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Ditopological texture spaces and semi-regularity axioms

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Abstract. In the present paper, we introduce and study a new class of ditopological separation axioms called semi-regularity axioms based on semi-open and semi-closed sets. Further, it is shown that the product of semi-R_0, semi-R_1 and semi-regular ditopological texture spaces are respectively, semi-R_0, semi-R_1 and semi-regular. Finally, the preservation of semi-regularity axioms under suitable surjective bi-irresolute difunctions is proved.

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1. Introduction

In Levine [11] introduced the concept of semi-open sets and semi-continuity in topological spaces. Further, semi-separation axioms are being studied by many mathematicians.

The theory of texture spaces is an alternative setting for fuzzy sets and therefore, lots of properties of Hutton algebras (known as fuzzy lattices) can be argued with regard to textures [1, 2, 3, 4]. Ditopologies on textures unify the fuzzy topologies, topologies and bitopologies in a non-complemented setting with the help of duality in the textural concepts [5]. It is studied R_0-, R_1- and regularity- separation axioms in ditopological texture spaces in [6]. On the other hand, the first generalization of open and closed sets and their some applications in ditopological texture spaces were given in [10]. Further the second author introduced and studied the concepts of semi-open, semi-closed sets and semi-bicontinuity in [9].

In the present paper, we introduce semi-R_0, semi-R_1 and semi-regular separation axioms in ditopological texture spaces and give some characterizations of them.

This paper is constructed as following. The next section contains a review of well-known properties of ditopological texture spaces. In Section 3, the concept of semi-open sets is re-examined on some additional properties and the concept of semi-dihomeomorphism is introduced. In Section 4, semi-regularity axioms in ditopological texture spaces are presented and some results of product ditopological texture spaces on semi-regularity axioms are given.

2. Preliminaries

This section contains the notions which are needed in the sequel. For more details see [1, 2, 3, 4, 5, 6].

Texture space: If S is a set, a texturing is a point-separating, complete, completely distributive lattice containing S and $\varnothing$ and for which meet coincides with intersection and finite joins with union. If $\mathcal{S}$ is textured by S, then we call $(S, \mathcal{S})$ a texture space or shortly, texture.
For a texture $(S, \mathcal{S})$, most properties are appropriately defined in terms of the p-sets $P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$ and, as a dually, the q-sets, $Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$.

Complementation: In general, a texturing of S need not be closed under set complementation, but there may exist a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$, $\forall A \in \mathcal{S}$ and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A)$, $\forall A, B \in \mathcal{S}$. In this case σ is called a complementation on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is said to be a complemented texture.

The following are some basic examples of textures.

Examples 2.1. (1) For any set X, $(X, \mathcal{P}(X), \pi)$, $\pi(Y) = X \setminus Y$ for $Y \subseteq X$, is the complemented *discrete* texture representing the usual set structure of X. Clearly, $P_x = \{x\}$, $Q_x = X \setminus \{x\}$ for all $x \in X$.

(2) Let $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$ and $\lambda((0, r]) = (0, 1 - r]$, $r \in [0, 1]$. Then $(L, \mathcal{L}, \lambda)$ is complemented texture space. Here $P_r = Q_r = (0, r]$ for all $r \in L$.

(3) For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$, $\iota([0, t]) = [0, 1-t)$ and $\iota([0, t)) = [0, 1 - t]$, $t \in [0, 1]$. $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, which we will refer to as the unit interval texture. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in \mathbb{I}$.

(4) If $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures, the product texturing $\mathcal{S} \otimes \mathcal{T}$ of $S \times T$ consists of arbitrary intersections of sets of the form $(A \times T) \cup (S \times B)$, $A \in \mathcal{S}, B \in \mathcal{T}$, and $(S \times T, \mathcal{S} \otimes \mathcal{T})$ is called the product of $(S, \mathcal{S})$ and $(T, \mathcal{T})$. For $s \in S, t \in T$ we clearly have $P_{(s,t)} = P_s \times P_t$ and $Q_{(s,t)} = (Q_s \times T) \cup (S \times Q_t)$.

Ditopological texture space: A ditopology on a texture $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of open sets τ and the set of closed sets κ satisfies

(i) $S, \varnothing \in \tau$; $\qquad S, \varnothing \in \kappa$,

(ii) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$; $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$,

(iii) $G_i \in \tau$, $i \in I \implies \bigvee_i G_i \in \tau$; $K_i \in \kappa$, $i \in I \implies \bigcap_i K_i \in \kappa$.

Therefore a ditopology is essentially a "topology" for which there is no a priori relation between the open and closed sets.

For $A \in \mathcal{S}$ we define the closure $cl(A)$ and the interior $int(A)$ of A under (τ, κ) by the equalities

$$cl(A) = \bigcap\{K \in \kappa \mid A \subseteq K\} \text{ and } int(A) = \bigvee\{G \in \tau \mid G \subseteq A\}.$$

If (τ, κ) is a ditopology on a complemented texture $(S, \mathcal{S}, \sigma)$ we say (τ, κ) is complemented if $\kappa = \sigma(\tau)$. In this case we have $\sigma(cl(A)) = int(\sigma(A))$ and $\sigma(int(A)) = cl(\sigma(A))$.

Product texture space: If $(S_j, \mathcal{S}_j)$, $j \in J$, are textures, $S = \prod_{j \in J} S_j$ and $A_k \in \mathcal{S}_k$ for some $k \in J$ we write

$$E(k, A_k) = \prod_{j \in J} Y_j \text{ where } Y_j = \begin{cases} A_j, & \text{if } j = k \\ S_j, & \text{otherwise.} \end{cases}$$

Then the product texturing $\mathcal{S} = \bigotimes_{j \in J} \mathcal{S}_j$ on S consists of arbitrary intersections of elements of the set $\mathcal{E} = \left\{ \bigcup_{j \in J_1} E(j, A_j) \mid J_1 \subseteq J, A_j \in \mathcal{S}_j \text{ for } j \in J_1 \right\}$.

Product ditopological texture spaces: Let $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)_{j \in J}$ be ditopological texture spaces and $(S, \mathcal{S})$ the product of the textures $(S_j, \mathcal{S}_j)_{j \in J}$. For each $j \in J$,

$$\pi_j = \bigvee \{ \overline{P}_{(s, s_j)} \mid s = (s_k) \in S \}, \ \Pi_j = \bigcap \{ \overline{Q}_{(s, s_j)} \mid s = (s_k) \in S^\flat \}$$

define the j-th projection difunction (π_j, Π_j) on $(S, \mathcal{S})$ to $(S_j, \mathcal{S}_j)$ [4, Lemma 3.9]. On the other hand the ditopology (τ, κ) on $(S, \mathcal{S})$ with subbase $\{ \Pi_j^{\leftarrow} G = E(j, G) \mid G \in \tau_j, j \in J \}$ and cosubbase $\{ \pi_j^{\leftarrow} K = E(j, K) \mid K \in \kappa_j, j \in J \}$ is called the product ditopology on $(S, \mathcal{S})$ [5]. Note that the product ditopology is the coarsest ditopology making the projection difunctions (π_j, Π_j), $j \in J$ are bicontinuous.

Direlation: Let us consider the product texture $\mathcal{P}(S) \otimes \mathcal{T}$ of the texture spaces $(S, \mathcal{P}(S))$ and $(T, \mathcal{T})$ and denote the p-sets and the q-sets by $\overline{P}_{(s,t)}$ and $\overline{Q}_{(s,t)}$, respectively. Clearly, $\overline{P}_{(s,t)} = \{s\} \times P_t$ and $\overline{Q}_{(s,t)} = (S \setminus \{s\} \times T) \cup (S \times Q_t)$ where $s \in S$ and $t \in T$. Then

(i) $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a relation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ if it satisfies
  [R1] $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$,
  [R2] $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.

(ii) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ if it satisfies
  [CR1] $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$,
  [CR2] $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.

(iii) A pair (r, R), where r is a relation and R a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, is called a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$.

Difunction: Let (f, F) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ if it satisfies the following two conditions :

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.
DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

Definition 2.2. Let $(f,F):(S,\mathcal{S})\to(T,\mathcal{T})$ be a difunction. For $A\in\mathcal{S}$ and $B\in\mathcal{T}$, the A-sections and the B-presections with respect to (f,F) are given as

$$\begin{aligned} f^{\to}A &= \bigcap\{Q_t \mid \forall s, f\not\subseteq \overline{Q}_{(s,t)} \Longrightarrow A\subseteq Q_s\},\\ F^{\to}A &= \bigvee\{P_t \mid \forall s, \overline{P}_{(s,t)}\not\subseteq F \Longrightarrow P_s\subseteq A\},\\ \text{and} & \\ f^{\leftarrow}B &= \bigvee\{P_s \mid \forall t, f\not\subseteq \overline{Q}_{(s,t)} \Longrightarrow P_t\subseteq B\},\\ F^{\leftarrow}B &= \bigcap\{Q_s \mid \forall t, \overline{P}_{(s,t)}\not\subseteq F \Longrightarrow B\subseteq Q_t\},\end{aligned}$$

respectively.

For a given difunction, the inverse image and the inverse co-image are equal; and the image and co-image are usually not.

We note that $((f^{\leftarrow})^{\leftarrow})(A)=f^{\to}(A)$ and $((F^{\leftarrow})^{\leftarrow})(A)=F^{\to}(A)$ by [4, Lemma 2.9].

Definition 2.3. Let $(f,F):(S,\mathcal{S})\to(T,\mathcal{T})$ be a difunction. Then (f,F) is called surjective if it satisfies the condition

SUR. For $t,t'\in T$, $P_t\not\subseteq Q_{t'} \implies \exists\, s\in S$ with $f\not\subseteq\overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)}\not\subseteq F$.

Likewise, (f,F) is called injective if it satisfies the condition

INJ. For $s,s'\in S$ and $t\in T$, $f\not\subseteq\overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)}\not\subseteq F \implies P_s\not\subseteq Q_{s'}$.

If (f,F) is both injective and surjective then it is called bijective.

3. Semi-open sets and semi-closed sets

In [9], semi-open and semi-closed sets were introduced in view of ditopological texture spaces. In this section, these concepts are re-examined and some additional properties of them are given.

Definition 3.1. Let $(S,\mathcal{S},\tau,\kappa)$ be a ditopological texture space and $A\in\mathcal{S}$. Then A is called

(i) semi-open if there is $G\in\tau$ such that $G\subseteq A\subseteq cl(G)$,

(ii) semi-closed if there is $K\in\kappa$ such that $int(K)\subseteq A\subseteq K$.

We denote by $SO\,(S)$ the set of semi-open sets in $\mathcal{S}$. Likewise, $SC\,(S)$ will denote the set of semi-closed sets.

Remark 3.2. Every open (closed) set in a ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ is semi-open (semi-closed) set but not conversely.

Lemma 3.3. *Let $(S,\mathcal{S},\tau,\kappa)$ be a ditopological texture space and $A\in\mathcal{S}$.*

(1) *A is semi-open if and only if $A\subseteq cl\,(int(A))$.*

(2) *A is semi-closed if and only if $int\,(cl(A))\subseteq A$.*

Proof. (1) Let $A\in\mathcal{S}$ be semi-open. Then $G\subseteq A\subseteq cl(G)$ for some open set $G\in\tau$. But $G\subseteq int(A)$ and $cl\,(G)\subseteq cl\,(int(A))$. Thus $A\subseteq cl(G)\subseteq cl(int(A))$.

Conversely, let $A\subseteq cl\,(int(A))$. Then for $G=int(A)$, we have $G\subseteq A\subseteq cl(G)$.

(2) Let $A\in\mathcal{S}$ be semi-closed. Then $int(K)\subseteq A\subseteq K$ for some closed set $K\in\kappa$. But $cl(A)\subseteq cl(K)=K$ and $int(cl(A))\subseteq int(K)$. Thus $int(cl(A))\subseteq int(K)\subseteq A$.

Conversely, let $int(cl(A)) \subseteq A$. Then for $K = cl(A)$, we have $int(K) \subseteq A \subseteq K$. □

Lemma 3.4. (1) *Let* $\{A_j\}_{j\in J}$ *be a collection of semi-open sets in* $(S, \mathcal{S}, \tau, \kappa)$. *Then* $\bigvee_{j\in J} A_j$ *is semi-open.*
(2) *Let* $\{B_j\}_{j\in J}$ *be a collection of semi-closed sets in* $(S, \mathcal{S}, \tau, \kappa)$. *Then* $\bigcap_{j\in J} B_j$ *is semi-closed.*

Proof. (1) For each A_j, $j \in J$, we have $G_j \in \tau$ such that $G_j \subseteq A_j \subseteq cl(G_j)$. Then

$$\bigvee_{j\in J} G_j \subseteq \bigvee_{j\in J} A_j \subseteq \bigvee_{j\in J} cl\,(G_j) \subseteq cl \bigvee_{j\in J} G_j.$$

(2) For each B_j, $j \in J$, we have $K_j \in \kappa$ such that $int(K_j) \subseteq B_j \subseteq K_j$. Then

$$int\left(\bigcap_{j\in J} K_j\right) \subseteq \bigcap_{j\in J} int\,(K_j) \subseteq \bigcap_{j\in J} B_j \subseteq \bigcap_{j\in J} K_j.$$

Since $\bigcap_{j\in J} K_j = K$ is closed set, $\bigcap_{j\in J} B_j$ is semi-closed . □

Now, we recall [9] the definition of semi-interior and semi-closure of a set.

Definition 3.5. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$. We define
(i) The semi-interior $sint(A)$ of A is the set

$$sint(A) = \bigvee \{B \mid B \text{ is semi-open and } B \subseteq A\},$$

(ii) The semi-closure $scl(A)$ of A is the set

$$scl(A) = \bigcap \{B \mid B \text{ is semi-closed and } A \subseteq B\}.$$

By Lemma 3.4, we have $sint(A)$ is semi-open and $scl(A)$ is semi-closed. It follows that $int(A) \subseteq sint(A) \subseteq A \subseteq scl(A) \subseteq cl(A)$.
Clearly, for a complemented ditopological texture space $(S, \mathcal{S}, \sigma, \tau, \kappa)$: $A \in \mathcal{S}$ is semi-open if and only if $\sigma(A)$ is semi-closed.

Proposition 3.6. *Let* $(S, \mathcal{S}, \sigma, \tau, \kappa)$ *be a complemented ditopological texture space and* $A \in \mathcal{S}$. *Then*
(1) $\sigma\,(scl(A)) = sint\,(\sigma\,(A))$.
(2) $\sigma\,(sint\,(A)) = scl\,(\sigma\,(A))$.
(3) $sint\,(A) = \sigma\,(scl\,(\sigma\,(A)))$.

Proof. (1)
$$\begin{aligned}\sigma\,(scl(A)) &= \sigma\left(\bigcap \{B \mid B \text{ is semi-closed and } A \subseteq B\}\right)\\ &= \bigvee \{\sigma\,(B) \mid B \text{ is semi-closed and } A \subseteq B\}\\ &= \bigvee \{\sigma\,(B) \mid \sigma\,(B) \text{ is semi-open and } \sigma\,(B) \subseteq \sigma\,(A)\}\\ &= sint\,(\sigma\,(A)).\end{aligned}$$
(2) can be proved similarly.
(3) By part (2). □

Theorem 3.7. *Let* $(S, \mathcal{S}, \tau, \kappa)$ *be a ditopological texture space and* $A, B \in \mathcal{S}$. *Then*
(1) $scl\,(\varnothing) = \varnothing$ *and* $scl\,(S) = S$.
(2) B *is semi-closed if and only if* $B = scl\,(B)$.
(3) $scl\,(scl\,(B)) = scl\,(B)$.

(4) $B \subseteq A \Rightarrow scl(B) \subseteq scl(A)$.
(5) $scl(A \cap B) \subseteq scl(A) \cap scl(B)$.

Proof. (1) is obvious.

(2) If B is semi-closed set, then B is itself a semi-closed set which contains B. Thus B the smallest semi-closed set containing B and $B = scl(B)$.

Conversely, assume that $scl(B) = B$. Since $scl(B)$ is a semi-closed set, B is semi-closed set.

(3) Since $scl(B)$ is a semi-closed set, by part (2), we have $scl(scl(B)) = scl(B)$.

(4) Assume that $B \subseteq A$. Then every semi-closed superset of A also contain B. This means every semi-closed superset of A is also a semi-closed superset of B. Thus the intersection of semi-closed supersets of B is contained in the intersection of semi-closed supersets of A. So $scl(B) \subseteq scl(A)$.

(5) For each $A, B \in \mathcal{S}$, we have that $(A \cap B) \subseteq B$ and $(A \cap B) \subseteq A$. Then by part (4), it follows that $scl(A \cap B) \subseteq scl(B)$ and $scl(A \cap B) \subseteq scl(A)$. Thus $scl(A \cap B) \subseteq scl(A) \cap scl(B)$. □

The next results are dual of Theorem 3.7:

Theorem 3.8. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A, B \in \mathcal{S}$. Then*
(1) $sint(\varnothing) = \varnothing$ *and* $sint(S) = S$.
(2) *B is semi-open if and only if* $B = sint(B)$.
(3) $sint(sint(B)) = sint(B)$.
(4) $B \subseteq A \Rightarrow sint(B) \subseteq sint(A)$.
(5) $sint(A) \cup sint(B) \subseteq sint(A \cup B)$.

Theorem 3.9. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$. Then*
(1) $sint(int(A)) = int(sint(A)) = int(A)$.
(2) $scl(cl(A)) = cl(scl(A)) = cl(A)$.

Proof. We prove (1), leaving the dual proof (2) to the interested reader.
Since $int(A)$ is open, it is semi-open by Remark 3.2. Then we can get $sint(int(A)) = int(A)$ by Theorem 3.8 (2). Also, we have $int(A) \subseteq sint(A) \subseteq A$. Thus we can get $int(A) \subseteq int(sint(A)) \subseteq int(A)$. So $int(sint(A)) = int(A)$. This completes the proof. □

Before leaving this section, let us investigate semi-properties of difunctions. Firstly we recall [9] the next definition.

Definition 3.10. Let $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$, $j = 1, 2$ be ditopological texture spaces. A difunction $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ is said to be

(i) semi-continuous (semi-irresolute) if for each open (semi-open) set $A \in \mathcal{S}_2$, the inverse image $F^{\leftarrow}(A) \in \mathcal{S}_1$ is a semi-open set.

(ii) semi-cocontinuous (semi-co-irresolute) if for each closed (semi-closed) set $B \in \mathcal{S}_2$, the inverse image $f^{\leftarrow}(B) \in \mathcal{S}_1$ is a semi-closed set.

(iii) semi-bicontinuous (semi-bi-irresolute) if it is semi-continuous and semi-cocontinuous (semi-irresolute and semi-co-irresolute).

The concepts of semi-open and semi-closed difunctions can be given as a generalization of open and closed difunctions.

Definition 3.11. Let (f, F) be a difunction from $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ to $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. Then (f, F) is called

(i) semi-open if $A \in SO(S_1) \implies f^{\rightarrow}(A) \in SO(S_2)$.
(ii) semi-co-open $A \in SO(S_1) \implies F^{\rightarrow}(A) \in SO(S_2)$.
(iii) semi-closed if $A \in SC(S_1) \implies f^{\rightarrow}(A) \in SC(S_2)$.
(iv) semi-co-closed $A \in SC(S_1) \implies F^{\rightarrow}(A) \in SC(S_2)$.

Remark 3.12. If (f, F) bijective, then by [4, Corollary 2.33], we have $f^{\rightarrow}A = F^{\rightarrow}A$ for all $A \in \mathcal{S}_1$. In this case, semi-openness (semi-closedness) coincides with semi-co-openness (semi-co-closedness) for difunctions respectively.

Definition 3.13. Let (f, F) be a difunction from $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ to $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. Then (f, F) is called a semi-dihomeomorphism if (f, F) is bijective, and semi-bi-irresolute with its inverse $(F^{\leftarrow}, f^{\leftarrow})$.

The following result characterizes the semi-dihomeomorphisms in terms of semi-open (semi-co-open), and semi-closed (semi-co-closed) difunctions.

Theorem 3.14. *Let (f, F) be a bijective difunction from $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ to $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. Then the following conditions are equivalent:*

(1) *(f, F) is a semi-dihomeomorphism.*
(2) *(f, F) is semi-bi-irresolute, semi-open and semi-co-closed.*
(3) *(f, F) is semi-bi-irresolute, semi-closed and semi-co-open.*
(4) $f^{\rightarrow}A = F^{\rightarrow}A \in SO(S_2) \iff A \in SO(S_1)$
and

$f^{\rightarrow}A = F^{\rightarrow}A \in SC(S_2) \iff A \in SC(S_1)$.

(5) $f^{\leftarrow}A = F^{\leftarrow}A \in SO(S_1) \iff A \in SO(S_2)$
and

$f^{\leftarrow}A = F^{\leftarrow}A \in SC(S_1) \iff A \in SO(S_2)$.

Proof. (1) $\Longrightarrow$ (2): By the definition of semi-dihomeomorphism, the inverse $(F^{\leftarrow}, f^{\leftarrow})$ of (f, F) is semi-bi-irresolute. Thus,

$$A \in SO(S_1) \Longrightarrow f^{\rightarrow}(A) = ((f^{\leftarrow})^{\leftarrow})(A) \in SO(S_2)$$

which shows that (f, F) is semi-open. Likewise, the semi-co-irresolutness of $(F^{\leftarrow}, f^{\leftarrow})$ implies the semi-coclosedness of (f, F).

(2) $\Longrightarrow$ (3) From Remark 3.12, the proof is immediate.

(3) $\Longrightarrow$ (4) $A \in SO(S_1) \Longrightarrow F^{\rightarrow}(A) \in SO(S_2)$ since (f, F) is semi-co-open, and we have already noted that $f^{\rightarrow}(A) = F^{\rightarrow}(A)$, since (f, F) is bijective.

On the other hand, if $f^{\rightarrow}(A) = F^{\rightarrow}(A) \in SO(S_2)$, then $A = F^{\leftarrow}(f^{\rightarrow}(A)) \in SO(S_2)$, by the semi-irresolutness of (f, F), the injectivity of (f, F) and [4, Corollary 2.33 (2)]. The proof of the second result is dual to this and is omitted.

(4) $\Longrightarrow$ (5) Take $B \in \mathcal{S}_2$ and $A = F^{\leftarrow}(B)$. Then, since (f, F) is surjective, $B = f^{\rightarrow}(F^{\leftarrow}(B)) = f^{\rightarrow}(A)$, by [4, Corollary 2.33 (1)]. If $B \in \mathcal{S}O(S_2)$, then $f^{\rightarrow}(A) \in SO(S_2)$ and we deduce $F^{\leftarrow}(B) = A \in \mathcal{S}O(S_1)$, by (4), and we know that $F^{\leftarrow}(B) = f^{\leftarrow}(B) \in SO(S_1)$. Thus $A \in SO(S_1)$ and by (4), we have $B = f^{\rightarrow}(A) \in SO(S_2)$. The proof of the remaining results is dual to this, and is omitted.

(5) $\Longrightarrow$ (1) Using the equivalences, we immediately obtain that (f, F) is semi-bi-irresolute. Now we show that $(F^{\leftarrow}, f^{\leftarrow})$ is also semi-bi-irresolute. Take $A \in SO(S_1)$.

Then $A = F^{\leftarrow}(f^{\rightarrow}(A)) \in SO(S_1)$, by [4, Corollary 2.33 (2)]. Since (f, F) is injective, whence $f^{\rightarrow}(A) \in SO(S_2)$ by (5) applied to $B = f^{\rightarrow}(A)$. Since $f^{\rightarrow}(A) = (f^{\leftarrow})^{\leftarrow}(A)$, $(F^{\leftarrow}, f^{\leftarrow})$ is semi-irresolute. Likewise $(F^{\leftarrow}, f^{\leftarrow})$ is semi-co-irresolute and we have shown that (f, F) is a semi-dihomeomorphism. □

4. Semi-regularity axioms

In this section, semi-R_0, semi-R_1 and semi-regular axioms in ditopological texture spaces are introduced. It will be noted that the basic duality involving the p-sets and q-sets has been exploited to give two dual forms of each axiom.

We begin by recalling [12] that a topological space X is said to be semi-R_0 if for each semi-open set A in X and $x \in A$, $scl\{x\} \subseteq A$, or equivalently, a topological space X is said to be semi-R_0 if for each semi-closed set B and $x \in X \setminus B$, $B \subseteq sint(X \setminus \{x\})$. This leads to the following analogous concepts in a ditopological texture space.

Definition 4.1. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. Then (τ, κ) is called

(i) semi-R_0 if $A \in SO(S), A \not\subseteq Q_s \implies scl(P_s) \subseteq A$, and

(ii) semi-co-R_0 if $B \in SC(S), P_s \not\subseteq B \implies B \subseteq sint(Q_s)$.

Theorem 4.2. *Let $(S, \mathcal{S}, \tau, \kappa)$ be ditopological texture space.*

(a) *The following are equivalent :*

(1) *(τ, κ) is semi-R_0.*

(2) *$\forall A \in SO(S)$, there is a family $\{B_j : j \in J\} \subseteq SC(S)$ such that $A = \bigvee_{j\in J} B_j$.*

(3) *Given $A \in SO(S)$, $s \in S$ with $A \not\subseteq Q_s$ there exists $B \in SC(S)$ with $B \subseteq A$ and $B \not\subseteq Q_s$.*

(b) *The following are equivalent :*

(1) *(τ, κ) is semi-co-R_0.*

(2) *$\forall B \in SC(S)$, there is a family $\{A_j : j \in J\} \subseteq SO(S)$ such that $B = \bigcap_{j\in J} A_j$.*

(3) *Given $B \in SC(S)$, $s \in S$ with $P_s \not\subseteq B$ there exists $A \in SO(S)$ with $A \subseteq B$ and $P_s \not\subseteq A$.*

Proof. We prove (a), leaving the dual proof (b) to the interested reader.

(1) $\implies$ (2): By [4, Theorem 1.2] and definition of semi-R_0, it is enough to observe that

$$A = \bigvee\{P_s \mid A \not\subseteq Q_s\} \subseteq \bigvee\{scl(P_s) \mid A \not\subseteq Q_s\} \subseteq A.$$

(2) $\implies$ (3): Let $A \in SO(S)$, $A \not\subseteq Q_s$ for some $s \in S$. Then there exists a family $\{B_j : j \in J\} \subseteq SC(S)$ such that $A = \bigvee_{j\in J} B_j$. Thus, $\bigvee_{j\in J} B_j \not\subseteq Q_s$. So, $B_{j_0} \not\subseteq Q_s$ for some $j_0 \in J$ and clearly, $B_{j_0} \subseteq A$.

(3) $\implies$ (1): Let $A \in SO(S), A \not\subseteq Q_s$. By (3), there exists $B \in SC(S)$ such that $B \subseteq A$ and $B \not\subseteq Q_s$. Then $P_s \subseteq B \subseteq A$, whence $scl(P_s) \subseteq A$. □

Corollary 4.3. *If $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is a complemented ditopological texture space, then the notions of semi-R_0 and semi-co-R_0 are equivalent.*

Proof. Assume that (τ, κ) be semi-R_0. Let $B \in SC(S)$. Then $\sigma(B) \in SO(S)$ and, by Theorem 4.2(a), there is a family $\{B_j \mid j \in J\} \subseteq SC(S)$ such that $\sigma(B) = \bigvee_{j \in J} B_j$. Now, we set $A_j = \sigma(B_j)$, $\forall j \in J$. Since $A_j \in SO(S)$, we have

$$B = \sigma(\sigma(B)) = \sigma(\bigvee_{j \in J}(B_j)) \quad = \bigcap_{j \in J} \sigma(B_j) = \bigcap_{j \in J} A_j.$$

This proves that (τ, κ) is semi-co-R_0 by Theorem 4.2(b). The proof of the sufficiency is similar. $\square$

Now we recall [7] that a topological space X is said to be semi-R_1 if for each semi-open set A in X, $x \in A$ and $y \in X \setminus A$ there exists a semi-open set G such that $x \in G$ and $y \notin scl(G)$. This leads to the following analogous concepts in a ditopological texture space.

Definition 4.4. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. Then (τ, κ) is called

(i) semi-R_1 if $A \in SO(S),\ A \not\subseteq Q_s,\ P_t \not\subseteq A \implies \exists G \in SO(S),\ G \not\subseteq Q_s, P_t \not\subseteq scl(G)$.

(ii) semi-co-R_1 if $B \in SC(S), P_s \not\subseteq B, B \not\subseteq Q_t \implies \exists K \in SC(S),\ P_s \not\subseteq K, sint(K) \not\subseteq Q_t$.

Theorem 4.5. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space.*

(a) *The following statements are equivalent :*

(1) *(τ, κ) is semi-R_1.*

(2) *For all $A \in SO(S)$ with $s, t \in S$, $A \not\subseteq Q_s, P_t \not\subseteq A \implies \exists G \in SO(S)$ with $P_s \subseteq G \subseteq scl(G) \subseteq Q_t$.*

(3) *For all $A \in SO(S)$, there is a family $\{G^i_j : i \in I,\ j \in J_i\} \subseteq SO(S)$ such that*

$$A = \bigvee_{i \in I} \bigcap_{j \in J_i} scl(G^i_j) = \bigvee_{i \in I} \bigcap_{j \in J_i} G^i_j.$$

(b) *The following statements are equivalent :*

(1) *(τ, κ) is semi-co-R_1.*

(2) *For all $B \in SC(S)$ with $s, t \in S$, $P_s \not\subseteq B$ and $B \not\subseteq Q_t \implies \exists K \in SC(S)$ with $P_t \subseteq sint(K) \subseteq K \subseteq Q_s$.*

(3) *For all $B \in SC(S)$, there is a family $\{K^i_j : i \in I,\ j \in J_i\} \subseteq SC(S)$ such that*

$$B = \bigcap_{i \in I} \bigvee_{j \in J_i} K^i_j = \bigcap_{i \in I} \bigvee_{j \in J_i} sint(K^i_j).$$

Proof. We prove (a), leaving the dual proof (b) to the reader.

(1) $\implies$ (2): It is clear.

(2) $\implies$ (3): Let $A \in SO(S)$ and $A \not\subseteq Q_s$ for some $s \in S$. For $P_t \not\subseteq A$, by (2) there exists $G^s_t \in SO(S)$ such that $P_s \subseteq G^s_t \subseteq scl(G^s_t) \subseteq Q_t$. Thus, by [4, Theorem 1.2(6)],

$$P_s \subseteq \bigcap_{P_t \not\subseteq A} G^s_t \subseteq \bigcap_{P_t \not\subseteq A} scl(G^s_t) \subseteq \bigcap_{P_t \not\subseteq A} Q_t = A.$$

So we obtain

$$A \subseteq \bigvee_{A \not\subseteq Q_s} P_s \subseteq \bigvee_{A \not\subseteq Q_s} \bigcap_{P_t \not\subseteq A} G_t^s \subseteq \bigvee_{A \not\subseteq Q_s} \bigcap_{P_t \not\subseteq A} scl(G_t^s) \subseteq A.$$

(3) $\Longrightarrow$ (1): Let $A \in SO(S)$, $A \not\subseteq Q_s$ and $P_t \not\subseteq A$. Then by the assumption, we have a family $\{G_j^i : i \in I,\ j \in J_i\} \subseteq SO(S)$ such that $A = \bigvee_{i \in I} \bigcap_{j \in J_i} scl(G_j^i)$. Thus, for all $i \in I$ we have $P_t \not\subseteq \bigcap_{j \in J_i} scl(G_j^i)$. So, for some $j_0 \in J_i$, we have $P_t \not\subseteq scl(G_{j_0}^i)$ and $G_{j_0}^i \not\subseteq Q_s$. □

Corollary 4.6. *If $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is a complemented ditopological texture space, then the notions of semi-R_1 and semi-co-R_1 are equivalent.*

Proof. Immediate from the point free characterization in Theorem 4.5(a-b-3). □

Now we recall [8] that a topological space X is said to be semi-regular if for each semi-open set A and $x \in A$, there exists a semi-open set G such that $x \in G$ and $scl(G) \subseteq A$. This leads to the following analogous concepts in a ditopological texture space.

Definition 4.7. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. Then (τ, κ) is called

(i) semi-regular if $A \in SO(S),\ A \not\subseteq Q_s \implies \exists G \in SO(S),\ G \not\subseteq Q_s,\ scl(G) \subseteq A$,

(ii) semi-co-regular if $B \in SC(S),\ P_s \not\subseteq B \implies \exists K \in SC(S),\ P_s \not\subseteq K,\ B \subseteq sint(K)$.

Theorem 4.8. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. Then*

(1) *(τ, κ) is semi-regular iff for all $A \in SO(S)$, there is a family $\{G_j : j \in J\} \subseteq SO(S)$ such that*

$$A = \bigvee_{j \in J} G_j = \bigvee_{j \in J} scl(G_j).$$

(2) *(τ, κ) is semi-co-regular iff for all $B \in SC(S)$, there is a family $\{K_j : j \in J\} \subseteq SC(S)$ such that*

$$B = \bigcap_{j \in J} K_j = \bigcap_{j \in J} sint(K_j).$$

Proof. We prove (1), leaving the dual proof (2) to the reader.

(1) ($\Longrightarrow$): Let $A \in SO(S)$ and $A \not\subseteq Q_s$ for some $s \in S$. By semi-regularity, there exists $G_s \in SO(S)$, $G_s \not\subseteq Q_s$ where $scl(G_s) \subseteq A$. Since $A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$, we have

$$A \subseteq \bigvee\{G_s \mid A \not\subseteq Q_s\} \subseteq \bigvee\{scl(G_s) \mid A \not\subseteq Q_s\} \subseteq A.$$

($\Longleftarrow$): It is clear. □

Corollary 4.9. *If $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is a complemented ditopological texture space, then the notions of semi-regular and semi-co-regular are coincide.*

Proof. Let (τ, κ) be a semi-regular ditopology and $B \in SC(S)$. Since $\sigma(B) \in SO(S)$, by semi-regularity, there is a family $\{G_j : j \in J\} \subseteq SO(S)$ such that

$$\sigma(B) = \bigvee_{j \in J} G_j = \bigvee_{j \in J} scl(G_j).$$

Thus we have

$$B = \sigma\left(\sigma\left(B\right)\right) = \sigma\left(\bigvee_{j\in J} G_j\right) = \sigma\left(\bigvee_{j\in J} scl(G_j)\right).$$

So

$$B = \bigcap_{j\in J} \sigma(G_j) = \bigcap_{j\in J} sint(\sigma(G_j)),$$

by Proposition 3.6. Considering $\sigma(G_j) \in SC(S)$, we deduce that (τ, κ) is semi-co-regular. The proof of sufficiency is similar. □

The following implications hold:

Theorem 4.10. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. Then*

$$\begin{aligned} &\textit{semi-regular} &&\implies \textit{semi} - R_1 \implies \textit{semi} - R_0, \\ &\textit{semi-co–regular} &&\implies \textit{semi-co-}R_1 \implies \textit{semi-co-}R_0. \end{aligned}$$

Proof. Let (τ, κ) be a semi-regular ditopological texture space. We show that it is semi-R_1. We take $A \in SO(S)$, $A \not\subseteq Q_s$, and $P_t \not\subseteq A$. By semi-regularity, we have $G \in SO(S)$ such that $G \not\subseteq Q_s$ and $scl(G) \subseteq A$. Since $P_t \not\subseteq A$, it is obtained $P_t \not\subseteq scl(G)$. Then (τ, κ) is semi-R_1.

Assume that (τ, κ) is semi-R_1. Now we show that it is semi-R_0. Let $A \in SO(S)$, $A \not\subseteq Q_s$, and $scl(P_s) \not\subseteq A$. Choose a point $t \in S$, where $scl(P_s) \not\subseteq Q_t$ and $P_t \not\subseteq A$. By the assumption, $P_t \not\subseteq scl(G)$ and $G \not\subseteq Q_s$ for some $G \in SO(S)$. Since $P_t \subseteq scl(P_s)$, we have $scl(P_s) \not\subseteq scl(G)$. Then, $P_s \not\subseteq G$, that is, $G \subseteq Q_s$. But this is a contradiction.

The other implications hold easily. □

Proposition 4.11. *Let $(S, \mathcal{S}, \tau, \kappa)$ be the product of the family $\{(S_j, \mathcal{S}_j, \tau_j, \kappa_j) \mid j \in J\}$ of ditopological texture spaces. Then we have the following.*

(1) $scl(\bigcap_{j\in J} E(j, A_j)) = \bigcap_{j\in J} E(j, scl_j(A_j))$. *That is,* $\prod_{j\in J} scl_j(A_j) = scl(\prod_{j\in J} A_j)$.

(2) $sint(\bigcup_{j\in J} E(j, A_j)) = \bigcup_{j\in J} E(j, sint_j(Aj))$.

Proof. (1) By definition of product ditopology, we have

$$scl(\prod_{j\in J} A_j) \subseteq \bigcap_{j\in J} E(j, scl_j(A_j)) = \prod_{j\in J} scl_j(A_j).$$

Let $A = \bigcap_{j\in J} E(j, A_j) = \prod_{j\in J} A_j$ and assume that

$$\bigcap_{j\in J} E(j, scl_j(A_j)) \not\subseteq scl(\prod_{j\in J} A_j) = scl(A).$$

Then we have $t = (t_j) \in S$ satisfying

$$\bigcap_{j\in J} E(j, scl_j(A_j)) \not\subseteq Q_t \text{ and } P_t \not\subseteq scl(\prod_{j\in J} A_j).$$

Also note that we must have $A_j \neq \varnothing$ for all $j \in J$ so we can choose $u_j \in S_j$ with $u_j \in A_j$. Since $P_t \not\subseteq scl(A)$, by definition of product ditopology, then there exists a family $\{C_j \mid j \in J\}$ such that

$$A \subseteq \bigcap_{j\in J} \pi_j^{\leftarrow} C_j \text{ and } P_t \not\subseteq \bigcap_{j\in J} \pi_j^{\leftarrow} C_j$$

whence $scl_j(C_j) = C_j$ for all $j \in J$. Also $P_t = \bigcap_{j\in J} E(j, P_{t_j}) \not\subseteq \bigcap_{j\in J} \pi_j^{\leftarrow} C_j$ gives $E(j, P_{t_j}) \not\subseteq \bigcap_{j\in J} \pi_j^{\leftarrow} C_j$ for all $j \in J$. Hence $E(j, P_{t_j}) \not\subseteq \pi_k^{\leftarrow} C_k$ for some $k \in J$ and so $P_{t_k} \not\subseteq C_k = scl_k(C_k)$.

On the other hand, $\bigcap_{j\in J} E(j, scl_j(A_j)) \not\subseteq \bigcup_{j\in J} E(j, Q_{t_j}) = Q_t$ gives $E(j, scl_j(A_j)) \not\subseteq E(j, Q_{t_j})$ for all $j \in J$. Then $scl_k(A_k) \not\subseteq Q_{t_k}$. Thus $scl_k(A_k) \not\subseteq scl_k(C_k)$. So $A_k \not\subseteq C_k$. Hence $A_k \not\subseteq Q_{a_k}$ and $P_{a_k} \not\subseteq C_k$ for some $a_k \in S_k$. Let $w = (w_j)$ defined by

$$w_j = \begin{cases} a_k, & \text{if } \ j = k \\ u_j, & \text{otherwise.} \end{cases}$$

Then $w = (w_j) \in S$ satisfies $w \in \bigcap_{j\in J} E(j, A_j)$ and $w \notin \bigcap_{j\in J} \pi_j^{\leftarrow} C_j$ which contradicts $\bigcap_{j\in J} E(j, A_j) \subseteq \bigcap_{j\in J} \pi_j^{\leftarrow} C_j$.

(2) By definition of product ditopology, clearly

$$\bigcup_{j\in J} E(j, sint_j(A_j)) \subseteq sint(\bigcup_{j\in J} E(j, A_j)).$$

Let $A = \bigcup_{j\in J} E(j, A_j)$ and assume that $sint(A) \not\subseteq \bigcup_{j\in J} E(j, sint(A_j))$. Then we have $t = (t_j) \in S$ satisfying $sint(A) \not\subseteq Q_t$ and $P_t \not\subseteq \bigcup_{j\in J} E(j, sint(A_j))$.
Since $sint(A) \not\subseteq Q_t$, by definition of product ditopology, there exists a family $\{B_j \mid j \in J\}$ such that

$$\bigvee_{j\in J} \Pi_j^{\leftarrow} B_j \subseteq A \ \text{ and } \ \bigvee_{j\in J} \Pi_j^{\leftarrow} B_j \not\subseteq Q_t$$

Thus $sint_j(B_j) = B_j$ for all $j \in J$. Since $\bigvee_{j\in J} \Pi_j^{\leftarrow} B_j \not\subseteq \bigcup_{j\in J} E(j, Q_{t_j}) = Q_t$, $\Pi_{j_0}^{\leftarrow} B_{j_0} \not\subseteq \bigcup_{j\in J} E(j, Q_{t_j})$ for some $j_0 \in J$. So $\Pi_{j_0}^{\leftarrow} B_{j_0} \not\subseteq E(j, Q_{t_j})$ for all $j \in J$. Hence $sint_{j_0}(B_{j_0}) = B_{j_0} \not\subseteq Q_{t_{j_0}}$.
Also $P_t = \bigcap_{j\in J} E(j, P_{t_j}) \not\subseteq \bigcup_{j\in J} E(j, sint_j(A_j))$ gives $P_{t_j} \not\subseteq sint_j(A_j)$ for all $j \in J$. Then $B_{j_0} \not\subseteq sint_j(A_j)$. Thus $sint_{j_0} B_{j_0} \not\subseteq sint_{j_0}(A_{j_0})$. So $B_{j_0} \not\subseteq A_{j_0}$, by the definition of semi-interior. But we have a contradiction by $\bigvee_{j\in J} \Pi_j^{\leftarrow} B_j \subseteq \bigcup_{j\in J} E(j, A_j) = A$. □

Theorem 4.12. *Let $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$, $j \in J$, be ditopological texture spaces, $(S, \mathcal{S})$ a texture and for each $j \in J$ let $(f_j, F_j) : (S, \mathcal{S}) \to (S_j, \mathcal{S}_j, \tau_j, \kappa_j)$ be a difunction. Consider the initial ditopological texture space (τ, κ) on $(S, \mathcal{S})$ generated by the difunctions (f_j, F_j) and the spaces $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$. Then if for each $j \in J$ the spaces $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$ are semi-R_0(semi-co-R_0, semi-R_1, semi-co-R_1, semi-regular, semi-co-regular), then the initial ditopological texture space (τ, κ) is semi-R_0 (semi-co-R_0, semi-R_1, semi-co-R_1, semi-regular, semi-co-regular).*

Proof. Firstly, note that the family $\beta_S = \{F_j^{\leftarrow}(G) \mid G \in SO(S_j), j \in J\}$ is a subbase for the family of semi-open sets in the initial ditopology, while the family $\gamma_S = \{f_j^{\leftarrow}(H) \mid H \in SC(S_j), j \in J\}$ is a co-subbase for the family of semi-closed sets in the initial ditopology.

Now, we prove the results for semi-R_0, semi-co-R_0 and semi-regular, leaving the other dual proofs interested reader.

Assume that for each $j \in J$, the spaces $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$ are semi-R_0. It will be sufficient to show that $F_j^{\leftarrow}(G)$, $G \in SO(S_j)$ can be written as a join element of $SC(S)$. However, (τ_j, κ_j) is semi-R_0 so by Theorem 4.2 (a) (2) we can write $G =$

$\bigvee_{i\in I} B_i$ for $B_i \in SC(S_j), i \in I$. Hence, $F_j^{\leftarrow}(G) = F_j^{\leftarrow}(\bigvee_{i\in I} B_i) = \bigvee_{i\in I} F_j^{\leftarrow}(B_i)$ by [4, Corollary 2.26], and $F_j^{\leftarrow}(B_i) = f_j^{\leftarrow}(B_i) \in SC(S)$, we deduce (τ,κ) is semi-R_0.

We now assume that the spaces are semi-R_1. Then we can write $G = \bigvee_{i\in I}\bigcap_{k\in K_i} H_k^i = \bigvee_{i\in I}\bigcap_{k\in K_i} scl(H_k^i)$ with $H_k^i \in SO(S_j)$ for all i,k. Hence,

$$F_j^{\leftarrow}(G) = \bigvee_{i\in I}\bigcap_{k\in K_i} F_j^{\leftarrow}(H_k^i) = \bigvee_{i\in I}\bigcap_{k\in K_i} F_j^{\leftarrow}(scl(H_k^i))$$

again using [4, Corollary 2.26]. Now, for each i,k we have $F_j^{\leftarrow}(H_k^i) \subseteq F_j^{\leftarrow}(scl(H_k^i))$ and so $\bigcap_{k\in K_i} F_j^{\leftarrow}(H_k^i) \subseteq \bigcap_{k\in K_i} F_j^{\leftarrow}(scl(H_k^i))$. Since the set on the right belongs to $SC(S)$, we obtain $\bigcap_{k\in K_i} F_j^{\leftarrow}(H_k^i) \subseteq \bigcap_{k\in K_i} scl(F_j^{\leftarrow}(H_k^i)) \subseteq \bigcap_{k\in K_i} F_j^{\leftarrow}(scl(H_k^i))$. Taking the join over $i \in I$ and using the above equalities for $F_j^{\leftarrow}(G)$ now leads to

$$F_j^{\leftarrow}(G) = \bigvee_{i\in I}\bigcap_{k\in K_i} F_j^{\leftarrow}(H_k^i) = \bigvee_{i\in I} scl(\bigcap_{k\in K_i} F_j^{\leftarrow}(H_k^i))$$

Since $F_j^{\leftarrow}(H_k^i) \in SO(S)$ for all i,k we deduce that (τ,κ) is semi-R_1.

Finally, let the given spaces be semi-regular, whence we can write $G = \bigvee_{k\in K} G_k = \bigvee_{k\in K} scl(G_k)$ for $G_k \in SO(S_j)$. Arguing as above, we obtain from $F_j^{\leftarrow}(G) = \bigvee_{k\in K} F_j^{\leftarrow}(G_k) = \bigvee_{k\in K} F_j^{\leftarrow} scl(G_k)$ that

$$F_j^{\leftarrow}(G) = \bigvee_{k\in K} F_j^{\leftarrow}(G_k) = \bigvee_{k\in K} scl(F_j^{\leftarrow}(G_k)),$$

whence (τ,κ) is semi-regular by Theorem 4.8. □

Theorem 4.13. *Let $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$, $j \in J$ be non-empty ditopological texture spaces and $(S, \mathcal{S}, \tau, \kappa)$ their product. Then*

(1) $(S, \mathcal{S}, \tau, \kappa)$ is semi-R_0 (semi-co-R_0) if and only if $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$ is semi-R_0 (respectively, semi-co-R_0) for all $j \in J$.

(2) $(S, \mathcal{S}, \tau, \kappa)$ is semi-R_1 (semi-co-R_1) if and only if $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$ is semi-R_1 (respectively, semi-co-R_1) for all $j \in J$.

(3) $(S, \mathcal{S}, \tau, \kappa)$ is semi-regular (semi-co-regular) if and only if $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$ is semi-regular (respectively, semi-co-regular) for all $j \in J$.

Proof. ($\Longleftarrow$): Since the product ditopology (τ,κ) is the initial ditopological texture space generated, by the projection difunctions (π_j, Π_j) and the spaces $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$, the sufficiency is clear from Theorem 4.12.

($\Longrightarrow$): We prove the result for semi-R_0, leaving the remaining cases to the interested reader. Assume that (τ,κ) is semi-R_0. Take $k \in J$, $G \in SO(S_k)$ and $G \not\subseteq Q_s^k$. For $j \in J\setminus\{k\}$ choose $a_j \in S_j^\flat$ and define $u = (u_j) \in S$ by

$$u_j = \begin{cases} s, \text{ if } j = k \\ a_j, \text{ otherwise.} \end{cases}$$

Since $Q_u = \bigcup_{j\in J} E(j, Q_{u_j})$, by [4, Proposition 1.3(2)] and $Q_{u_j} = Q_{a_j} \neq S_j$ for $j \in J\setminus\{k\}$, we have $E(k, sint_k(G)) \not\subseteq Q_u$. Here $sint(E(k,G)) = E(k, sint_k(G)) \not\subseteq Q_u$. Since (τ,κ) is semi-R_0, we have $scl(P_u) \subseteq sint(E(k,G))$. But $P_u = \prod_{j\in J} P_{u_j}$, by [5, Proposition 1.3(1)]. Thus $scl(P_u) = \prod_{j\in J} scl_j P_{u_j}$. So $scl_k(P_s) \subseteq sint_k(G)$. Hence we establish that (τ_k, κ_k) is semi-R_0. □

Finally, let us investigate preservation of the semi-regularity properties under difunctions.

Proposition 4.14. *Let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$ be ditopological texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ a surjective difunction.*

(1) *If (τ_1, κ_1) is semi-R_0 and (f, F) is semi-irresolute and semi-closed then (τ_2, κ_2) is semi-R_0.*

(2) *If (τ_1, κ_1) is semi-co-R_0 and (f, F) is semi-co-irresolute and semi-co-closed then (τ_2, κ_2) is semi-co-R_0.*

Proof. We prove (1), leaving the dual proof (2) to the reader.

Let $A \in SO(S_2)$. Since (f, F) is semi-irresolute, we have $F^{\leftarrow}(A) \in SO(S_1)$. By semi-$R_0$, we have $\{K_j \mid j \in J\} \subseteq SC(S_1)$ with $F^{\leftarrow}(A) = \bigvee_{j\in J} K_j$. Since (f, F) is surjective, by [4, Corollary 2.26 and 2.33(1)], we have

$$A = f^{\rightarrow}(F^{\leftarrow}(A)) = f^{\rightarrow}(\bigvee_{j\in J} K_j) = \bigvee_{j\in J} f^{\rightarrow}(K_j).$$

Since (f, F) is semi-closed $f^{\rightarrow}(K_j) \in SO(S_2)$ for all $j \in J$, whence $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is semi-R_0 by Theorem 4.2. □

Proposition 4.15. *Let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$ be ditopological texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ a bijective difunction.*

(1) *If (τ_1, κ_1) is semi-R_1 and (f, F) is semi-irresolute and semi-closed and semi-open then (τ_2, κ_2) is semi-R_1.*

(2) *If (τ_1, κ_1) is semi-co-R_1 and (f, F) is semi-co-irresolute and semi-co-closed and semi-co-open then (τ_2, κ_2) is semi-co-R_1.*

Proof. We prove (1), leaving the dual proof (2) to the reader.

Let $A \in SO(S_2)$. Since (f, F) is semi-irresolute, we have $F^{\leftarrow}(A) \in SO(S_1)$. By semi-$R_1$, we have $\{G^i_j \mid i \in I, j \in J_i\} \subseteq SO(S_1)$ with

$$F^{\leftarrow}(A) = \bigvee_{i\in I} \bigcap_{j\in J_i} scl(G^i_j) = \bigvee_{i\in I} \bigcap_{j\in J_i} G^i_j.$$

Since (f, F) is bijective, $f^{\rightarrow}G = F^{\rightarrow}G$ for all $G \in \mathcal{S}_1$. Using same arguments in the proof of above theorem, we have

$$\begin{aligned} A = f^{\rightarrow}(F^{\leftarrow}(A)) &= f^{\rightarrow}(\bigvee_{i\in I} \bigcap_{j\in J_i} scl(G^i_j)) \\ &= \bigvee_{i\in I} f^{\rightarrow}(\bigcap_{j\in J_i} scl(G^i_j)) = \bigvee_{i\in I} F^{\rightarrow}(\bigcap_{j\in J_i} scl(G^i_j)) \\ &= \bigvee_{i\in I} \bigcap_{j\in J_i} F^{\rightarrow}(scl(G^i_j)) = \bigvee_{i\in I} \bigcap_{j\in J_i} f^{\rightarrow}(scl(G^i_j)) \end{aligned}$$

Since (f, F) is semi-closed and semi-open, $f^{\rightarrow}(G^i_j) \in SO(S_2)$ and $f^{\rightarrow}(scl(G^i_j)) \in SC(S_2)$ for all $i \in I$ and $j \in J_i$. Thus $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is semi-R_1, by Theorem 4.5. □

Proposition 4.16. *Let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$ be ditopological texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ a surjective difunction.*

(1) *If (τ_1, κ_1) is semi-regular and (f, F) is semi-irresolute and semi-closed and semi-open then (τ_2, κ_2) is semi-regular.*

(2) *If (τ_1, κ_1) is semi-co-regular and (f, F) is semi-co-irresolute and semi-co-closed and semi-co-open, then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is semi-co-regular.*

Proof. We prove (1), leaving the dual proof (2) to the reader.

Let $A \in SO(S_2)$. Since (f, F) is semi-irresolute, we have $F^{\leftarrow}(A) \in SO(S_1)$. By semi-regularity, we have $\{G_j \mid j \in J\} \subseteq SO(S_1)$ with

$$F^{\leftarrow}(A) = \bigvee_{j \in J} G_j = \bigvee_{j \in J} scl(G_j)$$

Then we have

$$A = f^{\rightarrow}(F^{\leftarrow}(A)) = \bigvee_{j \in J} f^{\rightarrow}(G_j) = \bigvee_{j \in J} f^{\rightarrow}(scl(G_j)).$$

Since (f, F) is semi-closed and semi-open, $f^{\rightarrow}(G_j) \in SO(S_2)$ and $f^{\rightarrow}(scl(G_j)) \in SC(S_2)$ for all $j \in J$. Thus $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is semi-regular, by Theorem 4.8. □

From Propositions 4.14, 4.15, 4.16, we have immediately:

Corollary 4.17. *Let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$ be ditopological texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ a semi-dihomeomorphism. Then $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is semi-R_0 (-co-R_0, -R_1,-co-R_1, -regular, -co-regular) if and only if $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is semi-R_0 (respectively, -co-R_0, -R_1,-co-R_1, -regular, -co-regular).*

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