). Every fuzzy $\ddot{g}$-closed set is fuzzy αg-closed. Since (X, τ) be a fuzzy αT_b-space, A is fuzzy closed. Then (X, τ) be a fuzzy $T_{\ddot{g}}$-Space. □

Example 3.11. Let X = {a, b} and $\alpha : X \to [0,1]$ with $\tau = \{0_X,\ \alpha,\ 1_X\}$ where α is a fuzzy set in X defined by α(a)=α(b)=0.5. Then $F\ddot{G}$C(X)= $\{(\frac{a}{u},\ \frac{b}{v})\ |u = 0, v = 0$ and $u = 0.5, v = 0.5,\ u = 1, v = 1\}$ and FαGC(X)= $\{(\frac{a}{u},\ \frac{b}{v})\ |u \in [0, 1], v \in [0, 1],$ and $u = 1, v = 1\}$. Thus (X, τ) is a fuzzy topological space. Clearly (X, τ) be a fuzzy $T_{\ddot{g}}$-Space but not a fuzzy αT_b-space.

4. Fuzzy $gT_{\ddot{g}}$-space

Definition 4.1. A fuzzy topological spaces (X, τ) is called a fuzzy $gT_{\ddot{g}}$-Space if every fuzzy g-closed set in it is fuzzy $\ddot{g}$-closed set.

Example 4.2. Let X = {a, b} and α, β : X→[0, 1] with $\tau = \{0_X,\ \alpha,\ \beta,\ 1_X\}$ where α, β are fuzzy sets in X defined by α(a)=0.3, α(b)=0.3 and β(a)=0.5, β(b)=0.5. $F\ddot{G}$C(X)= $\{(\frac{a}{u},\ \frac{b}{v})\ |u = 0, v = 0,\ u = 0.5, v = 0.5,\ u = 0.7, v = 0.7$ and $u = 1, v = 1\}$. Then (X, τ) is a fuzzy topological space. Thus (X, τ) be a fuzzy $gT_{\ddot{g}}$-Space.

Example 4.3. Let X = {a, b} and $\alpha : X \to [0,1]$ with $\tau = \{0_X,\ \alpha,\ 1_X\}$ where α is a fuzzy set in X defined by α(a)=1, α(b)=0. Then $F\ddot{G}$C(X)= $\{(\frac{a}{u},\ \frac{b}{v})\ |u = 0, v = 0$ and $u \in [0, 1], v = 1\}$. Thus (X, τ) is a fuzzy topological space. clearly (X, τ) is not a fuzzy $gT_{\ddot{g}}$-Space.

Proposition 4.4. *Every fuzzy $T_{1/2}$-space is fuzzy $gT_{\ddot{g}}$-Space but not conversely.*

Proof. Let A be any fuzzy g-closed set of (X, τ). Since (X, τ) be a fuzzy $T_{1/2}$ space, A is fuzzy closed. Then (X, τ) be a fuzzy $gT_{\ddot{g}}$-Space. □

Remark 4.5. Fuzzy $T\ddot{g}$-space and fuzzy $gT_{\ddot{g}}$-Space are independent.

Example 4.6. Let X = {a, b} and α, β : X→[0, 1] with $\tau = \{0_X, \alpha, \beta, \alpha \vee \beta, 1_X\}$ where α, β are fuzzy sets in X defined by α(a)=0.6, α(b)=0 and β(a)=0, β(b)=0.3. $F\ddot{G}$C(X)= $\{(\frac{a}{u}, \frac{b}{v})\ |u = 0, v = 0$, and $u \in [0, 1], v = 1$, $u = 0.4, v = 0.7$, $u = 1, v = 0.7$ and $u = 1, v = 1\}$. Then (X, τ) is a fuzzy topological space. Thus (X, τ) be a fuzzy $gT_{\ddot{g}}$-Space but not a fuzzy $T_{\ddot{g}}$-Space.

Example 4.7. Let X = {a, b} and $\alpha : X \to [0, 1]$ with $\tau = \{0_X, \alpha, 1_X\}$ where α is a fuzzy set in X defined by α(a)=0.5, α(b)=0.5 Then $F\ddot{G}$C(X)= $\{(\frac{a}{u}, \frac{b}{v})\ |u = 0, v = 0$, $u = 0.5, v = 0.5$ and $u = 1, v = 1\}$. Then (X, τ) is a fuzzy topological space. Thus (X, τ) be a fuzzy $T_{\ddot{g}}$-Space but not fuzzy $gT_{\ddot{g}}$-Space.

Theorem 4.8. *A fuzzy topological spaces (X, τ) is a fuzzy $T_{1/2}$-space if and only if it is a fuzzy $T_{\ddot{g}}$-Space and a fuzzy $gT_{\ddot{g}}$-Space.*

Proof. Necessity: Follow directly from Proposition 3.4 and 4.4.
Suffiency: Suppose that X is both a fuzzy $T_{\ddot{g}}$-Space and fuzzy $gT_{\ddot{g}}$-Space. Let A be a fg-closed set of (X, τ). Since (X, τ) is fuzzy $gT_{\ddot{g}}$-Space, A is a fuzzy $\ddot{g}$-closed set of (X, τ). Since (X, τ) is a fuzzy $T_{\ddot{g}}$-Space, A is a fuzzy closed set of (X, τ). Then (X, τ) is a fuzzy $T_{1/2}$-space. □

References

[1] K. K. Azad, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981) 14–32.
[2] G. Balasubramanian and P. Sundaram, On some generalizations of fuzzy continuous functions, Fuzzy Sets and Systems 86 (1997) 93–100.
[3] G. Balasubramanian, On fuzzy pre-separation axioms, Bull. Cal. Math. Soc. 90 (6) (1998) 427–434.
[4] S. Bin shahna, On fuzzy strong semicontinuity and fuzzy precontinuity, Fuzzy Sets and Systems 44 (1991) 303–308.
[5] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[6] M. Jeyaraman, S. Vijayalakshmi, and O. Ravi, New Notions of closed sets in fuzzy topology, IJMA 4 (8) (2013) 67–73.
[7] N. Levine, Generalized Closed Sets in Topology, Rend. Circ. Math. Palermo 19 (1970)89–96.
[8] H. Maki et al., Generalized closed sets in fuzzy topological space, I, Meetings on topological space Theory and its Application (1998) 23–36.
[9] R. Prasad, S. S.Thakur and R. K. Saraf, Fuzzy α-irresolute mappings, J. Fuzzy Math. 2 (2) (1994) 335–339.
[10] R. K. Saraf, M. Caldas and S. Mishra, Results via Fgα-closed sets and Fαg-closed sets, (preprint).
[11] M. Sudha, E. Roja and M. k. Uma, Slightly fuzzy ω-continous mappings, Int. Journal of Math Analysis 5 (16) (2011) 779–787.
[12] P. Sundaram, Generalized continuity in fuzzy topology (preprint).
[13] T. H. Yalvac, Semi-interior and semi-closure of a fuzzy set, J. Math. Anal. Appl. 132 (1988) 356–364.
[14] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

M. Jeyaraman (jeya.math@gmail.com)
Department of Mathematics, Raja Duraisingam Govt. Arts College, Sivagangai-630561

S. Vijayalakshmi (vimalvishali@gmail.com)
Department of Mathematics, St. Michael College of Engg & Tech, Kalaiyarkovil-630551

R. Muthuraj (rmr1973@yahoo.co.in)
Department of Mathematics, H. H. The Rajah's College, Pudukkottai-622001