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ps-ro fuzzy α-continuous functions

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Abstract. In this paper, the notions of *ps-ro* α-open(closed) fuzzy sets on a fuzzy topological space are introduced and their basic properties are studied. It is shown that fuzzy α-open(closed) and *ps-ro* α-open(closed) fuzzy sets do not imply each other. Relations of these fuzzy sets with the existing concepts of both *ps-ro* open(closed) and *ps-ro* semiopen(closed) fuzzy sets are established. In terms of these fuzzy sets, *ps-ro* fuzzy α-open(closed) and *ps-ro* fuzzy α-continuous functions are defined. It is proved that the concept of *ps-ro* fuzzy α-open and fuzzy α-open functions are independent of each other. Interrelations of these functions with fuzzy α-continuous, *ps-ro* fuzzy continuous and *ps-ro* fuzzy semicontinuous functions are established along with their several characterizations.

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1. Introduction and preliminaries

Fuzzy α-open(closed) sets, fuzzy α-open(closed) functions and fuzzy α-continuity were introduced and their various characterizations were studied in [2, 13]. In [6], pseudo regular open fuzzy topology (in short, *ps-ro* fuzzy topology) was introduced. Based on this, a class of functions called *ps-ro* fuzzy continuous functions were introduced and explored in [7, 8]. In [5], a notion of *ps-ro* semiopen(closed) fuzzy sets, *ps-ro* fuzzy semiopen functions and *ps-ro* fuzzy semicontinuous functions were introduced and their different properties and interrelations with the existing allied concepts were studied.

In this paper, we initiate and explore the notions of *ps-ro* α-open(closed) fuzzy set, *ps-ro* fuzzy α-open(closed) functions and *ps-ro* fuzzy α-continuity. Several fruitful researches are carried out related to different types of fuzzy α-continuous types of functions in intuitionistic fuzzy topological spaces, such as, [3, 9], etc.

Let X and Y be two nonempty sets. If f is a function from X into Y and A, B are fuzzy sets on X and Y respectively, then $1 - A$ (called complement of A), $f(A)$ and $f^{-1}(B)$ are fuzzy sets on X, Y and X respectively, defined by $(1 - A)(x) = 1 - A(x) \forall x \in X$, $f(A)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} A(z), when f^{-1}(y) \neq \varnothing \\ 0,\ otherwise \end{cases}$ and $f^{-1}(B)(x) = B(f(x))$ [14]. Here, the product fuzzy set $A \times B$ on $X \times Y$ is defined by $(A \times B)(x, y) = inf\{A(x), B(y) : (x, y) \in X \times Y\}$ [10]. A collection $\tau \subseteq I^X$ is called a fuzzy topology on X if (i) $0, 1 \in \tau$ (ii) $\forall\ \mu_1, \mu_2, ..., \mu_n \in \tau \Rightarrow \wedge_{i=1}^{n} \mu_i \in \tau$ (iii) $\mu_\alpha \in \tau$, $\forall\ \alpha \in \Lambda$ (where Λ is an index set) $\Rightarrow \vee \mu_\alpha \in \tau$. Then (X, τ) is called a fts, the members of τ are called fuzzy open sets and their complements as fuzzy closed sets on X [4].

For a fuzzy set μ on X, the set $\mu^\alpha = \{x \in X : \mu(x) > \alpha\}$ is called the strong α-level set of X. In a fts (X, τ), the family $i_\alpha(\tau) = \{\mu^\alpha : \mu \in \tau\}$ for all $\alpha \in I_1 = [0, 1)$ forms a strong α-level topology on X [12, 11]. A fuzzy open set μ on a fts (X, τ) is said to be pseudo regular open fuzzy set if μ^α is regular open in $(X, i_\alpha(\tau)), \forall \alpha \in I_1$. The family of all pseudo regular open fuzzy sets form a fuzzy topology on X called *ps-ro* fuzzy topology on X, members of which are called *ps-ro* open fuzzy sets and their complements as *ps-ro* closed fuzzy sets on (X, τ) [6]. A function f from fts (X, τ_1) to fts (Y, τ_2) is pseudo fuzzy *ro* continuous (in short, *ps-ro* fuzzy continuous) if $f^{-1}(U)$ is *ps-ro* open fuzzy set on X for each pseudo regular open fuzzy set U on Y [7]. Equivalently, f is *ps-ro* fuzzy continuous if $f^{-1}(A)$ is *ps-ro* open fuzzy set on X for each *ps-ro* open fuzzy set A on Y [8]. Fuzzy *ps*-closure of A, *ps-cl*$(A) = \wedge\{B : A \leq B,\ B$ is *ps-ro* closed fuzzy set on $X\}$ and fuzzy *ps*-interior of A, *ps-int*$(A) = \vee\{B : B \leq A,\ B$ is *ps-ro* open fuzzy set on $X\}$ [7, 8]. A fuzzy set A on a fts (X, τ) is said to be *ps-ro* semiopen fuzzy set if there exist a *ps-ro* open fuzzy set U such that $U \leq A \leq$ *ps-cl*(U). Equivalently, A is *ps-ro* semiopen fuzzy set if $A \leq$ *ps-cl*(*ps-int*(A)). The complement of *ps-ro* semiopen fuzzy set is called *ps-ro* semiclosed fuzzy set. A function f from a fts (X, τ_1) to another fts (Y, τ_2) is called *ps-ro* fuzzy semiopen function if $f(A)$ is *ps-ro* semiopen fuzzy set on Y for each *ps-ro* open fuzzy set A on X. Here f is called *ps-ro* fuzzy semicontinuous if $f^{-1}(A)$ is *ps-ro* semiopen fuzzy set on X for each *ps-ro* open fuzzy set A on Y. [5].

A fuzzy set A on a fts (X, τ) is called fuzzy α-open if $A \leq int(cl(int(A)))$ and fuzzy α-closed if $A \geq cl(int(cl(A)))$. A function f between two fts (X, τ_1) and (Y, τ_2) is called fuzzy α-open(closed) function if $f(A)$ is fuzzy α-open(closed) set on Y, for each fuzzy open(closed) set A on X. f is called fuzzy α-continuous if $f^{-1}(A)$ is fuzzy α-open set on X for each fuzzy open A on Y [2, 13].

2. *ps-ro* α-OPEN(CLOSED) FUZZY SET

Definition 2.1. A fuzzy set A on a fuzzy topological space (X, τ) is called
(i) *ps-ro* α-open fuzzy set if $A \leq$ *ps-int*(*ps-cl*(*ps-int*(A))).
(ii) *ps-ro* α-closed fuzzy set if $A \geq$ *ps-cl*(*ps-int*(*ps-cl*(A))).

Clearly, *ps-ro* open(closed) fuzzy set implies *ps-ro* α-open(closed) fuzzy set but the converse is not true is shown by the following Example.

Example 2.2. Let $X = \{a, b, c\}$ and A, B and C be fuzzy sets on X defined by $A(t) = 0.6$, $B(t) = 0.8$, for all $t \in\ X$ and $C(a) = 0.3, C(b) = 0.3, C(c) = 0.4$. Then

$\tau = \{0, 1, A, B, C\}$ is a fuzzy topology on X. Also, C is not pseudo regular open fuzzy set for $0.3 \le \alpha < 0.4$. Also, the *ps-ro* fuzzy topology on X is $\{0, 1, A, B\}$. Let us define fuzzy set E by $E(t) = 0.7$, for all $t \in X$. E is *ps-ro* α-open but not *ps-ro* open fuzzy set on X. Also, $1 - E$ is *ps-ro* α-closed but not *ps-ro* closed fuzzy set on X.

Every *ps-ro* α-open(closed) fuzzy set is *ps-ro* semiopen(closed) fuzzy set but the converse is not true is shown by the following Example.

Example 2.3. Let $X = \{a, b, c\}$ and A, B and C be fuzzy sets on X defined by $A(t) = 0.1$, for all $t \in X$, $B(a) = 0.4, B(b) = 0.4, B(c) = 0.3$, $C(t) = 0.2$, for all $t \in X$. Then $\tau = \{0, 1, A, B, C\}$ is a fuzzy topology on X. Also, B is not pseudo regular open fuzzy set for $0.3 \le \alpha < 0.4$. Also, the *ps-ro* fuzzy topology on X is $\{0, 1, A, C\}$. Let us define fuzzy set D by $D(t) = 0.3$, for all $t \in X$. D is *ps-ro* semiopen but not *ps-ro* α-open fuzzy set on X. Also, $1 - D$ is *ps-ro* semiclosed but not *ps-ro* α-closed fuzzy set on X.

We see below that fuzzy α-open(closed) and *ps-ro* α-open(closed) fuzzy sets do not imply each other.

Remark 2.4. In Example 2.2, E is *ps-ro* α-open but not fuzzy α-open set and $1-E$ is *ps-ro* α-closed but not α-closed fuzzy set. In Example 2.3, D is α-open fuzzy set but not *ps-ro* α-open fuzzy set on X. $1 - D$ is α-closed fuzzy set but not *ps-ro* α-closed fuzzy set on X. Hence, fuzzy α-open(closed) and *ps-ro* α-open(closed) fuzzy sets are independent of each other.

Theorem 2.5. (1) *An arbitrary union of ps-ro α-open fuzzy sets is a ps-ro α-open fuzzy set.*

(2) *An arbitrary intersection of ps-ro α-closed fuzzy sets is a ps-ro α-closed fuzzy set.*

Proof. Straightforward. □

Theorem 2.6. *Let X and Y be fts. The product $A \times B$ is ps-ro α-open fuzzy set on the product space $X \times Y$ for A and B both ps-ro α-open fuzzy set on X and Y respectively.*

Proof. Let A and B be *ps-ro* α-open fuzzy sets on X and Y respectively. Then

$$A \le ps\text{-}int(ps\text{-}cl(ps\text{-}int(A))) \text{ and } B \le ps\text{-}int(ps\text{-}cl(ps\text{-}int(B))).$$

From Theorem 3.10 in [1],

$$ps\text{-}int(A \times B) = ps\text{-}int(A) \times ps\text{-}int(B)$$

and

$$ps\text{-}cl(A \times B) = ps\text{-}cl(A) \times ps\text{-}cl(B).$$

Thus

$$\begin{aligned} A \times B &\le ps\text{-}int(ps\text{-}cl(ps\text{-}int(A))) \times ps\text{-}int(ps\text{-}cl(ps\text{-}int(B))) \\ &= ps\text{-}int(ps\text{-}cl(ps\text{-}int(A \times B))). \end{aligned}$$

So $A \times B$ is a *ps-ro* α-open fuzzy set on $X \times Y$. □

Theorem 2.7. *If A is a fuzzy set on a fuzzy topological space X and B is ps-ro semiopen fuzzy set such that $B \leq A \leq$ ps-int(ps-cl(B)), then A is ps-ro α-open fuzzy set.*

Proof. Suppose B is *ps-ro* semiopen fuzzy set. Then $B \leq$ *ps-cl*(*ps-int*(B)). On one hand,

$$\begin{aligned} A &\leq ps\text{-}int(ps\text{-}cl(B)) \\ &\leq ps\text{-}int(ps\text{-}cl(ps\text{-}cl(ps\text{-}int(B)))) \\ &= ps\text{-}int(ps\text{-}cl(ps\text{-}int(B))) \\ &\leq ps\text{-}int(ps\text{-}cl(ps\text{-}int(A))). \end{aligned}$$

Thus A is *ps-ro* α-open fuzzy set on X. □

Theorem 2.8. *Let A be a fuzzy set of a fuzzy topological space (X, τ) then the following statements are equivalent :*

(1) *A is a ps-ro α-open fuzzy set.*
(2) *$(1 - A)$ is a ps-ro α-closed fuzzy set.*
(3) *$\exists$ ps-ro open fuzzy set B in X such that $B \leq A \leq$ ps-int(ps-cl(B)).*
(4) *$\exists$ ps-ro closed fuzzy set $(1 - B)$ in X such that*

$$ps\text{-}cl(ps\text{-}int(1 - B)) \leq (1 - A) \leq (1 - B).$$

Proof. (1)$\Leftrightarrow$ (2):

A is *ps-ro* α-open fuzzy set on X

$$\begin{aligned} &\Leftrightarrow A \leq ps\text{-}int(ps\text{-}cl(ps\text{-}int(A))) \\ &\Leftrightarrow (1 - A) \geq 1 - ps\text{-}int(ps\text{-}cl(ps\text{-}int(A))) \\ &\qquad = ps\text{-}cl(1 - ps\text{-}cl(ps\text{-}int(A))) \\ &\qquad = ps\text{-}cl(ps\text{-}int(1 - ps\text{-}int(A))) \\ &\qquad = ps\text{-}cl(ps\text{-}int(ps\text{-}cl(1 - A))) \end{aligned}$$

$\Leftrightarrow 1 - A$ is *ps-ro* α-closed fuzzy set.

(3)$\Leftrightarrow$ (4):

B is *ps-ro* open fuzzy set and $B \leq A \leq$ *ps-int*(*ps-cl*(B))

$$\begin{aligned} &\Leftrightarrow (1 - B) \geq (1 - A) \\ &\qquad \geq (1 - ps\text{-}int(ps\text{-}cl(B))) \\ &\qquad = ps\text{-}cl(1 - ps\text{-}cl(B)) \\ &\qquad = ps\text{-}cl(ps\text{-}int(1 - B)) \end{aligned}$$

$\Leftrightarrow 1 - B$ is *ps-ro* closed fuzzy set and
$ps\text{-}cl(ps\text{-}int(1 - B)) \leq (1 - A) \leq (1 - B)$.

(1)$\Leftrightarrow$ (3): Let A be *ps-ro* α-open fuzzy set. Then $A \leq$ *ps-int*(*ps-cl*(*ps-int*(A))). Let $B =$ *ps-int*(A). Then B is *ps-ro* open
and

$$B = ps\text{-}int(A) \leq A \leq ps\text{-}int(ps\text{-}cl(ps\text{-}int(A))) = ps\text{-}int(ps\text{-}cl(B)).$$

Thus $B \leq A \leq$ *ps-int*(*ps-cl*(B)).

Conversely, let B be *ps-ro* open fuzzy set such that $B \leq A \leq$ *ps-int*(*ps-cl*(B)). Then $B \leq$ *ps-int*(A)
and

$$ps\text{-}int(ps\text{-}cl(B)) \leq ps\text{-}int(ps\text{-}cl(ps\text{-}int(A))).$$

On the other hand,

$$B \leq ps\text{-}int(A) \leq A \leq ps\text{-}int(ps\text{-}cl(B)) \leq ps\text{-}int(ps\text{-}cl(ps\text{-}int(A))).$$

So A is *ps-ro* α-open fuzzy set. □

3. *ps-ro* FUZZY α-CONTINUOUS FUNCTION

Definition 3.1. A function f from a fts (X, τ_1) to another fts (Y, τ_2) is called *ps-ro* fuzzy α-continuous function if $f^{-1}(A)$ is *ps-ro* α-open fuzzy set on X, for each *ps-ro* open fuzzy set A on Y.

Clearly, *ps-ro* fuzzy continuous implies *ps-ro* fuzzy α-continuous but the converse is not true is shown by the following Example.

Example 3.2. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let A, B and G be fuzzy sets on X defined by $A(a) = 0.1, A(b) = 0.2$ and $A(c) = 0.2$, $B(t) = 0.3, \forall t \in X$ and $G(t) = 0.5, \forall t \in X$. Let C, D, E and F be fuzzy sets on Y defined by $C(t) = 0.3, \forall t \in Y$, $D(x) = 0.3, D(y) = 0.3$ and $D(z) = 0.4$, $E(t) = 0.4, \forall t \in Y$ and $F(x) = 0.1, F(y) = 0.1$ and $F(z) = 0.2$. $\tau_1 = \{0, 1, A, B, G\}$ and $\tau_2 = \{0, 1, C, D, E, F\}$ are fuzzy topologies on X and Y respectively. Clearly, A is not pseudo regular open fuzzy set for $0.1 \leq \alpha < 0.2$ on X. Also, the *ps-ro* fuzzy topology on X is $\{0, 1, B, G\}$. Again, D and F are not pseudo regular open fuzzy set for $0.3 \leq \alpha < 0.4$ and $0.1 \leq \alpha < 0.2$, respectively on Y . Also, the *ps-ro* fuzzy topology on Y is $\{0, 1, C, E\}$. Define a function f from the fts (X, τ_1) to fts (Y, τ_2) by $f(a) = x, f(b) = y$ and $f(c) = z$. Here, E is *ps-ro* open fuzzy set on Y and $f^{-1}(E)(t) = 0.4, \forall t \in X$ but $f^{-1}(E)$ is not *ps-ro* open fuzzy set on X proving that f is not *ps-ro* fuzzy continuous. It can be verified that $f^{-1}(U)$ is *ps-ro* α-open fuzzy set on X for every *ps-ro* open fuzzy set U on Y. Hence, f is *ps-ro* fuzzy α-continuous.

Also, Every *ps-ro* fuzzy α-continuous implies *ps-ro* fuzzy semicontinuous but the converse is not true is shown below.

Example 3.3. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let A, B and C be fuzzy sets on X defined by $A(a) = 0.1, A(b) = 0.2$ and $A(c) = 0.2$, $B(t) = 0.3, \forall t \in X$ and $C(a) = 0.4, C(b) = 0.5$ and $C(c) = 0.5$. Let D, E and F be fuzzy sets on Y defined by $D(t) = 0.3, \forall t \in Y$, $E(t) = 0.4, \forall t \in Y$ and $F(x) = 0.3, F(y) = 0.3$ $F(z) = 0.4$. $\tau_1 = \{0, 1, A, B, C\}$ and $\tau_2 = \{0, 1, D, E, F\}$ are fuzzy topologies on X and Y respectively. Clearly, A and C are not pseudo regular open fuzzy sets for $0.1 \leq \alpha < 0.2$ and $0.4 \leq \alpha < 0.5$ on X. Also, the *ps-ro* fuzzy topology on X is $\{0, 1, B\}$. Again, F is not pseudo regular open fuzzy set for $0.3 \leq \alpha < 0.4$ on Y . Also, the *ps-ro* fuzzy topology on Y is $\{0, 1, D, E\}$. Define a function f from the fts (X, τ_1) to fts (Y, τ_2) by $f(a) = x, f(b) = y$ and $f(c) = z$. Here, E is *ps-ro* open fuzzy set on Y and $f^{-1}(E)(t) = 0.4, \forall t \in X$ but $f^{-1}(E)$ is not *ps-ro* α-open fuzzy set on X proving that f is not *ps-ro* fuzzy α-continuous. It can be verified that $f^{-1}(U)$ is *ps-ro* semiopen fuzzy set on X for every *ps-ro* open fuzzy set U on Y. Hence, f is *ps-ro* fuzzy semicontinuous.

The concept of *ps-ro* fuzzy α-continuous and fuzzy α-continuous are totally independent of each other is shown below.

Remark 3.4. In Example 3.2, $f^{-1}(F)(a) = 0.1, f^{-1}(F)(b) = 0.1$ and $f^{-1}(F)(c) = 0.2$ is not fuzzy α-open set on X though F is open fuzzy set on Y proving that f is not fuzzy α-continuous but f is *ps-ro* fuzzy α-continuous. In Example 3.3, it can be verified that f is fuzzy α-continuous but f is not *ps-ro* fuzzy α-continuous. Hence, *ps-ro* fuzzy α-continuous and fuzzy α-continuous are totally independent of each other.

Theorem 3.5. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be function from fuzzy topological space X to fuzzy topological space Y. Then the following are equivalent :*

(1) *f is ps-ro fuzzy α-continuous.*

(2) *The inverse image of each ps-ro closed fuzzy set on Y is ps-ro α-closed fuzzy set on X.*

(3) *For each fuzzy point x_α on X and each ps-ro open fuzzy set B on Y and $f(x_\alpha) \in B$, there exist ps-ro α-open fuzzy set A on X such that $x_\alpha \in A$ and $f(A) \leq B$.*

(4) *$ps\text{-}cl(ps\text{-}int(ps\text{-}cl(f^{-1}(B)))) \leq f^{-1}(ps\text{-}cl(B)) \forall B \in Y$.*

(5) *$f(ps\text{-}cl(ps\text{-}int(ps\text{-}cl(A)))) \leq ps\text{-}cl(f(A)) \forall A \in X$.*

Proof. (1)$\Rightarrow$ (2): Let B be *ps-ro* closed fuzzy set on Y. Then $f^{-1}(1 - B)$ is *ps-ro* α-open fuzzy set on X. Thus $f^{-1}(1 - B) = 1 - f^{-1}(B)$. So the result follows.

(2)$\Rightarrow$ (1): Let B be *ps-ro* open fuzzy set on Y. Then, by given hypothesis, $f^{-1}(1-B)$ is *ps-ro* α-closed fuzzy set on X. Thus $f^{-1}(B)$ is *ps-ro* α-open fuzzy set on X. Hence, f is *ps-ro* fuzzy α-continuous.

(1)$\Rightarrow$ (3): Let x_α be any fuzzy point on X and B be any *ps-ro* open fuzzy set on Y such that $f(x_\alpha) \in B$. Since f is *ps-ro* fuzzy α-continuous, $f^{-1}(B)$ is *ps-ro* α-open fuzzy set on X which contains x_α. Let $f^{-1}(B) = A$. Then $x_\alpha \in A$ and $f^{-1}(B) = A$. Thus $f(A) \leq B$.

(3)$\Rightarrow$ (1): Let the given condition hold and B be any *ps-ro* open fuzzy set on Y. If $f^{-1}(B) = 0$, then the result is true. If $f^{-1}(B) \neq 0$, then there exist fuzzy point x_α on $f^{-1}(B)$, i.e., $f(x_\alpha) \in B$. Thus, by the given hypothesis, $\exists$ *ps-ro* α-open fuzzy set U_{x_α} on X which contains x_α such that $x_\alpha \in U_{x_\alpha} \leq f^{-1}(B)$. Since x_α is arbitrary, taking union of all such relations, we get

$$f^{-1}(B) = \vee\{x_\alpha : x_\alpha \in f^{-1}(B)\} \leq \vee\{U_{x_\alpha} : x_\alpha \in f^{-1}(B)\} \leq f^{-1}(B).$$

So $\vee\{U_{x_\alpha} : x_\alpha \in f^{-1}(B)\} = f^{-1}(B)$. This shows $f^{-1}(B)$ is *ps-ro* α-open fuzzy set. Hence f is *ps-ro* fuzzy α-continuous.

(2)$\Rightarrow$ (4): Let B be fuzzy set on Y. Then $ps\text{-}cl(B)$ is *ps-ro* closed fuzzy set on Y. Thus, by the given hypothesis, $f^{-1}(ps\text{-}cl(B)$ is *ps-ro* α-closed fuzzy set on X. So, $ps\text{-}cl(ps\text{-}int(ps\text{-}cl(f^{-1}(ps\text{-}cl(B))))) \leq f^{-1}(ps\text{-}cl(B))$.
Hence, $ps\text{-}cl(ps\text{-}int(ps\text{-}cl(f^{-1}(B)))) \leq f^{-1}(ps\text{-}cl(B))$.

(4)$\Rightarrow$ (5): Let A be fuzzy set on X and $f(A) = B$. Then $A \leq f^{-1}(B)$. Thus, by our hypothesis,

$$ps\text{-}cl(ps\text{-}int(ps\text{-}cl(f^{-1}(B)))) \leq f^{-1}(ps\text{-}cl(B)).$$

So,

$$ps\text{-}cl(ps\text{-}int(ps\text{-}cl(A))) \leq ps\text{-}cl(ps\text{-}int(ps\text{-}cl(f^{-1}(B))))$$

$$\leq f^{-1}(ps\text{-}cl(B)) = f^{-1}(ps\text{-}cl(f(A))).$$

Hence,

$$f(ps\text{-}cl(ps\text{-}int(ps\text{-}cl(A)))) \leq f(f^{-1}(ps\text{-}cl(f(A)))) \leq ps\text{-}cl(f(A)).$$

Therefore $f(ps\text{-}cl(ps\text{-}int(ps\text{-}cl(A)))) \leq ps\text{-}cl(f(A))$.

(5)$\Rightarrow$ (2): Let B be any *ps-ro* closed fuzzy set on Y and $A = f^{-1}(B)$. Then $f(A) \leq B$ and by given hypothesis,

$$f(ps\text{-}cl(ps\text{-}int(ps\text{-}cl(A)))) \leq ps\text{-}cl(f(A)) \leq ps\text{-}cl(B) = B.$$

Thus $f^{-1}(f(ps\text{-}cl(ps\text{-}int(ps\text{-}cl(A))))) \leq f^{-1}(B)$. So $ps\text{-}cl(ps\text{-}int(ps\text{-}cl(A))) \leq f^{-1}(B)$. Hence $f^{-1}(B)$ is *ps-ro* α-closed fuzzy set. □

4. *ps-ro* fuzzy α-open(closed) function

Definition 4.1. A function f from a fts (X, τ_1) to another fts (Y, τ_2) is called *ps-ro* fuzzy α-open(closed) function if $f(A)$ is *ps-ro* α-open(closed) fuzzy set on Y, for each *ps-ro* open(closed) fuzzy set A on X.

Clearly, every *ps-ro* fuzzy α-open function implies *ps-ro* semiopen but converse is not true is given below.

Remark 4.2. In Example 3.2, G is *ps-ro* open fuzzy set on X and $f(G)(t) = 0.5, \forall t \in Y$ but $f(G)$ is not *ps-ro* α-open fuzzy set on Y proving that f is not *ps-ro* fuzzy α-open function. Here, $f(U)$ is *ps-ro* semiopen fuzzy set on Y for every *ps-ro* open fuzzy set U on X. Hence, f is *ps-ro* fuzzy semiopen.

Every *ps-ro* fuzzy open function implies *ps-ro* fuzzy α-open function but the converse is not true is shown below.

Example 4.3. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let A and B be fuzzy sets on X defined by $A(a) = 0.7, A(b) = 0.7, A(c) = 0.8$ and $B(t) = 0.9, \forall t \in X$. Let C, D and E be fuzzy sets on Y defined by $C(t) = 0.6, \forall t \in Y$, $D(t) = 0.7, \forall t \in Y$ and $E(x) = 0.7, E(y) = 0.8$ and $E(z) = 0.8$. $\tau_1 = \{0, 1, A, B\}$ and $\tau_2 = \{0, 1, C, D, E\}$ are fuzzy topologies on X and Y respectively.
Clearly, A is not pseudo regular open fuzzy set for $0.7 \leq \alpha < 0.8$ on X. Also, the *ps-ro* fuzzy topology on X is $\{0, 1, B\}$. Again, E is not pseudo regular open fuzzy set for $0.7 \leq \alpha < 0.8$. Also, the *ps-ro* fuzzy topology on Y is $\{0, 1, C, D\}$. Define a function f from the fts (X, τ_1) to fts (Y, τ_2) by $f(a) = x, f(b) = y$ and $f(c) = z$. B is *ps-ro* open fuzzy set on X and $f(B)(t) = 0.9, \forall t \in Y$ but $f(B)$ is not *ps-ro* open fuzzy set on Y proving that f is not *ps-ro* fuzzy open function. It can be verified that $f(U)$ is *ps-ro* α-open fuzzy set on Y for every *ps-ro* open fuzzy set U on X. Hence, f is *ps-ro* fuzzy α-open.

The concept of *ps-ro* fuzzy α-open function and fuzzy α-open function are independent of each other is shown below.

Example 4.4. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let A and B be fuzzy sets on X defined by $A(a) = 0.2, A(b) = 0.2, A(c) = 0.3$ and $B(t) = 0.4, \forall t \in X$. Let C, D and E be fuzzy sets on Y defined by $C(t) = 0.3, \forall t \in Y$, $D(x) = 0.4, D(y) = 0.4$ and $D(z) = 0.5$ and $E(x) = 0.1, E(y) = 0.1$ and $E(z) = 0.2$. $\tau_1 = \{0, 1, A, B\}$

and $\tau_2 = \{0, 1, C, D, E\}$ are fuzzy topologies on X and Y respectively. Clearly, A is not pseudo regular open fuzzy set for $0.2 \leq \alpha < 0.3$ on X. Also, the *ps-ro* fuzzy topology on X is $\{0, 1, B\}$. Again, D and E are not pseudo regular open fuzzy set for $0.4 \leq \alpha < 0.5$ and $0.1 \leq \alpha < 0.2$ respectively. Also, the *ps-ro* fuzzy topology on Y is $\{0, 1, C\}$. Define a function f from the fts (X, τ_1) to fts (Y, τ_2) by $f(a) = x, f(b) = y$ and $f(c) = z$. B is *ps-ro* open fuzzy set on X and $f(B)(t) = 0.4, \forall t \in Y$ but $f(B)$ is not *ps-ro* α-open fuzzy set on Y proving that f is not *ps-ro* fuzzy α-open function. It can be verified that $f(U)$ is fuzzy α-open set on Y for every open fuzzy set U on X. Hence, f is fuzzy α-open.
In Example 3.3, C is open fuzzy set on X and $f(C)(a) = 0.4, f(C)(b) = 0.5$ and $f(C)(c) = 0.5$ but $f(C)$ is not fuzzy α-open set on Y proving that f is not fuzzy α-open function. It can be verified that $f(U)$ is *ps-ro* α-open fuzzy set on Y for every *ps-ro* open fuzzy set U on X. Hence, f is *ps-ro* fuzzy α-open.

Theorem 4.5. *For a function f from a fts (X, τ_1) to another fts (Y, τ_2) the following are equivalent :*
(1) *f is ps-ro fuzzy α-open.*
(2) $f(\text{ps-int}(A)) \leq \text{ps-int}(\text{ps-cl}(\text{ps-int}(f(A))))$ $\forall$ *fuzzy set A on X.*
(3) $\text{ps-int}(f^{-1}(B)) \leq f^{-1}(\text{ps-int}(\text{ps-cl}(\text{ps-int}(B))))$ $\forall$ *fuzzy set B on Y.*

Proof. (1)$\Rightarrow$ (2): Let *ps-int*(A) be *ps-ro* open fuzzy set on X for fuzzy set A on X. Since f is *ps-ro* fuzzy α-open, $f($*ps-int*(A) is *ps-ro* α-open fuzzy set on Y. Thus

$$f(\text{ps-int}(A)) \leq \text{ps-int}(\text{ps-cl}(\text{ps-int}(f(\text{ps-int}(A))))) \leq \text{ps-int}(\text{ps-cl}(\text{ps-int}(f(A)))).$$

So $f(\text{ps-int}(A)) \leq \text{ps-int}(\text{ps-cl}(\text{ps-int}(f(A))))$.

(2)$\Rightarrow$ (3): Let B be any fuzzy set on Y. Then $f^{-1}(B) = A$ is fuzzy set on X. By given hypothesis, $f(\text{ps-int}(A)) \leq \text{ps-int}(\text{ps-cl}(\text{ps-int}(f(A))))$. Thus

$$f(\text{ps-int}(f^{-1}(B))) \leq \text{ps-int}(\text{ps-cl}(\text{ps-int}(f(f^{-1}(B))))) \leq \text{ps-int}(\text{ps-cl}(\text{ps-int}(B))).$$

This gives

$$\text{ps-int}(f^{-1}(B)) \leq f^{-1}(f(\text{ps-int}(f^{-1}(B)))) \leq f^{-1}(\text{ps-int}(\text{ps-cl}(\text{ps-int}(B)))).$$

So $\text{ps-int}(f^{-1}(B)) \leq f^{-1}(\text{ps-int}(\text{ps-cl}(\text{ps-int}(B))))$.

(3)$\Rightarrow$ (1): Let A be *ps-ro* open fuzzy set on X and $B = f(A)$ be a fuzzy set on Y. Then, by given hypothesis, $\text{ps-int}(f^{-1}(B)) \leq f^{-1}(\text{ps-int}(\text{ps-cl}(\text{ps-int}(B))))$. On the other hand,

$$A = \text{ps-int}(A) \leq \text{ps-int}(f^{-1}(f(A))) \leq f^{-1}(\text{ps-int}(\text{ps-cl}(\text{ps-int}(f(A))))).$$

Thus

$$f(A) \leq f(f^{-1}(\text{ps-int}(\text{ps-cl}(\text{ps-int}(f(A)))))) \leq \text{ps-int}(\text{ps-cl}(\text{ps-int}(f(A)))).$$

So $f(A)$ is *ps-ro* α-open fuzzy set on Y. Hence f is *ps-ro* fuzzy α-open. □

Theorem 4.6. *Let (X, τ_1) and (Y, τ_2) be two fts and $f : (X, \tau_1) \to (Y, \tau_2)$ be a ps-ro fuzzy α-open(closed) function. If B be any fuzzy set on Y and A is a ps-ro closed(open) fuzzy set on X, containing $f^{-1}(B)$, then $\exists$ ps-ro α-closed(α-open) fuzzy set C on Y such that $B \leq C$ and $f^{-1}(C) \leq A$.*

Proof. Let B be any fuzzy set on Y and A be *ps-ro* closed(open) fuzzy set on X containing $f^{-1}(B)$. Then $1 - A$ is *ps-ro* open(closed) fuzzy set on X. Since f is *ps-ro* fuzzy α-open(closed), $f(1 - A)$ is *ps-ro* α-open(closed) fuzzy set on Y. Let $C = 1 - f(1 - A)$. Then C is *ps-ro* α-closed(open) fuzzy set on Y. Since $f^{-1}(B) \leq A$, $f^{-1}(1 - B) = 1 - f^{-1}(B) \geq (1 - A)$. So, $(1 - B) \geq f(f^{-1}(1 - B)) \geq f(1 - A)$ and $B \leq C$. On one hand,

$$f^{-1}(C) = f^{-1}(1 - f(1 - A)) = 1 - f^{-1}(f(1 - A)) \leq 1 - (1 - A) = A.$$

So $f^{-1}(C) \leq A$. This completes the proof. □

Theorem 4.7. *If a function f from a fts (X, τ_1) to another fts (Y, τ_2) be ps-ro fuzzy α-open then $f^{-1}(ps\text{-}cl(ps\text{-}int(ps\text{-}cl(B)))) \leq ps\text{-}cl(f^{-1}(B)) \forall$ fuzzy set $B \in Y$.*

Proof. For any fuzzy set B on Y, $f^{-1}(B)$ is fuzzy set on X. So, $ps\text{-}cl(f^{-1}(B))$ is *ps-ro* closed fuzzy set on X containing $f^{-1}(B)$. By Theorem 4.6, $\exists$ *ps-ro* α-closed fuzzy set C on Y such that $B \leq C$ and $f^{-1}(C) \leq ps\text{-}cl(f^{-1}(B))$. Since $B \leq C$,

$$\begin{aligned} f^{-1}(ps\text{-}cl(ps\text{-}int(ps\text{-}cl(B)))) &\leq f^{-1}(ps\text{-}cl(ps\text{-}int(ps\text{-}cl(C)))) \\ &\leq f^{-1}(C) \leq ps\text{-}cl(f^{-1}(B)). \end{aligned}$$

Thus $f^{-1}(ps\text{-}cl(ps\text{-}int(ps\text{-}cl(B)))) \leq ps\text{-}cl(f^{-1}(B))$. □

Theorem 4.8. *Let $f : X \to Y$ and $g : Y \to Z$ be functions, where X,Y and Z are fts. Then*

(1) $g \circ f$ is ps-ro fuzzy α-open if f is ps-ro fuzzy open and g is ps-ro fuzzy α-open.

(2) $g \circ f$ is ps-ro fuzzy α-closed if f is ps-ro fuzzy closed and g is ps-ro fuzzy α-closed.

Proof. Follows from the fact that $g \circ f(A) = g(f(A))$, for each fuzzy set A on X. □

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