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Fixed point theorems in fuzzy $\alpha-$norm spaces

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Abstract. We define fixed point in fuzzy $\alpha-$norm spaces and prove the existence theorems, we also consider Extensions of Banach's theorem and show its relation with fixed point in $\alpha-$norm spaces. The paper concludes with a result about Extensions of Banach's theorem in $\alpha-$norm spaces setting.

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1. Introduction

In this paper, starting from the article of Kirk [7], we study Extensions of Banach's theorem on fuzzy $\alpha-$norm spaces, and we give some fuzzy fixed points of such theorem. In 1992, Felbin [5] has offered in 1992 an alternative definition of a fuzzy norm on a linear space with an associated metric of the Kaleva and Seikkala type [6]. He has shown that every finite dimensional normed linear space has a completion. Then Xiao and Zhu [10] have modified the definition of this fuzzy norm and studied the topological properties of fuzzy normed linear spaces. Thereafter the concept of fuzzy norm space has been introduced and generalized in different ways by Bag and Samanta in [1], [2]. Moreover, some authors introduce some reasonable versions of fixed point theorems on fuzzy normed spaces (see [3, 4]).
Throughout this article, the symbols $\wedge$ and $\vee$ mean the **inf** and the **sup**, respectively.

2. Preliminaries

We start our work with the following definitions.

Definition 2.1 ([10]). A mapping $\eta : \mathbb{R} \longrightarrow [0,1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies the following conditions :

(N1) there exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.

(N2) for each $\alpha \in (0,1]$, there exist real numbers $\eta_\alpha^- \leq \eta_\alpha^+$ such that the $\alpha-$level set $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.

The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$.

Since each $r \in \mathbb{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbb{R})$ defined by

$$\tilde{r}(t) = \begin{cases} 1, & if\, t = r \\ 0, & if\, t \neq r, \end{cases}$$

it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$.

Definition 2.2 ([6]). The arithmetic operations $+$, $-$, $\times$ and $/$ on $F(\mathbb{R}) \times F(\mathbb{R})$ are defined by

$$\begin{aligned} (\eta + \gamma)(t) &= \sup_{t=x+y} (\min(\eta(x), \gamma(y))), \\ (\eta - \gamma)(t) &= \sup_{t=x-y} (\min(\eta(x), \gamma(y))), \\ (\eta \times \gamma)(t) &= \sup_{t=xy} (\min(\eta(x), \gamma(y))), \\ (\eta / \gamma)(t) &= \sup_{t=x/y} (\min(\eta(x), \gamma(y))), \end{aligned}$$

which are special cases of Zadeh's extension principle.

Definition 2.3 ([6]). Let $\eta \in F(\mathbb{R})$. If $\eta(t) = 0$, for all $t < 0$, then η is called a positive fuzzy real number. The set of all positive fuzzy real numbers is denoted by $F^+(\mathbb{R})$.

Definition 2.4 ([6]). Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0,1]$. Define a partial ordering by $\eta \leq \gamma$ if and only if $\eta_\alpha^- \leq \gamma_\alpha^-$ and $\eta_\alpha^+ \leq \gamma_\alpha^+$, for all $\alpha \in (0,1]$. Strict inequality in $F(\mathbb{R})$ is defined by $\eta < \gamma$ if and only if $\eta_\alpha^- < \gamma_\alpha^-$ and $\eta_\alpha^+ < \gamma_\alpha^+$, for all $\alpha \in (0,1]$.

Lemma 2.5. *Let $\eta \in F(\mathbb{R})$. Then $\eta \in F^+(\mathbb{R})$ if and only if $\tilde{0} \leq \eta$.*

Definition 2.6 ([1]). Let X be a linear space over $\mathbb{R}$. Let N be A fuzzy subset of $X \times \mathbb{R}$ such that for all $x, u \in X$ and $c \in \mathbb{R}$,

$(N1)$ $N(x,t) = 0$, for all $t \leq 0$,

$(N2)$ $x = 0$ if and only if $N(x,t) = 1$, for all $t > 0$,

$(N3)$ If $c \neq 0$ then $N(cx,t) = N(x, t/|c|)$, for all $t \in \mathbb{R}$,

$(N4)$ $N(x+u, s+t) \geq min\{N(x,s), N(u,t)\}$, for all $s, t \in \mathbb{R}$,

$(N5)$ $N(x,.)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

Then N is called a fuzzy norm on X.

Sometimes, we need two additional conditions as follows :

$(N6)$ $N(x,t) > 0$, for all $t > 0$ implies $x = 0$,

$(N7)$ For $x \neq 0$, $N(x,.)$ is a continuous function of $\mathbb{R}$ and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of $\mathbb{R}$.

Definition 2.7 ([1])**.** Let (X, N) be a fuzzy normed linear space.

(i) A sequence $\{x_n\} \subseteq X$ is said to converge to $x \in X$ ($\lim\limits_{n\to\infty} x_n = x$), if $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$.

(ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim\limits_{n,m\to\infty} N(x_n - x_m, t) = 1$, for all $t > 0$.

Definition 2.8 ([2])**.** Let (X, N) and (Y, N) be fuzzy normed linear spaces. Furthermore, let $f : X \longrightarrow Y$ be a function. The function f is said to be continuous at $x_0 \in X$, if for any sequence $\{x_n\} \subseteq X$ with $x_n \longrightarrow x_0$ implies $f(x_n) \longrightarrow f(x_0)$. f is said fuzzy continuous on X if f be fuzzy continuous at each point of X.

Definition 2.9. If X is a vector space over $\mathbb{R}$, a seminorm is a function $p : X \longrightarrow [0, \infty)$ having the properties :

(i) $p(cx) = |c|p(x)$, for all $c \in \mathbb{R}$ and $x \in X$.

(ii) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$.

Theorem 2.10 ([8])**.** *Let (X, N) be a fuzzy normed linear space. Define*

$$\|x\|_\alpha = \inf\{t > 0 : N(x, t) \geq \alpha\}, \ \alpha \in (0, 1).$$

Then $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of seminorms on X and they are called α-seminorms on X corresponding to the fuzzy norm N on X.

Example 2.11. Let $(X, \|.\|)$ be a normed linear space. Define a fuzzy norm N as follows:

$$N(x, t) = \begin{cases} t/(t + \|x\|), & if\, t > 0 \\ 0 \quad , & if\, t \leq 0. \end{cases}$$

We have

$$\begin{aligned} \|x\|_\alpha &= \wedge\{t > 0 : N(x, t) \geq \alpha\} \\ &= \wedge\{t > 0 : t/(t + \|x\|) \geq \alpha\} \\ &= \wedge\{t > 0 : t \geq \alpha\|x\|/(1 - \alpha)\} \\ &= (\alpha/(1 - \alpha))\|x\|. \end{aligned}$$

3. Fuzzy fixed point theorems

In this section, we will obtain the theorems and result about fuzzy α-norm fixed point.

Theorem 3.1. *Let (X, N) be a fuzzy Banach space and $f : X \to X$ a continuous map. Moreover, there exists a fuzzy real number $\tilde{0} < \eta < \tilde{1}$ such that*

$$N(x - f(x), t) \geq \alpha \textit{ implies that } N(f(x) - f^2(x), t\eta^-_\alpha) \geq \alpha, \textit{ for all } x \in X.$$

Then f has a fixed point in X.

Proof. Let $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of α-seminorms on X corresponding to the fuzzy norm N on X.

Suppose that $x_0 \in X$. If $N(x_0 - f(x_0), t) \geq \alpha$, then $N(f(x_0) - f^2(x_0), t\eta^-_\alpha) \geq \alpha$. Thus $\|f(x_0) - f^2(x_0)\|_\alpha \leq t\eta^-_\alpha$. So

$$\|f(x_0) - f^2(x_0)\|_\alpha \leq \eta^-_\alpha \|f(x_0) - x_0\|_\alpha, \text{ for all } \alpha \in (0, 1).$$

By induction, we obtain that

$$\|f^n(x_0) - f^{n+1}(x_0)\|_\alpha \leq (\eta^-_\alpha)^n \|f(x_0) - x_0\|_\alpha, \text{ for all } \alpha \in (0,1) \text{ and } n \in \mathbb{N}.$$

Hence

$$\|f^n(x_0) - f^m(x_0)\|_\alpha \leq (\sum_{k=n}^{m-1} (\eta^-_\alpha)^k) \|f(x_0) - x_0\|_\alpha, \text{ for all } \alpha \in (0,1) \text{ and } m > n.$$

Since $\tilde{0} < \eta < \tilde{1}$, it follows that $0 < \eta^-_\alpha < 1$, for all $\alpha \in (0,1)$. Thus $\sum_{n=1}^{\infty} (\eta^-_\alpha)^n$ is convergent, for all $\alpha \in (0,1)$. So for all $\epsilon > 0$ and $\alpha \in (0,1)$, there exists $N_\alpha > 0$ such that $\|f^n(x_0) - f^m(x_0)\|_\alpha < \epsilon$, for all $n, m \geq N_\alpha$. Let $t > 0$ and $0 < \epsilon < 1$ be given. Then there is $N > 0$ such that $\|f^n(x_0) - f^m(x_0)\|_{1-\epsilon} < t$, for all $n, m \geq N$. Thus $\inf\{s > 0 : N(f^n(x_0) - f^m(x_0), s) \geq 1 - \epsilon\} < t$, for all $n, m \geq N$. So $N(f^n(x_0) - f^m(x_0), t) \geq 1 - \epsilon$, for all $n, m \geq N$. Hence $\{f^n(x_0)\}$ is a Cauchy sequence. Since (X, N) is a Banach Space, there exists $y \in X$ such that $f^n(x_0) \longrightarrow y$. Since f is continuous, $f^{n+1}(x_0) \longrightarrow f(y)$. Therefore

$$f(y) = \lim_{n\to\infty} f^{n+1}(x_0) = \lim_{n\to\infty} f^n(x_0) = y.$$

□

Corollary 3.2. *Let (X, N) be a fuzzy Banach space and $f : X \longrightarrow X$ a continuous map. Moreover, there exists a fuzzy real number $\tilde{0} < \eta < \tilde{1}$ such that*

$$N(f(x) - f^2(x), t\eta^-_\alpha) \geq N(x - f(x), t), \text{ for all } x \in X.$$

Then f has a fixed point in X.

Theorem 3.3. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*

(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,

(2) $N(x - y, t) \geq \alpha$ implies that $N(f(x) - f(y), t\eta^-_\alpha) \geq \alpha$, for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \tilde{1}$.

Then f has a fixed point in $A \cap B$.

Proof. Let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of α-seminorms on X corresponding to the fuzzy norm N on X. Suppose that $x_0 \in X$. Then it readily follows that $N(x_0 - f(x_0), t) \geq \alpha$ implies that $N(f(x_0) - f^2(x_0), t\eta^-_\alpha) \geq \alpha$. Similar to proof of Theorem 3.1, $\{f^n(x_0)\}$ is a Cauchy sequence. Consequently $\{f^n(x_0)\}$ converges to some point $z \in A \cup B$. Since f is continuous,

$$f(z) = \lim_{n\to\infty} f^{n+1}(x_0) = \lim_{n\to\infty} f^n(x_0) = z.$$

If $z \in A$, then $z = f(z) \in B$. If $z \in B$, then $z = f(z) \in A$. Thus $z \in A \cap B$. So $A \cap B \neq \varnothing$. Since A and B are closed and X is complete, $A \cap B$ is a complete subspace of X. Since $f : A \cap B \longrightarrow A \cap B$ is a continuous and $N(x - f(x), t) \geq \alpha$,

$$N(f(x) - f^2(x), t\eta^-_\alpha) \geq \alpha, \text{ for all } x \in A \cap B.$$

Hence by Theorem 3.1, f has a fixed point in $A \cap B$. □

Now by example we show that these theorems are extension of classical analysis to fuzzy analysis.

Example 3.4. Let $(X, \|.\|)$ be a Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies

(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,

(2) $\|f(x)-f(y)\| \leq k\|x-y\|$, for all $x \in A$ and $y \in B$, where $k \in \mathbb{R}$ and $0 < k < 1$.

Define a fuzzy norm N as follows :

$$N(x,t) = \begin{cases} t/\|x\| & , \quad 0 < t \leq \|x\| \\ 1 & , \quad \|x\| < t \\ 0 & , \quad t \leq 0. \end{cases}$$

Let $\tilde{k} \in F(\mathbb{R})$ and $[\tilde{k}]_\alpha = [k, k]$. Assume that $x \in A$, $y \in B$ and $N(x - y, t) \geq \alpha$.

Case1: Let $t < \|x - y\|$. Then $t/\|x - y\| = N(x - y, t) \geq \alpha$. Thus

$$\alpha\|f(x) - f(y)\| \leq \alpha k\|x - y\| \leq kt.$$

So $kt/\|f(x) - f(y)\| \geq \alpha$. Hence $N(f(x) - f(y), kt) \geq \alpha$.

Case2: Let $t \geq \|x - y\|$. Then $kt \geq k\|x - y\| \geq \|f(x) - f(y)\|$. This implies that

$$N(f(x) - f(y), tk) \geq \alpha.$$

Thus f has a fixed point in $A \cap B$.

Example 3.5. Let $\mathbb{R}$ be a real number set and $A = [-1, 0]$, $B = [0, 1]$. Suppose that map $f : A \cup B \longrightarrow A \cup B$ defined by $f(x) = (-1/2)x$. Define a fuzzy norm N as follows:

$$N(x,t) = \begin{cases} t/|x| & , \quad 0 < t \leq |x| \\ 1 & , \quad |x| < t \\ 0 & , \quad t \leq 0. \end{cases}$$

It is clear that f satisfies

(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,

(2) $|f(x) - f(y)| = (1/2)|x - y|$, for all $x \in A$ and $y \in B$.

By Example 3.4, f has a fixed point in $A \cap B$.

Corollary 3.6. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*

(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,

(2) $N(f(x) - f(y), t\eta^-_\alpha) \geq N(x - y, t)$, for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \tilde{1}$.

Then f has a fixed point in $A \cap B$.

Corollary 3.7. *Let (X, N) be a fuzzy Banach space and A, B be two non-empty closed subsets of X. Suppose that $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be two functions such that*

$$N(x - y, t) \geq \alpha \text{ implies that } N(f(x) - g(y), t\eta^-_\alpha) \geq \alpha, \text{ for all } x \in A, \ y \in B,$$

where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \tilde{1}$. Then f has a fixed point in $A \cap B$.

Corollary 3.8. *Let (X, N) be a fuzzy Banach space and A, B be two non-empty closed subsets of X. Suppose that $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be two functions such that*

$$N(f(x) - g(y), t\eta^-_\alpha) \geq N(x - y, t), \text{ for all } x \in A, \ y \in B,$$

where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \tilde{1}$. Then f has a fixed point in $A \cap B$.

Theorem 3.9. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*
(1) *$f(A) \subseteq B$ and $f(B) \subseteq A$,*
(2) *$N(f(x) - x, t) \geq \alpha$ and $N(f(y) - y, s) \geq \alpha$ imply that $N(f(x) - f(y), (t + s)\eta_\alpha^-) \geq \alpha$, for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \widetilde{1/4}$.*
Then f has a unique fixed point in $A \cap B$.

Proof. Let $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of α-seminorms on X corresponding to the fuzzy norm N on X.

Assume that $N(f(x) - x, t) \geq \alpha$ and $N(f(y) - y, s) \geq \alpha$. Then

$$N(f(x) - f(y), (t + s)\eta_\alpha^-) \geq \alpha, \text{ for all } x \in A \text{ and } y \in B.$$

Thus $\|f(x) - f(y)\|_\alpha \leq (t + s)\eta_\alpha^-$. So

$$\|f(x) - f(y)\|_\alpha \leq (\|f(x) - x\|_\alpha + \|f(y) - y\|_\alpha)\eta_\alpha^-, \text{ for all } x \in A, y \in B.$$

Suppose that $x \in A$. Then we have

$$\|f^2(x) - f(x)\|_\alpha \leq (\|f^2(x) - f(x)\|_\alpha + \|f(x) - x\|_\alpha)\eta_\alpha^-,$$

which implies that $\|f^2(x) - f(x)\|_\alpha \leq t_\alpha\|f(x) - x\|_\alpha$ where $t_\alpha = \eta_\alpha^-/(1 - \eta_\alpha^-) \in (0, 1)$. By induction, we obtain that $\|f^{n+1}(x) - f^n(x)\|_\alpha \leq t_\alpha^n\|f(x) - x\|_\alpha$, for all $n \in \mathbb{N}$. Thus

$$\|f^n(x) - f^m(x)\|_\alpha \leq (\sum_{k=n}^{m-1} t_\alpha^k)\|f(x) - x\|_\alpha, \text{ for all } \alpha \in (0, 1) \text{ and } m > n.$$

Since $0 < t_\alpha < 1$, for all $\alpha \in (0, 1)$, $\sum_{n=1}^{\infty} t_\alpha^n$ is convergent, for all $\alpha \in (0, 1)$. So for all $\epsilon > 0$ and $\alpha \in (0, 1)$, there exists $N_\alpha > 0$ such that

$$\|f^n(x) - f^m(x)\|_\alpha < \epsilon, \text{ for all } n, m \geq N_\alpha.$$

Let $x_0 \in A$, $t > 0$ and $0 < \epsilon < 1$ be given. Then there is $N > 0$ such that

$$\|f^n(x_0) - f^m(x_0)\|_{1-\epsilon} < t, \text{ for all } n, m \geq N.$$

Thus

$$\inf\{s > 0 : N(f^n(x_0) - f^m(x_0), s) \geq 1 - \epsilon\} < t, \text{ for all } n, m \geq N.$$

So

$$N(f^n(x_0) - f^m(x_0), t) \geq 1 - \epsilon, \text{ for all } n, m \geq N.$$

Hence $\{f^n(x_0)\}$ is a Cauchy sequence. Since (X, N) is a Banach Space, there exists $z \in X$ such that $f^n(x_0) \longrightarrow z$. By (1), we get $z \in A \cap B$.

Let $t > 0$ fixed. Then we have

$$N(f(z) - z, t) \geq \min\{N(f(z) - f^n(x_0), t/2), N(f^n(x_0) - z, t/2)\}.$$

By (2), we observe that

$$N(f(z) - f^n(x_0), t/2) \geq \min\{N(f^{n-1}(z) - f^n(x_0), t/(4\eta_\alpha^-)), N(f(z) - z, t/(4\eta_\alpha^-))\},$$

for all $n \in \mathbb{N}$. Since $\{f^n(x_0)\}$ is a Cauchy sequence and $f^n(x_0) \longrightarrow z$, there exists $N > 0$ such that

$$\min\{N(f^{n-1}(x)-f^n(x_0),t/(4\eta_\alpha^-)),N(f^n(x_0)-z,t/2)\}\geq\alpha,\text{ for all }n,m\geq N.$$

Thus $N(f(z)-z,t)\geq\min\{\alpha,N(f(z)-z,t/(4\eta_\alpha^-))\}$. So we obtain that

$$N(f(z)-z,t)\geq\min\{\alpha,N(f(z)-z,t/(4\eta_\alpha^-)^n)\},\text{ for all }n\in N.$$

Since $\tilde{0}<\eta<\widetilde{1/4}$, it follows that $\lim_{n\to\infty}t/(4\eta_\alpha^-)^n=\infty$. By (N5), we have $\lim_{n\to\infty}N(f(z)-z,t/(4\eta_\alpha^-)^n)=1$. Hence $N(f(z)-z,t)\geq\alpha$, for all $\alpha\in(0,1)$. Which implies that $N(f(z)-z,t)=1$, for all $t>0$, so $f(z)=z$.

To prove the uniqueness of z, assume that there exists $w\in A\cap B$ such that $f(w)=w$. Then $N(f(z)-z,t/(2\eta_\alpha^-))=1=N(f(w)-w,t/(2\eta_\alpha^-))$, for all $\alpha\in(0,1)$ and all $t>0$, by (2), $N(f(z)-f(w),t)\geq\alpha$, for all $\alpha\in(0,1)$. Thus $N(f(z)-f(w),t)=1$, for all $t>0$. So $z=f(z)=f(w)=w$. □

Example 3.10. Let $(X,\|.\|)$ be a Banach space and A, B be non-empty closed subsets of X. Suppose that map $f:A\cup B\longrightarrow A\cup B$ satisfies

(1) $f(A)\subseteq B$ and $f(B)\subseteq A$,

(2) $\|f(x)-f(y)\|\leq k(\|f(x)-x\|+\|f(y)-y\|)$, for all $x\in A$ and $y\in B$, where $k\in\mathbb{R}$ and $0<k<1/4$.

Define a fuzzy norm N as follows :

$$N(x,t)=\begin{cases}t/\|x\| & , \quad 0<t\leq\|x\|\\ 1 & , \quad \|x\|<t\\ 0 & , \quad t\leq 0.\end{cases}$$

Let $\tilde{k}\in F(\mathbb{R})$ and $[\tilde{k}]_\alpha=[k,k]$. Assume that $x\in A$, $y\in B$ and $N(f(x)-x,t)\geq\alpha$, $N(f(y)-y,s)\geq\alpha$.

Case1: Let $t<\|f(x)-x\|$ and $s<\|f(y)-y\|$. Then

$$t/\|f(x)-x\|=N(f(x)-x,t)\geq\alpha$$

and

$$s/\|f(y)-y\|=N(f(y)-y,t)\geq\alpha.$$

Thus

$$\alpha\|f(x)-f(y)\|\leq\alpha k(\|f(x)-x\|+\|f(y)-y\|)\leq k(t+s).$$

So $k(t+s)/\|f(x)-f(y)\|\geq\alpha$. Hence $N(f(x)-f(y),k(t+s))\geq\alpha$.

Case2: Let $t\geq\|f(x)-x\|$ and $s<\|f(y)-y\|$. Then

$$kt\geq k\|f(x)-x\|\geq\alpha k\|f(x)-x\|\text{ and }ks\geq\alpha k\|f(y)-y\|.$$

Thus $\alpha\|f(x)-f(y)\|\leq\alpha k(\|f(x)-x\|+\|f(y)-y\|)\leq k(t+s)$.
So $N(f(x)-f(y),k(t+s))\geq\alpha$.

Case3: Let $t<\|f(x)-x\|$ and $s\geq\|f(y)-y\|$. Similar to case2,

$$N(f(x)-f(y),k(t+s))\geq\alpha.$$

case4: Let $t\geq\|f(x)-x\|$ and $s\geq\|f(y)-y\|$. Then

$$kt\geq k\|f(x)-x\|\text{ and }ks\geq k\|f(y)-y\|.$$

Thus

$$\|f(x)-f(y)\|\leq k(\|f(x)-x\|+\|f(y)-y\|)\leq k(t+s).$$

So

$$N(f(x)-f(y),k(t+s))\geq\alpha.$$

Hence f has a unique fixed point in $A\cap B$.

Example 3.11. Let $\mathbb{R}$ be a real number set and $A=B=[0,1]$. Suppose that map $f:A\cup B\longrightarrow A\cup B$ defined by

$$f(x)=\begin{cases}1/6 & , \quad x=1\\ 1/3 & , \quad x\neq 1.\end{cases}$$

Define a fuzzy norm N as follows :

$$N(x,t)=\begin{cases}t/|x| & , \quad 0<t\leq|x|\\ 1 & , \quad |x|<t\\ 0 & , \quad t\leq 0.\end{cases}$$

It is clear that f satisfies

(1) $f(A)\subseteq B$ and $f(B)\subseteq A$,

(2) $|f(x)-f(y)|=(1/5)(|f(x)-x|+|f(y)-y|)$, for all $x\in A$ and $y\in B$.

By Example 3.10, f has a unique fixed point in $A\cap B$.

Corollary 3.12. *Let (X,N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f:A\cup B\longrightarrow A\cup B$ satisfies*

(1) $f(A)\subseteq B$ and $f(B)\subseteq A$,

(2) $N(f(x)-f(y),(t+s)\eta_\alpha^-)\geq 1/2(N(f(x)-x,t)+N(f(y)-y,s))$, for all $x\in A$ and $y\in B$, where $\eta\in F(\mathbb{R})$ and $\tilde{0}<\eta<\widetilde{1/4}$.

Then f has a unique fixed point in $A\cap B$.

Corollary 3.13. *Let (X,N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f:A\cup B\longrightarrow A\cup B$ satisfies*

(1) $f(A)\subseteq B$ and $f(B)\subseteq A$,

(2) $N(f(x)-f(y),(t+s)\eta_\alpha^-)\geq\min\{N(f(x)-x,t),N(f(y)-y,s)\}$, for all $x\in A$ and $y\in B$, where $\eta\in F(\mathbb{R})$ and $\tilde{0}<\eta<\widetilde{1/4}$.

Then f has a unique fixed point in $A\cap B$.

Theorem 3.14. *Let (X,N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f:A\cup B\longrightarrow A\cup B$ satisfies*

(1) $f(A)\subseteq B$ and $f(B)\subseteq A$,

(2) $N(f(x)-y,t)\geq\alpha$ and $N(f(y)-x,s)\geq\alpha$ implies that $N(f(x)-f(y),(t+s)\eta_\alpha^-)\geq\alpha$, for all $x\in A$ and $y\in B$, where $\eta\in F(\mathbb{R})$ and $\tilde{0}<\eta<\widetilde{1/2}$.

Then f has a unique fixed point in $A\cap B$.

Proof. Let $x\in A$. Similar to proof of Theorem 3.9. Since $\{f^n(x)\}$ is a Cauchy sequence, there exists $z\in A\cap B$ such that $f^n(x)\longrightarrow z$.

Now, we show that $f(z)=z$. Let $t>0$ and $\epsilon>0$ fixed. Then, by (2), we have

$$N(f^n(x)-f(z),t)\geq\min\{N(f^n(x)-z,t/(2\eta_\alpha^-)),N(f^{n-1}(x)-f(z),t/(2\eta_\alpha^-))\}.$$

Since $\tilde{0}<\eta<\widetilde{1/2}$,

$$N(f^n(x)-f(z),t)\geq\min\{N(f^n(x)-z,t/2),N(f^{n-1}(x)-f(z),t/(2\eta_\alpha^-))\}.$$

Thus

$N(f^n(x)-f(z),t) \geq \min\{N(f^n(x)-z,t/2), N(f(z)-x,t/((2\eta_\alpha^-)^n))\}$, for all $n \in N$.

Since $f^n(x) \longrightarrow z$, there exists $N_1 > 0$ such that

$$N(f^n(x) - z, t/2) \geq 1 - \epsilon, \text{ for all } n \geq N_1.$$

Since $\tilde{0} < \eta < \widetilde{1/2}$, $\lim_{n\to\infty} t/(2\eta_\alpha^-)^n = \infty$. Thus $\lim_{n\to\infty} N(f(z)-z, t/(2\eta_\alpha^-)^n) = 1$. So there is $N_2 > 0$ such that $N(f(z) - z, t/(2\eta_\alpha^-)^n) > 1 - \epsilon$, for all $n \geq N_2$. Which implies that $N(f^n(x) - f(z), t) \geq 1 - \epsilon$, for all $n \geq N = \max(N_1, N_2)$. Hence $f^n(x) \longrightarrow f(z)$ and thus $f(z) = z$.

To prove the uniqueness of z, assume that there exists $w \in A \cap B$ such that $f(w) = w$. Then, by (2), we have

$$N(f(z) - f(w), t) \geq \min\{N(f(z) - w, t/(2\eta_\alpha^-)), N(f(w) - z, t/(2\eta_\alpha^-))\}.$$

Thus

$$\begin{aligned} N(z-w,t) &\geq \min\{N(z-w,t/(2\eta_\alpha^-)), N(w-z,t/(2\eta_\alpha^-))\} \\ &= N(w-z,t/(2\eta_\alpha^-)) \\ &\geq N(z-w,t), \end{aligned}$$

So $N(z - w, t) = N(w - z, t/(2\eta_\alpha^-)) = N(w - z, t/((2\eta_\alpha^-)^n))$, for all $n \in N$. Since $\lim_{n\to\infty} t/(2\eta_\alpha^-)^n = \infty$, it follows that $N(z - w, t) = 1$, for all $t > 0$. Hence $z = w$. □

Example 3.15. Let $(X, \|.\|)$ be a Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies

(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,

(2) $\|f(x) - f(y)\| \leq k(\|f(x) - y\| + \|f(y) - x\|)$, for all $x \in A$ and $y \in B$, where $k \in \mathbb{R}$ and $0 < k < 1/2$.

Define a fuzzy norm N as follows :

$$N(x,t) = \begin{cases} t/\|x\| & , \quad 0 < t \leq \|x\| \\ 1 & , \quad \|x\| < t \\ 0 & , \quad t \leq 0. \end{cases}$$

Let $\tilde{k} \in F(\mathbb{R})$ and $[\tilde{k}]_\alpha = [k, k]$. Assume that $x \in A$, $y \in B$ and $N(f(x) - y, t) \geq \alpha$, $N(f(y) - x, s) \geq \alpha$.

Case1: Let $t < \|f(x) - y\|$ and $s < \|f(y) - x\|$. Then

$$t/\|f(x) - y\| = N(f(x) - y, t) \geq \alpha$$

and

$$s/\|f(y) - x\| = N(f(y) - x, t) \geq \alpha.$$

Thus

$$\alpha\|f(x) - f(y)\| \leq \alpha k(\|f(x) - y\| + \|f(y) - x\|) \leq k(t + s).$$

So $\eta(t + s)/\|f(x) - f(y)\| \geq \alpha$. Hence $N(f(x) - f(y), \eta_\alpha^-(t + s)) \geq \alpha$.

Case2: Let $t \geq \|f(x) - y\|$ and $s < \|f(y) - x\|$. Then

$$kt \geq k\|f(x) - y\| \geq \alpha k\|f(x) - y\| \text{ and } ks \geq \alpha k\|f(y) - x\|.$$

Thus $\alpha\|f(x)-f(y)\| \leq \alpha k(\|f(x)-y\|+\|f(y)-x\|) \leq k(t+s)$. So

$$N(f(x)-f(y), k(t+s)) \geq \alpha.$$

Case3: Let $t < \|f(x)-y\|$ and $s \geq \|f(y)-x\|$. Similar to case2,

$$N(f(x)-f(y), k(t+s)) \geq \alpha.$$

case4: Let $t \geq \|f(x)-y\|$ and $s \geq \|f(y)-x\|$. Then

$$kt \geq \eta\|f(x)-y\| \text{ and } ks \geq \eta\|f(y)-x\|.$$

Thus

$$\|f(x)-f(y)\| \leq k(\|f(x)-y\|+\|f(y)-x\|) \leq k(t+s).$$

So

$$N(f(x)-f(y), k(t+s)) \geq \alpha.$$

Hence f has a unique fixed point in $A \cap B$.

Corollary 3.16. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*
(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,
(2) $N(f(x)-f(y), (t+s)\eta_\alpha^-) \geq 1/2(N(f(x)-y, t)+N(f(y)-x, s))$, for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \widetilde{1/2}$.
Then f has a unique fixed point in $A \cap B$.

Corollary 3.17. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*
(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,
(2) $N(f(x)-f(y), (t+s)\eta_\alpha^-) \geq \min\{N(f(x)-y, t), N(f(y)-x, s)\}$, for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \widetilde{1/2}$.
Then f has a unique fixed point in $A \cap B$.

Theorem 3.18. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*
(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,
(2) $N(f(x)-x, t) \geq \alpha$, $N(f(y)-y, s) \geq \alpha$ and $N(x-y, r) \geq \alpha$ implies that $N(f(x)-f(y), (t+s+r)\eta_\alpha^-) \geq \alpha$, for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \widetilde{1/6}$.
Then f has a unique fixed point in $A \cap B$.

Proof. Let $x \in A$. Similar to proof of Theorem 3.9, $\{f^n(x)\}$ is a Cauchy sequence. Then there exists $z \in A \cap B$ such that $f^n(x) \longrightarrow z$.

Now, we show that $f(z) = z$. Let $t > 0$ and $0 < \epsilon < 1$. We have

$$N(f(z)-z, t) \geq \min\{N(f(z)-f^n(x), t/2), N(f^n(x)-z, t/2)\}.$$

Now by (2), we observe that

$$\begin{aligned} N(f(z)-f^n(x), t/2) \quad &\geq \quad \min\{N(f^{n-1}(x)-f^n(x), t/(6\eta_\alpha^-)), N(f(z)-z, t/(6\eta_\alpha^-)), \\ &\qquad N(f^{n-1}(x)-z, t/(6\eta_\alpha^-))\}, \end{aligned}$$

for all $n \in N$. Since $\{f^n(x)\}$ is a Cauchy sequence and $f^n(x) \longrightarrow z$, there exists $N > 0$ such that $\min\{N(f^{n-1}(x) - f^n(x), t/2), N(f^n(x) - z, t/2)\} \geq 1 - \epsilon$, for all $n \geq N$. Thus $N(f(z) - z, t) \geq \min\{1 - \epsilon, N(f(z) - z, t/(6\eta_\alpha^-))\}$. So we obtain that

$$N(f(z) - z, t) \geq \min\{1 - \epsilon, N(f(z) - z, t/(6\eta_\alpha^-)^n)\}, \text{ for all } n \in N.$$

Since $\tilde{0} < \eta < \widetilde{1/6}$, $\lim_{n\to\infty} t/(6\eta_\alpha^-)^n = \infty$. Thus $\lim_{n\to\infty} N(f(z)-z, t/(6\eta_\alpha^-)^n) = 1$. So $N(f(z) - z, t) \geq 1 - \epsilon$, for all $\epsilon \in (0, 1)$. Which implies that $N(f(z) - z, t) = 1$, for all $t > 0$, so $f(z) = z$.

To prove the uniqueness of z, assume that there exists $w \in A \cap B$ such that $f(w) = w$. By (2), we have

$$\begin{aligned} N(w - z, t) &= N(f(z) - f(w), t) \\ &\geq \min\{N(f(z) - z, t/(3\eta_\alpha^-)), N(f(w) - w, t/(3\eta_\alpha^-)), \\ &\quad N(w - z, t/(3\eta_\alpha^-))\} \\ &\geq N(w - z, t/(3\eta_\alpha^-)) \\ &\geq N(w - z, t). \end{aligned}$$

Thus $N(z - w, t) = N(w - z, t/(3\eta_\alpha^-)) = N(w - z, t/(3\eta_\alpha^-)^n)$, for all $n \in N$. Since $\lim_{n\to\infty} t/(3\eta_\alpha^-)^n = \infty$, it follows that $N(z - w, t) = 1$, for all $t > 0$. Hence $z = w$. □

Corollary 3.19. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*
(1) *$f(A) \subseteq B$ and $f(B) \subseteq A$,*
(2) *$N(f(x)-f(y), (t+s+r)\eta_\alpha^-) \geq 1/3(N(f(x)-x, t)+N(f(y)-y, s)+N(x-y, r))$,*
for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \widetilde{1/6}$.
Then f has a unique fixed point in $A \cap B$.

Corollary 3.20. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*
(1) *$f(A) \subseteq B$ and $f(B) \subseteq A$,*
(2) *$N(f(x)-f(y), (t+s+r)\eta_\alpha^-) \geq \min\{N(f(x)-x, t), N(f(y)-y, s), N(x-y, r)\}$,*
for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \widetilde{1/6}$.
Then f has a unique fixed point in $A \cap B$.

Theorem 3.21. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*
(1) *$f(A) \subseteq B$ and $f(B) \subseteq A$,*
(2) *$N(f(x) - x, t) \geq \alpha$, $N(f(y) - y, s) \geq \alpha$ and $N(x - y, r) \geq \alpha$ implies that $N(f(x) - f(y), \max\{t, s, r\}\eta_\alpha^-) \geq \alpha$, for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \widetilde{1/2}$.*
Then f has a unique fixed point in $A \cap B$.

Proof. Let $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ be an increasing family of α-seminorms on X corresponding to the fuzzy norm N on X.

Assume that $N(f(x) - x, t) \geq \alpha$, $N(f(y) - y, s) \geq \alpha$ and $N(x - y, r) \geq \alpha$. Then

$$N(f(x) - f(y), \max\{t, s, r\}\eta_\alpha^-) \geq \alpha, \text{ for all } x \in A \text{ and } y \in B.$$

Thus

$$\|f(x) - f(y)\|_\alpha \leq \max\{t, s, r\}\eta_\alpha^-.$$

So $\|f(x)-f(y)\|_\alpha \leq \max\{\|f(x)-x\|_\alpha, \|f(y)-y\|_\alpha, \|x-y\|_\alpha\}\eta_\alpha^-$, for all $x \in A, y \in B$.

Let $x \in A$. If $\max\{\|f(x) - x\|_\alpha, \|f^2(x) - f(x)\|_\alpha\} = \|x - f(x)\|_\alpha$, then

$$\|f^2(x) - f(x)\|_\alpha \leq \eta_\alpha^- \|f(x) - x\|_\alpha.$$

If $\max\{\|f(x) - x\|_\alpha, \|f^2(x) - f(x)\|_\alpha\} = \|f^2(x) - f(x)\|_\alpha$, then

$$\|f^2(x) - f(x)\|_\alpha \leq \eta_\alpha^- \|f^2(x) - f(x)\|_\alpha.$$

Since $\tilde{0} < \eta < \widetilde{1/2}$, this is contradiction.Thus $\|f^2(x) - f(x)\|_\alpha \leq \eta_\alpha^- \|f(x) - x\|_\alpha$. By induction, we have $\|f^{n+1}(x) - f^n(x)\|_\alpha \leq \eta_\alpha^n \|f(x) - x\|_\alpha$. Similar to proof of Theorem 3.1, $\{f^n(x)\}$ is a Cauchy sequence. So there exists $z \in A \cap B$ such that $f^n(x) \longrightarrow z$.

Now, we show that $f(z) = z$. Let $t > 0$ and $0 < \epsilon < 1$. We have

$$N(f(z) - z, t) \geq \min\{N(f(z) - f^n(x), t/2), N(f^n(x) - z, t/2)\}.$$

By (2), we observe that

$$\begin{aligned} N(f(z) - f^n(x), t/2) \quad \geq \quad & \min\{N(f^{n-1}(x) - f^n(x), t/(2\eta_\alpha^-)), N(f(z) - z, t/(2\eta_\alpha^-)), \\ & N(f^{n-1}(x) - z, t/(2\eta_\alpha^-))\}, \end{aligned}$$

for all $n \in N$. Since $\{f^n(x)\}$ is a Cauchy sequence and $f^n(x) \longrightarrow z$, it follows that there exists $N > 0$ such that

$$\min\{N(f^{n-1}(x)-f^n(x), t/(2\eta_\alpha^-), N(f^{n-1}(x)-z, t/(2\eta_\alpha^-)), N(f^n(x)-z, t/2)\} \geq 1-\epsilon,$$

for all $n \geq N$. Then $N(f(z) - z, t) \geq \min\{1 - \epsilon, N(f(z) - z, t/(2\eta_\alpha^-))\}$. Thus we obtain that

$$N(f(z) - z, t) \geq \min\{1 - \epsilon, N(f(z) - z, t/(2\eta_\alpha^-)^n)\}, \text{ for all } n \in N.$$

Since $\tilde{0} < \eta < \widetilde{1/2}$, it follows that $\lim_{n\to\infty} t/(2\eta_\alpha^-)^n = \infty$. So $\lim_{n\to\infty} N(f(z) - z, t/(2\eta_\alpha^-)^n) = 1$. Hence $N(f(z) - z, t) \geq 1 - \epsilon$, for all $\epsilon \in (0, 1)$. Which implies that $N(f(z) - z, t) = 1$, for all $t > 0$. Therefore $f(z) = z$.

Similar to proof of Theorem 3.18, z is a unique fixed point of f in $A \cap B$. □

Corollary 3.22. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*

(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,

(2) $N(f(x) - f(y), \max\{t, s, r\}\eta_\alpha^-) \geq 1/3(N(f(x) - x, t) + N(f(y) - y, s) + N(x - y, r))$, for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \widetilde{1/6}$.

Then f has a unique fixed point in $A \cap B$.

Corollary 3.23. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : A \cup B \longrightarrow A \cup B$ satisfies*

(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,

(2) $N(f(x) - f(y), \max\{t, s, r\}\eta_\alpha^-) \geq \min\{N(f(x) - x, t), N(f(y) - y, s), N(x - y, r)\}$, for all $x \in A$ and $y \in B$, where $\eta \in F(\mathbb{R})$ and $\tilde{0} < \eta < \widetilde{1/6}$.

Then f has a unique fixed point in $A \cap B$.

Definition 3.24. The function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is called an altering distance function if the following properties are satisfied :
(i) φ is continuous and nondecreasing,
(ii) $\varphi(t) = 0$ if and only if $t = 0$.

Theorem 3.25. *Let (X, N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f : X \longrightarrow X$ satisfies*
(1) $f(A) \subseteq B$ and $f(B) \subseteq A$,
(2) if $N(f(x)-x, t_1) \geq \alpha$, $N(f(y)-y, t_2) \geq \alpha$, $N(x-y, t_3) \geq \alpha$, $N(f(x)-y, t_4) \geq \alpha$ and $N(f(y) - x, t_5) \geq \alpha$ then there exists $t > 0$ such that $N(f(x) - f(y), t) \geq \alpha$, and $\varphi(t) \leq \varphi(\max\{t_1, t_2, t_3, 1/2(t_4+t_5)\}) - \phi(\max\{t_3, t_2\})$, for all $x \in A$ and $y \in B$, where φ and ϕ are altering distance functions.
Then f has a unique fixed point in $A \cap B$.

Proof. Let $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ be an increasing family of α-seminorms on X corresponding to the fuzzy norm N on X.
Let $N(f(x) - x, t_1) \geq \alpha$, $N(f(y) - y, t_2) \geq \alpha$, $N(x - y, t_3) \geq \alpha$, $N(f(x) - y, t_4) \geq \alpha$ and $N(f(y) - x, t_5) \geq \alpha$. Then there exists $t > 0$ such that $N(f(x) - f(y), t) \geq \alpha$, and $\varphi(t) \leq \varphi(\max\{t_1, t_2, t_3, 1/2(t_4 + t_5)\}) - \phi(\max\{t_3, t_2\})$, for all $x \in A$ and $y \in B$. Thus $\varphi(\|f(x) - f(y)\|_\alpha) \leq \varphi(t) \leq \varphi(\max\{t_1, t_2, t_3, 1/2(t_4 + t_5)\}) - \phi(\max\{t_3, t_2\})$. Since φ and ϕ are continuous,

$$\varphi(\|f(x) - f(y)\|_\alpha) \leq$$
$$\varphi(\max\{\|f(x) - x\|_\alpha, \|f(y) - y\|_\alpha, \|x - y\|_\alpha, 1/2(\|f(x) - y\|_\alpha + \|f(y) - x\|_\alpha)\})-$$
$$\phi(\max\{\|f(y) - y\|_\alpha, \|x - y\|_\alpha\}),$$

for all $x \in A, y \in B$. Let $x_0 \in A$. Since $f(A) \subseteq B$, we choose $f(x_0) = x_1 \in B$. Continuing this process, we can construct sequences $\{x_n\}$ in X such that $x_{2n} \in A$, $x_{2n+1} \in B$. If $x_{2n_0+1} = x_{2n_0}$, for some $n_0 \in N$, then x_{2n_0+1} is a fixed point of f in $A \cap B$. Thus, we may assume that $x_{2n+1} \neq x_{2n}$, for all $n \in N$. Similar to proof of Theorem 5 in [9], $\lim_{n\to\infty} \|x_n - x_{n+1}\|_\alpha = 0$, for all $\alpha \in (0, 1)$. And for all $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists $N = N(\alpha, \epsilon) > 0$ such that $\|x_n - x_m\|_\alpha < \epsilon$, for all $n, m \geq N$. Let $t > 0$ and $0 < \epsilon < 1$ be given. Then there exists $N > 0$ such that

$$\|x_n - x_m\|_{1-\epsilon} < t, \text{ for all } n, m \geq N.$$

Thus $N(x_n - x_m, t) \geq 1 - \epsilon$, for all $n, m \geq N$. So $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to $x \in A \cap B$. Now we have

$$\begin{aligned}
\varphi(\|x_{2n+1} - f(x)\|_\alpha) &= \varphi(\|f(x_{2n}) - f(x)\|_\alpha) \\
&\leq \varphi(\max\{\|f(x_{2n}) - x_{2n}\|_\alpha, \|f(x) - x\|_\alpha, \|x_{2n} - x\|_\alpha, \\
&\quad 1/2(\|f(x) - x_{2n}\|_\alpha + \|f(x_{2n}) - x\|_\alpha)\}) - \\
&\quad \phi(\max\{\|f(x) - x\|_\alpha, \|x - x_{2n}\|_\alpha\}) \\
&= \varphi(\max\{\|x_{2n+1} - x_{2n}\|_\alpha, \|f(x) - x\|_\alpha, \|x_{2n} - x\|_\alpha, \\
&\quad 1/2(\|f(x) - x_{2n}\|_\alpha + \|x_{2n+1} - x\|_\alpha)\}) - \\
&\quad \phi(\max\{\|f(x) - x\|_\alpha, \|x - x_{2n}\|_\alpha\}),
\end{aligned}$$

for all $\alpha \in (0, 1)$. As $n \longrightarrow \infty$, we get that

$$\varphi(\|f(x) - x\|_\alpha) \leq \varphi(\|f(x) - x\|_\alpha) - \phi(\|f(x) - x\|_\alpha), \text{ for all } \alpha \in (0, 1).$$

Then $\phi(\|f(x)-x\|_\alpha)=0$. Since ϕ is an altering distance function,

$$\|f(x)-x\|_\alpha=0, \text{ for all } \alpha\in(0,1).$$

Let $t>0$. If $N(f(x)-x,t)\neq 1$. Then there exists $\alpha\in(0,1)$ such that $N(f(x)-x,t)<\alpha$. Thus $N(f(x)-x,s)<\alpha$, for all $s\leq t$. Thus

$$\|f(x)-x\|_\alpha=\inf\{s>0: N(f(x)-x,s)\geq\alpha\}\geq t>0,$$

which is a contradiction. Therefore $N(f(x)-x,t)=1$, for all $t>0$. So $f(x)=x$.

To prove the uniqueness of the fixed point, we let y be any other fixed point of f in $A\cap B$. Now we have

$$\begin{aligned}\varphi(\|x-y\|_\alpha) &= \varphi(\|f(x)-f(y)\|_\alpha)\\ &\leq \varphi(\max\{\|y-x\|_\alpha,\|y-y\|_\alpha,\|x-x\|_\alpha\})-\\ &\quad \phi(\max\{\|x-y\|_\alpha,\|y-y\|_\alpha\})\\ &= \varphi(\|x-y\|_\alpha)-\phi(\|x-y\|_\alpha),\end{aligned}$$

for all $\alpha\in(0,1)$. Thus $\phi(\|x-y\|_\alpha)=0$ and $\|x-y\|_\alpha=0$, for all $\alpha\in(0,1)$. Similar to above $x=y$. □

Corollary 3.26. *Let (X,N) be a fuzzy Banach space and A, B be non-empty closed subsets of X. Suppose that map $f:X\longrightarrow X$ satisfies*

(1) $f(A)\subseteq B$ and $f(B)\subseteq A$,

(2) if $N(f(x)-x,t_1)\geq\alpha$, $N(f(y)-y,t_2)\geq\alpha$, $N(x-y,t_3)\geq\alpha$, $N(f(x)-y,t_4)\geq\alpha$ and $N(f(y)-x,t_5)\geq\alpha$ then $N(f(x)-f(y),\max\{t_1,t_2,t_3,1/2(t_4+t_5)\}-\phi(\max\{t_3,t_2\}))\geq\alpha$, for all $x\in A$ and $y\in B$, where ϕ is an altering distance functions.

Then f has a unique fixed point in $A\cap B$.

4. Conclusion

The theory of fuzzy approximate fixed points is not less interesting than that of fuzzy fixed points and many results formulated in the latter can be adapted to a less restrictive framework in order to guarantee the existence of the fuzzy $\alpha-$norm fixed points. We proved results about fuzzy fixed points on $\alpha-$fuzzy norm spaces. we think that this paper could be of interest to the researchers working in the field fuzzy functional analysis in particular, fuzzy fixed point theory are used.

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