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On fuzzy H-closedness of fuzzy topological spaces

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Abstract. In the present paper, we have initiated fuzzy covering axiom weakly fuzzy H-closedness using the fuzzy closure operator which is a "good extension" of celebrated notion H-closedness of general topological space. We have achieved several characterizations of this notion via. fuzzy upper weak-θ-limits, fuzzy upper $\mathcal{I}$-θ-limits of fuzzy nets of fuzzy closed sets. We have also shown that weakly fuzzy H-closedness is preserved under fuzzy θ-continuous closed surjection. Besides these, we have studied another "good extension" of H-closed spaces, namely fuzzy $H^\star$-closedness along with their several characterizations in terms of net, α-net and α-filter. We have shown that closure of fuzzy open sets of fuzzy $H^\star$-closed spaces is fuzzy $H^\star$-closed. We have established that the concepts of fuzzy $H^\star$-closedness and fuzzy compactness of fuzzy topological space are equivalent in presence of the fuzzy regularity.

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1. Introduction

The fuzzy mathematics started its journey with the introduction of "fuzzy sets" by Zadeh [19] in 1965 and now it becomes a vast in collaborate with its different branches. One of its important branch is fuzzy topology which was first introduced by Chang [8]. Fuzzy compactness introduced by Chang [8] is not a "good extension" of compactness of ordinary topological space and it fails to satisfy most important properties of compactness of ordinary topology. Then Lowen introduced several notions of fuzzy covering axiom, namely weakly fuzzy compactness[12], strongly fuzzy compactness [13], fuzzy ultra-compactness[13] and fuzzy compactness [12] which are "good extension" of compactness of ordinary topology. Some of the recent reseach works related to fuzzy covering axioms are found in the papers [1, 2, 3, 4, 5, 6, 18].

The covering property H-closedness is the first celebrated generalization of compactness of ordinary topology. The book of Porter and Wood is a nice reference for H-closed spaces. This covering property satisfy most of the desire properties like :

(a) Product of H-closed spaces is H-closed.

(b) H-closedness is preserved under θ-continuous closed surjection.

(c) Closure of open sets of a H-closed space is H-closed, but closed sets need not be H-closed.

(d) Every regular H-closed space is compact.

In fuzzy topology, several mathematicians tried to fuzzyfy the concept of H-closedness of ordinary topology and the notions of almost compact fuzzy sets [14], fuzzy almost compact [9] and θ-compact [7] were initiated and investigated. But none of these notions is a good extension of H-closed topological space.

Observing this serious drawback of these notions, in this paper, we have introduced fuzzy covering axioms weakly fuzzy H-closedness and fuzzy $H^\star$-closed which are "good extension" of H-closed spaces. We have shown that the notions weakly fuzzy H-closedness and fuzzy $H^\star$-closedness have generalized the notions of weakly fuzzy compact and fuzzy compactness of Lowen [12] and in presence fuzzy regularity, the concepts of weakly fuzzy H-closedness (resp. fuzzy $H^\star$-closedness) and fuzzy compactness (resp. weakly fuzzy compactness) of fuzzy topological space are equivalent. Several characterizations of these notions via. fuzzy upper weak-θ-limits, fuzzy upper $\mathcal{I}$-θ-limits, θ-cluster point of fuzzy nets of fuzzy closed sets have been achieved. The concepts of fuzzy α-nets and α-filters were used by Guojun [11] to study the N-compactness of fuzzy topology. We have given characterizations of weakly fuzzy H-closed spaces and fuzzy $H^\star$-closed spaces in terms of θ-cluster points of fuzzy α-nets and α-filters. We have also shown that our new covering axioms are preserved under fuzzy θ-continuous closed surjection. Further, we have established that the product of two weakly fuzzy H-closed spaces is weakly fuzzy H-closed.

2. Preliminaries

Throughout this paper, spaces (X, σ) and (Y, δ) (or simply X and Y) represent non-empty fuzzy topological spaces due to Chang [8] and the symbols I and I^X have been used for the unit closed interval $[0, 1]$ and the set of all functions with domain X and codomain I respectively. The support of a fuzzy set A is the set $\{x \in X : A(x) > 0\}$ and is denoted by $supp(A)$. A fuzzy set with only non-zero value $\lambda \in (0, 1]$ at only one element $x \in X$ is called a fuzzy point and is denoted by x_λ and the set of all fuzzy points of a fuzzy topological space is denoted by $FP(X)$. For any two fuzzy sets A, B of X, $A \leq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$. A fuzzy point x_λ is said to be in a fuzzy set A (denoted by $x_\lambda \in A$) if $x_\lambda \leq A$, that is, if $\lambda \leq A(x)$. The constant fuzzy set of X with value $\alpha \in I$ is denoted by $\underline{\alpha}$. A fuzzy set A is said to be quasi-coincident with B (written as $A\hat{q}B$) [16] if $A(x) + B(x) > 1$ for some $x \in X$. A fuzzy open set A of X is called fuzzy quasi-neighborhood of a fuzzy point x_λ if $x_\lambda \hat{q} A$.

It is well-known that a function $\psi : X \to Y$ is fuzzy θ-continuous [15] if for every fuzzy point x_λ and every fuzzy quasi-neighborhood V of a fuzzy point $\psi(x_\lambda)$, there exists a fuzzy quasi-neighborhood U of a fuzzy point x_λ such that $\psi(cl(U)) \leq cl(V)$.

Throughout the paper, $\mathcal{D}$ stands for a directed set. An ideal on a non-empty set S is defined as a non-empty family $\mathcal{I}$ of subsets of S satisfying:

(i) $\varnothing \in \mathcal{I}$,

(ii) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$ and

(iii) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.

An ideal $\mathcal{I}$ on $\mathcal{D}$ is called non-trivial if $\mathcal{D} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ on $\mathcal{D}$ is called admissible [10] if $\mathcal{D} - M_\lambda \in \mathcal{I}$, where $M_\lambda = \{n \in \mathcal{D} : n \geq \lambda\}$ for all $\lambda \in \mathcal{D}$. Throughout this paper, $\mathcal{I}$ stands for an admissible ideal on $\mathcal{D}$.

Lowen [13] defined the notion of prefilter. A non-empty collection $\mathcal{F}$ of non-zero fuzzy subsets of X is called a prefilter(=filter) if

(i) $\mu \wedge \nu \in \mathcal{F}$ for all $\mu, \nu \in \mathcal{F}$ and

(ii) $\mu \leq \nu$ and $\mu \in \mathcal{F}$ implies $\nu \in \mathcal{F}$.

A non-empty collection $\mathcal{B}$ of non-zero fuzzy subsets of X is called a filterbase if for all $\mu, \nu \in \mathcal{F}$, there exists an $\eta \in \mathcal{F}$ such that $\eta \leq \mu \wedge \nu$. Clearly, the family $< \mathcal{B} >= \{\nu \in I^X : \mu \leq \nu, \mu \in \mathcal{B}\}$ is a filter.

A prefilter $\mathcal{F}$ is called prime if $\mu \vee \nu \in \mathcal{F}$ implies either $\mu \in \mathcal{F}$ or $\nu \in \mathcal{F}$. The collection all prime prefilters finer than a given prefilter $\mathcal{F}$ is denoted by $\mathcal{P}(\mathcal{F})$. The set of minimal elements of $\mathcal{P}(\mathcal{F})$ is denoted by $\mathcal{P}_m(\mathcal{F})$.

A filter $\mathcal{G}$ on X and a prefilter $\mathcal{F}$ on X are said to be compatible if $\mu(x) \neq 0$ for some $x \in F$ for all $\mu \in \mathcal{F}$ and $F \in \mathcal{G}$. Suppose μ_F be the fuzzy set on X defined by $\mu_F(x) = \mu(x)$ for all $x \in F$ and $\mu_F(x) = 0$ for all $x \in X - F$. We use the symbol $(\mathcal{F}, \mathcal{G}) =< \{\mu_F : \mu \in \mathcal{F}, F \in \mathcal{G}\} >$. Lowen [13] shaw that $\mathcal{P}_m(\mathcal{F}) = \{(\mathcal{F}, \mathcal{U}) : \mathcal{U}$ is an ultrafilter on X and is compatible with $\mathcal{F}\}$. The adherence of a prefilter $\mathcal{F}$ on X is the fuzzy set $ad\mathcal{F}$ defined by $ad\mathcal{F}(x) = \inf\{cl(\mu)(x) : \mu \in \mathcal{F}\}$ for all $x \in X$ and the limit of $\mathcal{F}$ is the fuzzy set $\lim \mathcal{F}$ defined by $\lim \mathcal{F}(x) = \inf\{ad\mathcal{B}(x) : \mathcal{B} \in \mathcal{P}_m(\mathcal{F})\}$ for all $x \in X$.

Let $\mathcal{F}$ be a prefilter on X and $\mu \in I^X$. Then the characteristic set of $\mathcal{F}$ with respect to μ is the set $\mathcal{C}^\mu(\mathcal{F}) = \{a \in I : \forall \nu \in \mathcal{F}, \exists x \in X$ such that $\nu(x) > \mu(x) + a\}$ and the spermium of this set is called the characteristic value of $\mathcal{F}$ with respect to μ is the set $c^\mu(\mathcal{F})$. A characteristic set is one of the form $\varnothing$, $\{0\}$, $[0, c]$ for some $c \in I - \{1\}$ or $[0, c[$ for some $c \in I$. We use the following notations:

(i) $\mathcal{W}(X)$, the set of all pefilters on X,

(ii) $\mathcal{W}^\mu(X) = \{\mathcal{F} \in \mathcal{W}(X) : \mathcal{C}^\mu(\mathcal{F}) \neq \varnothing\}$,

(iii) $\mathcal{W}^\mu_+(X) = \{\mathcal{F} \in \mathcal{W}(X) : c^\mu(\mathcal{F}) > 0\}$ and

(iv) $\mathcal{W}^\mu_K(X) = \{\mathcal{F} \in \mathcal{W}(X) : c^\mu(\mathcal{F}) = K\}$,

where K is some nonempty characteristic set.

A member of $\mathcal{W}^\mu_K(X)$ is called K-filter. A prefilter $\mathcal{F}$ is called ϵ-prefilter if $\mu \not\leq \epsilon$ for each $\mu \in \mathcal{F}$.

3. Weakly fuzzy H-closed topological spaces

We first recall the definition of fuzzy almost compact set due to M. N. Mukherjee and S. P. Sinha [14]. A fuzzy subset $\eta \in I^X$ is called fuzzy almost compact if for each family $\{\mu_k \in \delta : k \in \mathbb{N}\}$ satisfying $\bigvee_{i=1}^{\infty} \mu_i \geq \eta$, there exist finite number of indices $k_1, k_2, ..., k_p \in \mathbb{N}$ such that $\bigvee_{i=1}^{p} cl(\mu_{k_i}) \geq \eta$. Here $\{\mu_{k_i} : i = 1, 2, ..., p\}$ is called a fuzzy finite proximate subcover of the cover $\{\mu_k \in \delta : k \in \mathbb{N}\}$ for η. If $\eta = \underline{1}$ and η is a fuzzy almost compact set, then X is called a fuzzy almost compact space due to A. Di Concilio and G. Gerla [9].

The following example shows that fuzzy almost compactness due to A. Di Concilio and G. Gerla [9] and θ-compactness due to M. Caldas and S. Jafari [7] of fuzzy topological space are not "good extensions" of H-compactness of general topological space.

Example 3.1. Let X be the closed unit interval $[0,1]$ with usual topology τ. Then (X,τ) is compact and so H-closed. But since the cover $\{\mu_n : n \in \mathbb{N}\}$, where $\mu_n(x) = 1 - \frac{1}{n}$ for $(X,\omega(\tau))$ consists of fuzzy clopen sets that has no finite subcover for $(X,\omega(\tau))$, X is neither fuzzy almost compact nor fuzzy θ-compact.

Now we propose a new definition, namely weakly fuzzy H-closedness following weakly fuzzy compactness due to Lowen [12] that is a "good extension" of H-closedness of general topological space.

Definition 3.2. A fuzzy subset $\eta \in I^X$ is called weakly fuzzy H-set if for each family $\{\mu_\alpha \in \delta : \alpha \in \Delta\}$ satisfying $\bigvee\{\mu_\alpha \in \delta : \alpha \in \Delta\} \geq \eta$ and for each $\epsilon > 0$, there exist finite number of indices $\alpha_1, \alpha_2, ..., \alpha_p \in \Delta$ such that $\bigvee_{i=1}^{p} cl(\mu_{\alpha_i}) \geq \underline{\eta - \epsilon}$. If $\eta = \underline{1}$ and η is a weakly fuzzy H-set, then X is called a weakly fuzzy H-closed space.

Following Theorem shows that weakly H-closedness of fuzzy topological space is a "good extension" of H-closedness of general topological space.

Theorem 3.3. (1) *If (X,τ) is an H-closed topological space, every constant fuzzy subset η of the fuzzy topological space $(X,\omega(\tau))$ is weakly fuzzy H-set.*

(2) *If $(X,\omega(\tau))$ is a weakly fuzzy H-closed space, (X,τ) is H-closed.*

Proof. (1) Let (X,τ) be a H-closed space, η be a constant fuzzy subset of the fuzzy topological space $(X,\omega(\tau))$ with value h and $\epsilon \in (0,h)$. Let $\Omega = \{\mu_\alpha \in \delta : \alpha \in \Delta\}$ be a family satisfying $\bigvee_{\alpha\in\Delta} \mu_\alpha \geq \eta$. Define $[0,h] = I_h$ and $\Omega(\mu_\alpha) = \{(x,r) \in X \times \mathbb{R} : \mu_\alpha(x) > r - \epsilon\}$ for each $\alpha \in \Delta$. Then $\{\Omega(\mu_\alpha) : \alpha \in \Delta\}$ is a cover of $X \times I_h$ consists of open sets of $X \times \mathbb{R}$. Since I_h is H-closed and so $X \times I_h$ is H-closed, there exist finite number of indices $\alpha_1, \alpha_2, ..., \alpha_p \in \Delta$ such that $\bigcup_{i=1}^{p} cl(\Omega(\mu_{\alpha_i})) \supset X \times I_h$. We claim that $\bigvee_{i=1}^{p} cl(\mu_{\alpha_i}) \geq \underline{\eta - \epsilon}$. If possible, let there exists an $x \in X$ such that $cl(\mu_{\alpha_i})(x) < \eta - \epsilon$ for each $i = 1, 2, ..., p$. We shall show that $(x,\epsilon) \notin cl(\Omega(\mu_{\alpha_i}))$ for each $i = 1, 2, ..., p$. We fix an $i \in \{1, 2, ..., p\}$. Since $\underline{1} - cl(\mu_{\alpha_i}) \in (X,\omega(\tau))$, there exists an open set $U \in \tau$ containing x such that $1 - cl(\mu_{\alpha_i})(y) > 1 - cl(\mu_{\alpha_i})(x) - \epsilon$, i.e. $cl(\mu_{\alpha_i})(y) < cl(\mu_{\alpha_i})(x) + \epsilon$ for all $y \in U$. Then it is easy to verify that $U \times (h-\epsilon, h)$ is an open set in $X \times \mathbb{R}$ containing (x,ϵ) with $(U \times (h-\epsilon,h)) \bigcap \Omega(\mu_{\alpha_i}) = \varnothing$.

(2) let $(X,\omega(\tau))$ be weakly fuzzy H-closed. Suppose $\{U_\alpha : \alpha \in \Delta\}$ be an open cover of X. Then clearly, $\bigvee\{\chi_{U_\alpha} : \alpha \in \Delta\} = \underline{1}$. Then for each $\epsilon > 0$, there exist finite number of indices $\alpha_1, \alpha_2, ..., \alpha_p \in \Delta$ such that $\bigvee_{i=1}^{p} cl(\chi_{U_{\alpha_i}}) \geq \underline{1 - \epsilon}$. We claim that $\{cl(U_{\alpha_i}) : i = 1, 2, ..., p\}$ is a subcover of X as desired. In fact, if $x \in X - \bigcup\{cl(U_{\alpha_i}) : i = 1, 2, ..., p\}$, then for each $\epsilon \in (0,1)$, $x_{1-\epsilon} \notin cl(\chi_{U_{\alpha_i}})$ and so $cl(\chi_{U_{\alpha_i}})(x) < 1 - \epsilon$ for each $i = 1, 2, ..., p$, which is impossible. □

Corollary 3.4. *The fuzzy topological space $(X,\omega(\tau))$ is weakly fuzzy H-closed if and only if (X,τ) is H-closed.*

Remark 3.5. (1) Every fuzzy almost compact space due to A. Di Concilio and G. Gerla [9] is weakly fuzzy H-closed.

(2) Every weakly fuzzy compact space due to Lowen [12] is weakly fuzzy H-closed.

Following example shows that none of above statements are reversible.

Example 3.6. (1) Let (X, τ) be any infinite H-closed space. Then by Corollary 3.4, $(X, \omega(\tau))$ is weakly fuzzy H-closed. But the space $(X, \omega(\tau))$ is not fuzzy almost compact as the fuzzy clopen cover $\{\mu_\alpha : \alpha \in [0,1]\}$, $\mu_\alpha(x) = \alpha$ for all $x \in X$, has no finite fuzzy subcover for X.

(2) Consider the topological space $Z = \{(\frac{1}{p}, \frac{1}{q}) : p \in \mathbf{N}, \mid q \mid \in \mathbf{N}\} \cup \{(\frac{1}{p}, 0) : p \in \mathbf{N}\}$ equipped with the subspace usual topology inherited from the usual Euclidean space $\mathbf{R}^2$. Consider the space $X = Z \cup \{s, t\}$ with the topology τ defined by $U \in \tau$ if $U \cap Z$ is open in Z and $\{(\frac{1}{p}, \frac{1}{q}) : p \geq r, q \in \mathbf{N}\} \subset U$ for some $r \in \mathbf{N}$ when $s \in U$ and $\{(\frac{1}{p}, \frac{1}{q}) : p \geq r, -q \in \mathbf{N}\} \subset U$ for some $r \in \mathbf{N}$ when $t \in U$. Then (X, τ) is non-compact H-closed space [17]. Applying Corollary 3.4 and Theorem 4.1 [12], we can conclude that $(X, \omega(\tau))$ is weakly fuzzy H-closed, but not weakly fuzzy compact.

Theorem 3.7. *Let X be a fuzzy regular space. Then X is weakly fuzzy H-closed if and only if X is weakly fuzzy compact.*

Proof. Let X be fuzzy regular and weakly fuzzy H-closed. Let $\{\mu_\alpha \in \delta : \alpha \in \Delta\}$ be a family satisfying $\bigvee\{\mu_\alpha \in \delta : \alpha \in \Delta\} = \underline{1}$ and $\epsilon > 0$. Since X is regular, for each $\alpha \in \Delta$, there exists a family $\{\nu^\alpha_\beta \in \delta : \beta \in \Delta_\alpha\}$ such that $\mu_\alpha = \bigvee\{\nu^\alpha_\beta \in \delta : \beta \in \Delta_\alpha\}$ and $cl(\nu_{\alpha_\beta}) \leq \mu_\alpha$ for each $\beta \in \Delta_\alpha$. Since $\bigvee\{\nu^\alpha_\beta \in \delta : \beta \in \Delta_\alpha, \alpha \in \Delta\} = \underline{1}$ and X is weakly fuzzy H-closed, there exist finite subsets $\Delta_0 \subset \Delta$ and $\Delta^0_\alpha \subset \Delta_\alpha$ for each $\alpha \in \Delta_0$ such that $\bigvee\{cl(\nu^\alpha_\beta) \in \delta : \beta \in \Delta^0_\alpha, \alpha \in \Delta_0\} \geq \underline{1 - \epsilon}$, i.e. $\bigvee\{\mu_\alpha \in \delta : \alpha \in \Delta_0\} \geq \bigvee\{\bigvee\{cl(\nu^\alpha_\beta) \in \delta : \beta \in \Delta^0_\alpha\}, \alpha \in \Delta_0\} \geq \underline{1 - \epsilon}$. Hence X is weakly fuzzy compact.

Converse follows from remark 3.5 (ii). □

Theorem 3.8. *Let X be a weakly fuzzy H-closed topological space, Y be a fuzzy Hausdorff space and $\psi : X \to Y$ be a fuzzy θ-continuous surjection. Then Y is weakly fuzzy H-closed.*

Proof. Let $\{\mu_\alpha \in \sigma : \alpha \in \Delta\}$ be a family satisfying $\bigvee\{\mu_\alpha \in \sigma : \alpha \in \Delta\} = \underline{1}_Y$ and $\epsilon > 0$. Let $x_\lambda \in Pt(X)$. Then there exists $\alpha \in \Delta$ such that $\psi(x)_\lambda \in \mu_\alpha$. Since ψ is fuzzy θ-continuous, there exists an $\nu_\alpha \in \delta$ such that $\psi(cl(\nu_\alpha)) \leq cl(\mu_\alpha)$. Here $\{\nu_\alpha \in \delta : \alpha \in \Delta\}$ form a cover for X and so there exist finite number of indices $\alpha_1, \alpha_2, ..., \alpha_p \in \Delta$ such that $\bigvee_{i=1}^p cl(\nu_{\alpha_i}) \geq \underline{1_X - \epsilon}$. Since ψ is surjective, $\bigvee_{i=1}^p cl(\mu_{\alpha_i}) \geq \underline{1_Y - \epsilon}$. □

Definition 3.9. Let $\{A_n : n \in \mathcal{D}\}$ be a net of fuzzy sets of a fuzzy topological space X. The fuzzy upper weak-θ-limit of $\{A_n : n \in \mathcal{D}\}$ is defined and denoted by $FUWL_\theta(A_n) = \vee\{x_\lambda \in FP(X)$: for every fuzzy quasi-neighborhood U of x_λ and for every $n_0 \in \mathcal{D}$, there exists an $n(\geq n_0) \in \mathcal{D}$ such that $int(A_n)\hat{q}cl(U)\}$.

Theorem 3.10. *A fuzzy topological space X is weakly fuzzy H-closed if and only if for every net $\{F_n : n \in \mathcal{D}\}$ of fuzzy closed sets with $FUWL_\theta(F_n) = \bar{0}$ and for every $\epsilon > 0$, there exists an $n_0 \in \mathcal{D}$ such that $int(F_n) \leq \underline{\epsilon}$ for all $n \geq n_0$.*

Proof. Let X be weakly fuzzy H-closed and $\{F_n : n \in \mathcal{D}\}$ be a net of fuzzy closed sets with $FUWL_\theta(F_n) = \bar{0}$ and $\epsilon > 0$. Then for each fuzzy point x_λ of X, there exist a fuzzy quasi-neighborhood U_{x_λ} of x_λ and an $n(x_\lambda)$ such that $int(F_n)\bar{q}cl(U_{x_\lambda})$ for all $n \geq n(x_\lambda)$. Since X is weakly fuzzy H-closed and $\{U_{x_\lambda} : x_\lambda \in FP(X)\}$ is a fuzzy open cover of X, there exists finite number of fuzzy points $e_1, e_2, ..e_p \in FP(X)$ such that $\bigvee_{i=1}^{p} cl(U_{e_i}) \geq \underline{1-\epsilon}$. Choose an $n_0 \in \mathcal{D}$ such that $n \geq n(e_i)$ for all $i = 1, 2, ..., p$. Then $int(F_n)\bar{q}cl(U_{e_i})$ for all $n \geq n_0$ and for each $i = 1, 2, ..., p$ and so $int(F_n)\bar{q}\bigvee_{i=1}^{p} cl(U_{e_i})$ for all $n \geq n_0$. Thus for all $x \in X$ and for all $n \geq n_0$, $int(F_n)(x) + \bigvee_{i=1}^{p} cl(U_{e_i})(x) \leq 1$, i.e. $int(F_n)(x) \leq \epsilon$.

Conversely, let the condition of the theorem holds and Ω be a fuzzy open cover of X. Let the set $\mathcal{D} = \{n \subset \Omega :| n |< \aleph_0\}$ be directed by the relation "$\leq$" defined by $n_1 \leq n_2$ if and only if $n_2 \subset n_1$. Consider the fuzzy net $\{F_n : n \in \mathcal{D}\}, F_n = \bigwedge_{\mu \in n} \mu'$. We claim that $FUWL_\theta(F_n) = \bar{0}$. First we note that $\{F'_n : n \in \mathcal{D}\}$ is a fuzzy cover of X. Let $x_\lambda \in FP(X)$. Then there exists an $n(x_\lambda) \in \mathcal{D}$ such that $x_\lambda \notin F_{n(x_\lambda)}$ and so $F'_{n(x_\lambda)}$ is a fuzzy quasi-neighborhood of x_λ and for every $n(> n(x_\lambda)) \in \mathcal{D}$, $F'_{n(x_\lambda)}\bar{q}F_n$. So, for all $x \in X$ and for all $n > n(x_\lambda)$, $F'_{n(x_\lambda)}(x) + F_n(x) \leq 1$, i.e. $F'_{n(x_\lambda)}(x) \leq 1 - F_n(x)$, i.e. $cl(F'_{n(x_\lambda)})\bar{q}int(F_n)$. So $FUWL_\theta(F_n) = \bar{0}$. Then by the condition of the theorem, for every $\epsilon > 0$, there exists an $n_0 \in \mathcal{D}$ such that $int(F_n) \leq \underline{\epsilon}$ for all $n \geq n_0$. Now select the family $\{\mu \in \Omega : \mu \in n_0\}$. We claim that $\bigvee\{cl(\mu) \in \Omega : \mu \in n_0\} \geq \underline{1-\epsilon}$. In fact, $\underline{1-\epsilon} \leq \underline{1} - int(F_{n_0}) = \underline{1} - int(\bigwedge_{\mu \in n_0} \mu') = \bigvee_{\mu \in n_0} cl(\mu)$. Thus X is weakly fuzzy H-closed. □

Definition 3.11. Let $\{A_n : n \in \mathcal{D}\}$ be a fuzzy net of fuzzy sets of a fuzzy topological space X. Then the fuzzy upper $\mathcal{I}$-θ-limit of $\{A_n : n \in \mathcal{D}\}$ is defined and denoted by $FIUL_\theta(A_n) = \vee\{x_\lambda \in FP(X)$: for every fuzzy quasi-neighborhood U of x_λ, $\{n \in \mathcal{D} : A_n\hat{q}cl(U)\} \notin \mathcal{I}\}$.

Theorem 3.12. *Let X be a weakly fuzzy H-closed topological space. Then for every fuzzy net $\{F_n : n \in \mathcal{D}\}$ of fuzzy closed sets, for every ideal $\mathcal{I}$ on $\mathcal{D}$ with $FIUL_\theta(F_n) = \bar{0}$ and for every $\epsilon > 0$, $\{n \in \mathcal{D} : F_n \not\leq \underline{\epsilon}\} \in \mathcal{I}$.*

Proof. Let X be weakly fuzzy H-closed and $\{F_n : n \in \mathcal{D}\}$ be a fuzzy net of fuzzy closed sets, $\mathcal{I}$ be an ideal on $\mathcal{D}$ with $FIUL_\theta(F_n) = \bar{0}$ and $\epsilon > 0$. Then for each fuzzy point x_λ of X, there exists a fuzzy quasi-neighborhood U_{x_λ} of x_λ such that $\{n \in \mathcal{D} : F_n\hat{q}cl(U_{x_\lambda})\} \in \mathcal{I}$. Since X is weakly fuzzy H-closed and $\{U_{x_\lambda} : x_\lambda \in FP(X)\}$ is a fuzzy open cover of X, there exists finite number of fuzzy points $e_1, e_2, ..e_p \in FP(X)$ such that $\bigvee_{i=1}^{p} cl(U_{e_i}) \geq \underline{1-\epsilon}$. Here $\{n \in \mathcal{D} : F_n\hat{q}\bigvee_{i=1}^{p} cl(U_{e_i})\} = \bigcup_{i=1}^{p}\{n \in \mathcal{D} : F_n\hat{q}cl(U_{e_i})\} \in \mathcal{I}$. Since $\{n \in \mathcal{D} : F_n \not\leq \underline{\epsilon}\} \subset \{n \in \mathcal{D} : F_n\hat{q}\bigvee_{i=1}^{p} cl(U_{e_i})\}$, $\{n \in \mathcal{D} : F_n \not\leq \underline{\epsilon}\} \in \mathcal{I}$. □

Theorem 3.13. *Let (X, δ) be a weakly fuzzy H-closed topologically generated space and $U \in \delta$. Then $cl(U)$ is weakly fuzzy H-closed.*

Proof. Let τ be a topology on X that generates δ, i.e. $\omega(\tau) = \delta$. Suppose $\Sigma = \{U_\alpha \in \delta : \alpha \in \Delta\}$ is a family satisfying $\bigvee\{U_\alpha \in \delta : \alpha \in \Delta\} \geq \underline{cl(U)}$. Since $\underline{1} - cl(U) \in \delta$, $G = \{(x, r) : U(x) < r\}$ is an open set in H-closed space $\overline{X \times I}$, $cl(G)$ is H-closed. Now let $\epsilon > 0$ and define

$$\Omega(U_\alpha) = \{(x, r) \in X \times \mathbb{R} : U_\alpha(x) > r - \epsilon\} \text{ for each } \alpha \in \Delta.$$

Then $\bigcup\{\Omega(U_\alpha) : \alpha \in \Delta\} \supset cl(G)$. Since $cl(G)$ is H-closed, there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \Delta$ such that $\bigcup\{cl(\Omega(U_{\alpha_i})) : i = 1, 2, ..., n\} \supset cl(G)$. Thus $\bigvee\{cl(U_{\alpha_i}) : i = 1, 2, ..., n\} \supset cl(U) - \epsilon$. So $cl(U)$ is weakly fuzzy H-closed. □

Theorem 3.14. *Let X be a fuzzy almost compact topological space. Then for every fuzzy net $\{F_n : n \in \mathcal{D}\}$ of fuzzy closed sets, for every ideal $\mathcal{I}$ on $\mathcal{D}$ with $FIUL_\theta(F_n) = \bar{0}$, $\{n \in \mathcal{D} : F_n \neq \underline{0}\} \in \mathcal{I}$.*

Proof. Let X be a fuzzy almost compact space and $\{F_n : n \in \mathcal{D}\}$ be a fuzzy net of fuzzy closed sets, $\mathcal{I}$ be an ideal on $\mathcal{D}$ with $FIUL_\theta(F_n) = \underline{0}$. Then for each fuzzy point x_λ of X, there exists a fuzzy quasi-neighborhood U_{x_λ} of x_λ such that $\{n \in \mathcal{D} : F_n \hat{q} cl(U_{x_\lambda})\} \in \mathcal{I}$. Since X is fuzzy almost compact and $\{U_{x_\lambda} : x_\lambda \in FP(X)\}$ is a fuzzy open cover of X, there exists finite number of fuzzy points $e_1, e_2, ..., e_p \in FP(X)$ such that $\bigvee_{i=1}^{p} cl(U_{e_i}) = \underline{1}_X$. Here $\{n \in \mathcal{D} : F_n \hat{q} \bigvee_{i=1}^{p} cl(U_{e_i})\} = \bigcup_{i=1}^{p}\{n \in \mathcal{D} : F_n \hat{q} cl(U_{e_i})\} \in \mathcal{I}$. Since $\{n \in \mathcal{D} : F_n \neq \underline{0}\} \subset \{n \in \mathcal{D} : F_n \hat{q} \bigvee_{i=1}^{p} cl(U_{e_i})\}$, $\{n \in \mathcal{D} : F_n \neq \underline{0}\} \in \mathcal{I}$. □

Theorem 3.15. *A fuzzy topological space X is weakly fuzzy H-closed if and only if for each fuzzy open ϵ-prefilter $\mathcal{F}$, $ad\mathcal{F} \neq \underline{0}$.*

Proof. Let X be weakly fuzzy H-closed and $\mathcal{F} \in \mathcal{W}_{+}^{\alpha}(X)$ be a fuzzy open ϵ-prefilter such that $ad\mathcal{F} = \underline{0}$. Consider the family $\Sigma = \{\underline{1} - cl(\mu) : \mu \in \mathcal{F}\}$. Then

$$\bigvee\{\underline{1} - cl(\mu) : \mu \in \mathcal{F}\} = \underline{1}.$$

Let $\epsilon > 0$. Then there exists finite number of members $\mu_1, \mu_2, ..., \mu_n \in \mathcal{F}$ such that $\bigvee_{i=1}^{n} cl(\underline{1} - cl(\mu_i)) \geq \underline{1 - \epsilon}$. On one hand,

$$\bigvee_{i=1}^{n} cl(\underline{1} - cl(\mu_i)) = \bigvee_{i=1}^{n}(\underline{1} - int(cl(\mu_i))) = \underline{1} - \bigvee_{i=1}^{n} int(cl(\mu_i)) \leq \underline{1} - \bigvee_{i=1}^{n} \mu_i.$$

Thus $\bigvee_{i=1}^{n} \mu_i \leq \epsilon$. So $\mathcal{F}$ is not a fuzzy open ϵ-prefilter.

Conversely, let X be not weakly fuzzy H-closed. Then there exists a family $\{\mu_\alpha \in \delta : \alpha \in \Delta\}$ satisfying $\bigvee\{\mu_\alpha \in \delta : \alpha \in \Delta\} = \underline{1}$
and an $\epsilon > 0$ such that for each finite set $A \subset \Delta$,

$$\bigvee\{cl(\mu_\alpha) : \alpha \in A\} \not\geq \underline{1 - \epsilon}.$$

Consider the family $\mathcal{F} = \{\mu \in \delta : \mu \geq \underline{1} - \bigvee\{cl(\mu_\alpha) : \alpha \in A\}\}$. Then $\mathcal{F}$ is a fuzzy open ϵ-prefilter on X. Thus

$$\begin{aligned} ad\mathcal{F} &= \bigwedge\{cl(\mu) : \mu \in \mathcal{F}\} \\ &\leq \bigwedge\{cl(\underline{1} - \bigvee\{cl(\mu_\alpha) : \alpha \in A\}) : A \subset \Delta \text{ is finite}\} \\ &\leq \underline{1} - \bigvee\{\mu_\alpha : \alpha \in \Delta\}. \end{aligned}$$

So $ad\mathcal{F} = \underline{0}$. □

Theorem 3.16. *$X \times Y$ is weakly fuzzy H-closed if and only if X and Y are weakly fuzzy H-closed.*

Proof. Let $X \times Y$ be weakly fuzzy H-closed. Since the projections on X and Y are continuous (and so θ-continuous) surjection, by Theorem 3.8, X and Y are weakly fuzzy H-closed.

Conversely, let X and Y be weakly fuzzy H-closed. Let $\mathcal{F}$ be a fuzzy open ϵ-prefilter on $X \times Y$. We claim that $\mathcal{F}_X = \{\pi_X(\mu) : \mu \in \mathcal{F}\}$ and $\mathcal{F}_Y = \{\pi_Y(\mu) : \mu \in \mathcal{F}\}$ are fuzzy open ϵ-prefilters on X and Y respectively. Let $\pi_X(\mu) \in \mathcal{F}_X$ and $\eta \in I^X$ such that $\pi_X(\mu) \leq \eta$. Then $\mu \leq \pi_X^{-1}(\eta)$. Since $\mathcal{F}$ is a prefilter on $X \times Y$, $\pi_X^{-1}(\eta) \in \mathcal{F}$. Thus $\eta = \pi_X(\pi_X^{-1}(\eta)) \in \mathcal{F}_X$. Let $\mu, \nu \in \mathcal{F}$. Then $\mu \wedge \nu \in \mathcal{F}$. Now $\pi_X(\mu \wedge \nu) \leq \pi_X(\mu) \wedge \pi_X(\nu)$ implies that $\pi_X(\mu) \wedge \pi_X(\nu) \in \mathcal{F}_X$. Thus $\mathcal{F}_X$ is fuzzy open ϵ-prefilter on X. Similarly, $\mathcal{F}_Y$ is fuzzy open ϵ-prefilter on Y. Since X and Y are weakly fuzzy H-closed, $ad\mathcal{F}_X \neq \underline{0}_X$ and $ad\mathcal{F}_Y \neq \underline{0}_Y$. Now

$$\begin{aligned} ad\mathcal{F} &= \bigwedge\{cl(\mu \times \nu) : \mu \times \nu \in \mathcal{F}\} \\ &= \bigwedge\{cl_X(\mu) \times cl_Y(\nu) : \mu \times \nu \in \mathcal{F}\} \\ &\geq \bigwedge\{cl_X(\mu) : \mu \in \mathcal{F}_X\} \times \bigwedge\{cl_X(\mu) : \mu \in \mathcal{F}_X\} \\ &= ad\mathcal{F}_X \times ad\mathcal{F}_Y \neq \underline{0}_{X \times Y}. \end{aligned}$$

So $X \times Y$ is weakly fuzzy H-closed. □

4. Fuzzy $H^\star$-closed spaces

Definition 4.1. A fuzzy topological space X is called fuzzy $H^\star$-closed if for each $\alpha \in (0, 1]$, for each $\beta \in (0, \alpha)$ and for each family $\mathcal{U}_{\alpha\beta}$ of fuzzy open sets with the property that $\bigvee \mathcal{U}_{\alpha\beta} \geq \underline{\alpha}$, there exists finite subfamily $\mathcal{U}^0_{\alpha\beta}$ satisfying $\bigvee cl(\mathcal{U}^0_{\alpha\beta}) \geq \underline{\alpha - \beta}$.

Remark 4.2. A fuzzy topological space X is fuzzy $H^\star$-closed if and only if each constant fuzzy subset of X is weakly fuzzy H-set of X.

Theorem 4.3. *A topological space (X, τ) is H-closed if and only if the fuzzy topological space $(X, \omega(\tau))$ is fuzzy $H^\star$-closed.*

Proof. The proof follows from the result of Theorem 3.3. □

Remark 4.4. Clearly, every fuzzy compact space due to Lowen [12] is $H^\star$-closed, but the converse is not true in general. In fact, the fuzzy topological space $(X, \omega(\tau))$ of Example 3.6(2) is fuzzy $H^\star$-closed, but not fuzzy compact due to Lowen [12].

Theorem 4.5. *A fuzzy topological space X is fuzzy $H^\star$-closed if and only if for each $\alpha \in [0, 1)$, for each fuzzy net $\{F_n : n \in \mathcal{D}\}$ of fuzzy closed sets with $FUWL_\theta(F_n) \leq \underline{1 - \alpha}$ and for each $\beta \in (0, \alpha)$, there exists an $n_0 \in \mathcal{D}$ such that $int(F_n) \leq \underline{1 - \alpha + \beta}\}$ for all $n \geq n_0$.*

Proof. Let X be fuzzy $H^\star$-closed, $\alpha \in [0, 1)$ and $\{F_n : n \in \mathcal{D}\}$ be a net of fuzzy closed sets with $FUWL_\theta(F_n) \leq \underline{1 - \alpha}$ and $\beta \in (0, \alpha)$. Then for each fuzzy point x_λ of X, there exist $U_{x_\lambda} \in \mathcal{Q}(X, x_\lambda)$ for which $FUWL_\theta(F_n) < x_\lambda \leq \underline{1 - \alpha + \frac{\beta}{2}}$ and an $n(x_\lambda)$ such that $int(F_n)\bar{q}cl(U_{x_\lambda})$, i.e. $cl(U_{x_\lambda}) \leq \underline{1} - int(F_n)$ for all $n \geq n(x_\lambda)$. Since $U_{x_\lambda} \in \mathcal{Q}(X, x_\lambda)$, $U_{x_\lambda}(x) > 1 - \lambda \geq \alpha - \frac{\beta}{2}$. Consider the family $\mathcal{U}_\alpha = \{U_{x_\lambda} : x_\lambda \in Pt(X), FUWL_\theta(F_n) < x_\lambda \leq \underline{1 - \alpha + \frac{\beta}{2}}\}$. Since $\bigvee\{F_{n(x_\lambda)} : x_\lambda \in Pt(X)\} \geq$

$\alpha - \frac{\beta}{2}$, the fuzzy $H^\star$-closedness of X ensures the existence of finite number of points $x^1_{\lambda_1}, x^2_{\lambda_2}, ..., x^n_{\lambda_n} \in Pt(X)$ such that $\bigvee\{cl(U_{x^i_{\lambda_i}} : i = 1, 2, ..., n\} \geq \underline{\alpha - \beta}$. Consider $n_0 \in \mathcal{D}$ such that $n_0 \geq n(x^i_{\lambda_i})$ for each $i \in \{1, 2, ...n\}$. So, $\underline{1} - int(F_n) \geq \bigvee\{cl(U_{x^i_{\lambda_i}}) : i = 1, 2, ..., n\} > \underline{\alpha - \beta}$ for all $n \geq n_0$. Hence $int(F_n) < \underline{1 - \alpha + \beta}$ for all $n \geq n_0$.

Conversely, let the condition of the theorem holds. Suppose $\alpha \in (0, 1]$, $\beta \in (0, \alpha)$ and $\mathcal{U}_{\alpha\beta}$ be a family of fuzzy open sets with the property that $\bigvee \mathcal{U}_{\alpha\beta} \geq \underline{\alpha}$. Let the $\mathcal{D} = \{n \subset \mathcal{U}_\alpha :| n |< \aleph_0\}$ be directed by the relation "$\leq$" defined by $n_1 \leq n_2$ if and only if $n_2 \subset n_1$. Consider the fuzzy net $\{F_n : n \in \mathcal{D}\}, F_n = \bigwedge_{U \in n} U'$. It is clear that $FUWL_\theta(F_n) < \underline{1 - \alpha}$. Then by the condition, there exists an $n_0 \in \mathcal{D}$ such that $int(F_n) < \underline{1 - \alpha + \beta}$ for all $n \geq n_0$. So,

$$\underline{\alpha - \beta} < 1 - int(F_{n_0}) = 1 - \bigwedge_{U \in n_0} int(U') = \bigvee_{U \in n_0} cl(U).$$

Hence X fuzzy $H^\star$-closed. □

Theorem 4.6. *Let X be a fuzzy $H^\star$-closed topological space, $\alpha \in [0, 1)$ and $\beta \in (0, \alpha)$. Further suppose $\{F_n : n \in \mathcal{D}\}$ be a fuzzy net of fuzzy closed sets such that $FIUL_\theta(F_n) \leq \underline{1 - \alpha}$ for every ideal $\mathcal{I}$ on $\mathcal{D}$. Then $\{n \in \mathcal{D} : F_n \not\leq \underline{1 - \alpha + \beta}\} \in \mathcal{I}$.*

Proof. Let X be fuzzy $H^\star$-closed and $\{F_n : n \in \mathcal{D}\}$ be a fuzzy net of fuzzy closed sets, $\mathcal{I}$ be an ideal on $\mathcal{D}$ with $FIUL_\theta(F_n) \leq \underline{1 - \alpha}$ and an $\alpha \in [0, 1)$. Then for each fuzzy point x_λ of X satisfying $FIUL_\theta(F_n) < x_\lambda \leq \underline{1 - \alpha + \frac{\beta}{2}}$, there exists a fuzzy quasi-neighborhood U_{x_λ} of x_λ such that $\{n \in \mathcal{D} : F_n \hat{q} cl(U_{x_\lambda})\} \in \mathcal{I}$. Since X is fuzzy $H^\star$-closed and $\{U_{x_\lambda} : x_\lambda \in FP(X), FIUL_\theta(F_n) < x_\lambda \leq \underline{1 - \alpha + \frac{\beta}{2}}\}$ satisfies the property that for each $x \in X$, $U_{x_\lambda}(x) > 1 - \lambda \geq \alpha - \frac{\beta}{2}$, there exist finite number of fuzzy points $e_1, e_2, ..e_p \in FP(X)$ such that $\bigvee_{i=1}^p cl(U_{e_i}) \geq \underline{\alpha - \frac{\beta}{2} - \frac{\beta}{2}}$, that is, $\bigvee_{i=1}^p cl(U_{e_i}) \geq \underline{\alpha - \beta}$. Here,

$$\{n \in \mathcal{D} : F_n \hat{q} \vee_{i=1}^p cl(U_{e_i})\} = \bigcup_{i=1}^p \{n \in \mathcal{D} : F_n \hat{q} cl(U_{e_i})\} \in \mathcal{I}.$$

Since $\{n \in \mathcal{D} : F_n \not\leq \underline{1 - \alpha + \beta}\} \subset \{n \in \mathcal{D} : F_n \hat{q} \vee_{i=1}^p cl(U_{e_i})\}$,

$$\{n \in \mathcal{D} : F_n \not\leq \underline{1 - \alpha + \beta}\} \in \mathcal{I}.$$

□

Recall that a fuzzy net $\mathcal{S} = \{s_k : k \in \mathcal{D}\}$ is called an α-net [11], $\alpha \in (0, 1]$ if the net (called value net) $\hat{\mathcal{S}} = \{\lambda_k : k \in \mathcal{D}\}$, where λ_k is the value of the fuzzy point x_k converges to α.

Theorem 4.7. *For a fuzzy topological space X, following conditions are equivalent:*
(1) *X is fuzzy $H^\star$-closed space.*
(2) *For each $\alpha \in (0, 1]$ and for each $\beta \in (0, \alpha)$, every α-net has a fuzzy θ-cluster point with value $\alpha - \beta$.*

Proof. (1)⇒(2): Let X be fuzzy $H^\star$-closed space. If possible, let $\mathcal{S} = \{s_k : k \in \mathcal{D}\}$ be an α-net which has no fuzzy θ-cluster point with value $\alpha - \beta$. Then for each $x \in X$, there exist $k_x \in \mathcal{D}$ and $V_x \in \mathcal{R}(X, x_{\alpha-\beta})$ such that $s_k \in int(V_x)$ for all $k \geq k_x$. Consider the family $\Sigma = \{V'_x : x \in X\}$. Then for each $x \in X$, there exists $V'_x \in \Sigma$ such that $V_x(x) < \alpha - \beta$, i.e. $V'_x(x) > 1 - \alpha + \beta$, i.e. $\bigvee\{V'_x : x \in X\} \geq$

$\underline{1-\alpha+\beta}$. Since X is a fuzzy $H^{\star}$-closed space, there exists finite number of points $x^1, x^2, ..., x^n \in X$ such that for each $x \in X$, there exists $i \in \{1,2,...,n\}$ such that $\bigvee\{cl(V'_{x_i}) : i = 1,2,...,n\} \geq \underline{1-\alpha+\beta-\beta}$, i.e. $\bigwedge\{int(V_{x_i}) : i = 1,2,...,n\} \leq \underline{\alpha}$. Thus $int(V_{x_i})(x) \leq \alpha$ for all $x \in X$ and for all $i \in \{1,2,...,n\}$. Consider a $k_0 \in \mathcal{D}$ with the property $k_0 \geq k_{x_i}$ for each $i \in \{1,2,...,n\}$. Then for each $i \in \{1,2,...,n\}$, $s_k \in int(V_{x_i})$ for all $k \geq k_0$. For each $k \in \mathcal{D}$, suppose $supp(s_k) = x^k$. So we get $int(V_{x_i})(x^k) > \alpha$ for all $k \geq k_0$ and for all $i \in \{1,2,...,n\}$, which is a contradiction.

(2)⇒(1): Let X be not fuzzy $H^{\star}$-closed. Then there exists an $\alpha \in (0,1]$ and a $\beta \in (0,\alpha)$ and a family $\mathcal{U}$ of fuzzy open sets of X with the properties :

(a) $\bigvee\{U : U \in \mathcal{U}\} \geq \underline{1-\alpha+\beta}$ and

(b) for each finite subfamily $\mathcal{U}^0$ of $\mathcal{U}$, $\bigvee\{cl(U_{x_i}) : i = 1,2,...,n\} \ngeq \underline{1-\alpha+\beta-\beta}$, i.e. $\bigwedge\{int(U'_{x_i}) : i = 1,2,...,n\} \nleq \underline{\alpha}$.

Now consider the set $\mathcal{D} = \{\mathcal{U}'^0 : \mathcal{U}^0$ is a finite subfamily of $\mathcal{U}\}$. Then $\mathcal{D}$ becomes to a directed set by the order " $\geq$ " defined by $\mathcal{U}'^0 \geq \mathcal{U}'^1$ if and only if $\mathcal{U}'^0 \supset \mathcal{U}'^1$. Now consider $\mathcal{U}'^0 \in \mathcal{D}$. Then there exists an $x \in X$ such that $int(U')(x) \geq \alpha$ for each $U \in \mathcal{U}^0$. Thus for each $\mathcal{U}'^0 \in \mathcal{D}$, we can select a fuzzy point $s(\mathcal{U}'^0) = x_\alpha$ such that $x_\alpha \in int(U')$ for each $U \in \mathcal{U}^0$. Thus we get a constant fuzzy α-net $\mathcal{S} = \{s(\mathcal{U}'^0) : \mathcal{U}'^0 \in \mathcal{D}\}$. We claim that $\mathcal{S}$ has no fuzzy θ-cluster point with value $\alpha-\beta$. let $y_{\alpha-\beta}$ be any fuzzy point with value $\alpha-\beta$. Then there exists an $U_0 \in \mathcal{U}$ satisfying $U_0(y) \geq 1-\alpha+\beta$, i.e. $U'(x) \leq \alpha-\beta$. So $U'_0 \in \mathcal{R}(X, x_{\alpha-\beta})$. Clearly, $\{U'_0\} \in D$. Therefore for all $\mathcal{U}'^0 \in D$ with $\mathcal{U}'^0 \geq \{U'_0\}$, we get $U'_0 \in \mathcal{U}'^0$ and so $s(\mathcal{U}'^0) \in int(U'_0)$. Therefore y_α is not fuzzy θ-cluster point of $\mathcal{S}$. □

Corollary 4.8. *For a fuzzy topological space X, following conditions are equivalent:*

(1) *X is weakly fuzzy H-closed space.*

(2) *For each $\epsilon > 0$, every 1-net has a fuzzy θ-cluster point with value $1-\epsilon$.*

For a fuzzy set F of a fuzzy topological space X, $h(F) = \sup\{F(x) : x \in X\}$ is called the hight of the fuzzy set F. Let $\mathcal{F}$ be a fuzzy filter on X and $\inf\{h(F) : F \in \mathcal{F}\} = \alpha$. Then the fuzzy filter $\mathcal{F}$ is called an α-filter [11]. If for each $F \in \mathcal{F}$, there exists an $x \in X$ such that $F(x) \geq \alpha$, then $\mathcal{F}$ is called a constant α-filter [11]. A fuzzy point x_λ is called its θ-adherent point if for each $U \in \mathcal{R}(X, x_\lambda)$ and each $F \in \mathcal{F}$, $F \nleq int(U)$.

Theorem 4.9. *For a fuzzy topological space X, then following conditions are equivalent:*

(1) *X is fuzzy $H^{\star}$-closed.*

(2) *For each $\alpha \in (0,1]$ and for each $\beta \in (0,\alpha)$, every α-filter $\mathcal{F}$ has a fuzzy θ-adherent point with value $\alpha-\beta$.*

Proof. (1)⇒(2): Suppose $\mathcal{F}$ is an α-filter such that $A \in \mathcal{F}$. Consider the directed set $\mathcal{D} = \{(F,n) : F \in \mathcal{F}, n \in \mathbb{N}\}$ ordered by " $\geq$ " that is defined by $(n_1, F_1) \geq (n_2, F_2)$ if and only if $F_1 \leq F_2$ and $n_1 \geq n_2$. For each $(F,n) \in \mathcal{D}$, we can select a fuzzy point $x^{nF}_{\lambda_{nF}} \in F$ such that $\alpha - \frac{1}{n} < \lambda_{nF} < \alpha + \frac{1}{n}$. Here $\mathcal{S} = \{x^{nF}_{\lambda_{nF}} : (F,n) \in \mathcal{D}\}$ is an α-net in X [11]. Since X is $H^{\star}$-closed, there exists a fuzzy point $x_{\alpha-\beta} \in \Theta(\mathcal{S})$. We claim that $x_{\alpha-\beta}$ is a fuzzy θ-adherent point of $\mathcal{F}$. Let $U \in \mathcal{R}(X, x_{\alpha-\beta})$ and $F \in \mathcal{F}$. Since $(F,1) \in \mathcal{D}$, there exists $(E,k) \in \mathcal{D}$ such that $(E,k) \geq (F,1)$ and $x^{kE}_{\lambda_{kE}} \notin int(U)$.

Since $x^{kE}_{\lambda_{kE}} \in E$, $x^{kE}_{\lambda_{kE}} \in F$. So $F \not\leq int(U)$. Thus x_α is a fuzzy θ-adherent point of $\mathcal{F}$.

(2)⇒(1): Let $\alpha \in (0,1]$ and $\beta \in (0,\alpha)$. Let $\mathcal{S} = \{s_k : k \in \mathcal{D}\}$ be an α-net in X. For each $n \in \mathcal{D}$, let $F_n = \vee\{s_k : k \geq n\}$. Then $\mathcal{F} = \{F \in I^X : F \geq F_n$ for some $n \in \mathcal{D}\}$ is a fuzzy α-filter [11]. Thus there exists an $x_{\alpha-\beta} \in Pt(X)$ which is a fuzzy θ-adherent point of $\mathcal{F}$. Now we shall show that $x_{\alpha-\beta} \in \Theta(\mathcal{S})$. Let $U \in \mathcal{R}(X, x_{\alpha-\beta})$ and $n \in D$. Since $F_n \not\leq int(U)$, we get a $k \in \mathcal{D}$ such that $k \geq n$ and $s_k \notin int(U)$. □

Corollary 4.10. *For a fuzzy topological space X, following conditions are equivalent:*

(1) *X is weakly fuzzy H-closed space.*

(2) *For each $\epsilon > 0$, every 1-filter $\mathcal{F}$ has a fuzzy θ-adherent point with value $1-\epsilon$.*

Theorem 4.11. *Let (X, δ) be a fuzzy $H^\star$-closed topologically generated space and $U \in \delta$. Then $cl(U)$ is weakly fuzzy H-closed.*

Proof. The proof follows from Theorem 3.13. □

Theorem 4.12. *Let X be a fuzzy regular space. Then X is fuzzy $H^\star$-closed if and only if X is fuzzy compact due to Lowen* [12].

Proof. The proof is analogous to the proof of Theorem 3.7 and is thus omitted. □

Theorem 4.13. *Let X be a fuzzy $H^\star$-closed topological space, Y be a fuzzy Hausdorff space and $\psi : X \to Y$ be a fuzzy θ-continuous surjection. Then Y is fuzzy $H^\star$-closed.*

Proof. Let X be a fuzzy $H^\star$-closed topological space. Let $\alpha \in (0,1]$, $\beta \in (0,\alpha)$ and $\mathcal{V}_{\alpha\beta}$ be a family of fuzzy open sets with the property that $\bigvee \mathcal{V}_{\alpha\beta} \geq \underline{\alpha}$. Then for each $y \in Y$, there exists a $V \in \mathcal{V}_{\alpha\beta}$ such that $V(y) \geq \alpha > \alpha - \epsilon$. Suppose $x \in X$ such that $\psi(x) = y$. Let $\epsilon > 0$ be an arbitrary. Then $V \in \mathcal{Q}(Y, y_{1-\alpha+\epsilon})$. Since ψ is fuzzy θ-continuous, there exists an $U_V \in \mathcal{Q}(X, x_{1-\alpha+\epsilon})$ such that $\psi(cl(U_V)) \leq cl(V)$. Consider the family $\mathcal{U}_{\alpha\beta} = \{U_V : V \in \mathcal{V}_{\alpha\beta}\}$. We note that $\bigvee \mathcal{U}_{\alpha\beta} \geq \underline{\alpha}$. Since X is fuzzy $H^\star$-closed, there exist finite number of fuzzy sets $V_1, V_2, ..., V_n \in \sigma$ such that

$$\bigvee\{cl(U_{V_i}) : i = 1, 2, ..., n\} \geq \underline{\alpha - \beta}.$$

Thus $\bigvee\{cl(V_i) : i = 1, 2, ..., n\} \geq \underline{\alpha - \beta}$. So Y is fuzzy $H^\star$-closed. □

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