Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 12, No. 1, (July 2016), pp. 139–153
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

The relationships between degree rough sets and matroids

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Received 13 November 2015; Revised 18 December 2015; Accepted 2 February 2016

Abstract. There are many similarities between rough set theory and matroid theory. Recently, many researchers explore the connections between them. In this paper, we apply matroid theory to rough set theory through approximation operators. Firstly, a series of matroidal structures induced by equivalence relations are proposed, then we combine degree rough sets with matroids, and characterize the circuit, independent set, rank function, base, closure and so on of these matroids by degree approximation operators. Secondly, we discuss the connections between generalized degree rough sets and matroids. Finally, conditions of the same matroid induced by different binary relations are studied.

2010 AMS Classification: 52B40, 68T37

Keywords: Rough set, Matroid, Degree rough set, Approximation number function.

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1. Introduction

Rough set theory has been proposed by Pawlak [18] as a tool for dealing with uncertainty, and it has been widely applied to pattern recognition, attribute reduction, rule extraction, granular computing and data mining, etc [2, 3, 4, 6, 7, 8, 16, 19, 25]. The classical rough set theory is based on equivalence relations, however, it has been extended to generalized rough sets based on relations [28], covering-based rough sets [1, 15, 24], fuzzy rough sets [22], and degree rough sets [23, 26].

Matroid theory [9, 17] are a generalization of linear independence in vector spaces, which provide a unifying abstract treatment of dependence in linear algebra and graph theory. Matroidal structures are important structures from theory as well as application. In theory, matroids can be defined in many different ways, they have been combined with other theories deeply. Some authors have connected matroid theory with classical rough sets [11, 20],covering-based rough sets [21], generalized rough sets based on relations [29] and so on. In application, it has widely been applied to many fields, such as combinatorial optimization [10] and algorithm design [5].

Recently, matroidal approaches to rough set theory is becoming popular. The powerful axiomatic system and wide applications of matroid theory will be benefit to the development of rough set theory. Liu et al. [11] proposed a parametric matroid of the rough set through defining a parametric set family based on the lower approximation operator, thereby obtaining significant results by combining Pawlak's rough set theory and matroid theory. Moreover, in order to take advantages of both rough set theory and matroid theory, Liu et al. [12, 13, 14] also proposed k-rank matroids, partition-circuit matroids and a matroidal structure of rough sets based on a serial and transitive relation. Zhu et al. [29] established a matroidal structures through the upper approximation number and studied generalized rough sets with matroid approaches, where they obtained some new characteristics of rough sets. Wang et al. [20] induced a matroid (2-circuit matroid) by equivalence relations, and they studied some properties of this matroid. Moreover, matroidal approaches were used to study rough sets and information systems in that paper. Inspired by them, we study 2-circuit matroid from the viewpoint of upper approximation number based on equivalence relations, it can be extended to a series of matroids. Therefore, 2-circuit matroid is a special situation. It is interesting that we find the connections between this kind of matroids and degree rough sets are closed, and we can characterize the circuit, independent set, rank function, base, closure and so on of these matroids by degree approximation operators, in other words, these matroids can also induced by degree rough sets. Besides, we extend classical degree rough sets to generalized degree rough sets. By this way, generalized degree approximation operators can also induce matroid structures. Furthermore, this kind of matroids can be determined by generalized degree approximation operators uniquely, that is, only the same degree approximation operator can induce the same matroid.

The rest of this paper is arranged as follows. In Section 2, we review some fundamental knowledge of rough sets and matroids. In section 3, we propose a series of matroids induced by upper approximation number functions based on equivalence relations, and we discuss the relationships between these matroids and degree rough sets. In section 4, we extend equivalence relations to generalized binary relations, and then we study the connections between generalized degree rough sets and matroids. In section 5, we propose the conditions of the same matroids induced by different binary relations. Finally, In, section 6, we conclude this paper and points out further works.

2. Preliminaries

To facilitate our discussion, in this section, we recall some fundamental definitions related to rough sets, degree rough sets and matroids.

Classical rough sets are based on equivalence relations. In rough sets, a pair of approximation operators are used to describe an object.

Definition 2.1 ([18]). (Approximation space). Let U be a nonempty and finite set called universe. Let R be an equivalence relation on U, that is, R is reflexive, symmetric and transitive. The ordered pair (U, R) is called an approximation space.

Definition 2.2 ([18]). (Approximation operator). Let R be an equivalence relation on U. A pair of approximation operators $\underline{R}, \overline{R} : P(U) \to P(U)$, are defined as follows: for all $X \subseteq U$,

$\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\} = \bigcup\{K \in U/R \mid K \subseteq X\}$,

$\overline{R}(X) = \{x \in U \mid [x]_R \cap X \neq \varnothing\} = \bigcup\{K \in U/R \mid K \cap X \neq \varnothing\}$,

where $[x]_R = \{y \in U \mid (x, y) \in R\}$, $U/R = \{[x]_R \mid x \in U\}$. They are called the lower and upper approximation operators with respect to R, respectively.

Definition 2.3 ([18]). (R-precise and R-rough set). Let R be an equivalence relation on U. For all $X \subseteq U$, if $\overline{R}(X) = \underline{R}(X)$, then we say X is a R-precise set; otherwise, we say X is a R-rough set.

Definition 2.4 ([23]). Let R be an equivalence relation on U. For all $X \subseteq U$,

$\underline{R}_k(X) = \bigcup\{K \in U/R \mid |K| - |K \cap X| \leq k\}$,

$\overline{R}_k(X) = \bigcup\{K \in U/R \mid |K \cap X| > k\}$,

are called the lower and upper degree approximations of X respectively, k is an arbitrary natural number, which is called degree. If $\overline{R}_k(X) \neq \underline{R}_k(X)$, then we say X is an R-rough set with respect to the degree of k; otherwise, we say X is an R-precise set.

In classical rough sets, R is an equivalence relation. However, it is very limit in practical problems. Therefore, we often extend it to the generalized binary relation based rough sets.

Definition 2.5 ([26]). Let R be a binary relation on U. The ordered pair (U, R) is called a generalized approximation space. Denote $R_s(x) = \{y \in U \mid (x, y) \in R\}$, $R_s(x)$ is called the successor neighbourhood of x. For all $X \subseteq U$, the lower approximation $\underline{apr}(X)$ and upper approximation $\overline{apr}(X)$ of X with respect to R are defined as follows respectively:

$\underline{apr}(X) = \{x \in U \mid R_s(x) \subseteq X\}$,

$\overline{apr}(X) = \{x \in U \mid R_s(x) \cap X \neq \varnothing\}$.

Definition 2.6 ([26]). Let (U, R) is a generalized approximation space. For all $X \subseteq U$,

$\underline{apr}_k(X) = \{x \in U \mid |R_s(x)| - |R_s(x) \cap X| \leq k\}$,

$\overline{apr}_k(X) = \{x \in U \mid |R_s(x) \cap X| > k\}$,

are called the lower and upper generalized degree approximations of X respectively, k is an arbitrary natural number.

Proposition 2.7 ([27]). *The properties of binary relation R can be described by it's neighbourhood operators :*

R is a serial relation $\Leftrightarrow \forall\, x \in U, R_s(x) \neq \varnothing$.

R is an inverse serial relation $\Leftrightarrow R_s(U) = U$.

R is a reflexive relation $\Leftrightarrow \forall\, x \in U, x \in R_s(x)$.

R is a symmetric relation $\Leftrightarrow \forall\, x, y \in U, x \in R_s(y) \Rightarrow y \in R_s(x)$.

R is a transitive relation $\Leftrightarrow \forall\, x, y \in U, y \in R_s(x) \Rightarrow R_s(y) \subseteq R_s(x)$.

R is an Euclid relation $\Leftrightarrow \forall\ x, y \in U, y \in R_s(x) \Rightarrow R_s(x) \subseteq R_s(y)$.

Matroids can be regarded as an algebra structure which generated by linearly independent sets in vector spaces, and it also can be regarded as some dependent sets which determined by the circle subgraph which generated by edge sets. Therefore, one of main characteristics of matroids is that there are many equivalent ways to define them.

Definition 2.8 ([9]). (Matroid). A matroid is an ordered pair $M = (U, \mathcal{I})$, where U (the ground set) is a finite set, and $\mathcal{I}$ (the independent sets) a family of subsets of U with the following properties :

(I1) $\varnothing \in \mathcal{I}$,

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$,

(I3) If I_1, $I_2 \in \mathcal{I}$, and $|I_1| < |I_2|$, then there exists $u \in I_2 - I_1$ such that $I_1 \cup \{u\} \in \mathcal{I}$, where $|I|$ denotes the cardinality of I.

Definition 2.9 ([9]). Let $\mathcal{A}$ be a family of subsets of U, we denote

$Upp(\mathcal{A}) = \{X \subseteq U \mid \exists\ A \in \mathcal{A}$, such that $A \subseteq X\}$.

$Low(\mathcal{A}) = \{X \subseteq U \mid \exists\ A \in \mathcal{A}$, such that $X \subseteq A\}$.

$Max(\mathcal{A}) = \{X \in \mathcal{A} \mid \forall\ Y \in \mathcal{A}, X \subseteq Y \Rightarrow X = Y\}$.

$Min(\mathcal{A}) = \{X \in \mathcal{A} \mid \forall\ Y \in \mathcal{A}, Y \subseteq X \Rightarrow X = Y\}$.

The dependent set of a matroid generalizes the linear dependence in vector spaces and the cycle in graphs. The circuit of a matroid is a minimal dependent set.

Definition 2.10 ([9]). (Circuit). Let $M = (U, \mathcal{I})$ be a matroid. A minimal dependent set in M is called a circuit of M, and we denote the family of all circuits of M by $\mathcal{C}(M)$, that is, $\mathcal{C}(M) = Min(\mathcal{I}^c)$, where $\mathcal{I}^c$ is the complement of $\mathcal{I}$ in $P(U)$.

In Ref. [9], there is an axiom about circuit in matroids, which shows that a matroid and it's circuits are determined by each other.

Proposition 2.11 ([9]). *(Circuit axiom). Let $\mathcal{C}$ be a family of subsets of U. Then there exists $M = (U, \mathcal{I})$ such that $\mathcal{C} = \mathcal{C}(M)$ if and only if $\mathcal{C}$ satisfies the following three conditions :*

(C1) $\varnothing \notin \mathcal{C}$.

(C2) *If $C_1, C_2 \in \mathcal{C}$, and $C_1 \subseteq C_2$, then $C_1 = C_2$.*

(C3) *If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, and $u \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2 - \{u\}$.*

The base is another important feature of a matroid, which generalized from the maximal linearly independent group in vector space and the spanning tree in graph.

Definition 2.12 ([9]). (Base). Let $M = (U, \mathcal{I})$ be a matroid. A maximal independent set of M is called a base of M, and the family of all bases of M is denoted by $\mathcal{B}(M)$, that is, $\mathcal{B}(M) = Max(\mathcal{I})$.

Similarly, an axiom of the matroid is constructed from the viewpoint of the base.

Proposition 2.13 ([9]). *(Base axiom). Let $\mathcal{B}$ be a family of subsets of U. Then there exists a matroid $M = (U, \mathcal{I})$ such that $\mathcal{B} = \mathcal{B}(M)$ if and only if $\mathcal{B}$ satisfies the following two conditions :*

(B1) $\mathcal{B} \neq \varnothing$.

(B2) *If* $B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 - B_2$, *then there exists* $y \in B_2 - B_1$ *such that* $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

The rank function of a matroid is a generalization of the dimension of a vector space and the rank of a matrix, which is a very important concept. It can be used for define the closure operator, which reflects the dependency between a set and elements.

Definition 2.14 ([9, 17])**.** (Rank function). Let $M = (U, \mathcal{I})$ be a matroid and $X \subseteq U$. We define the rank $r(X)$ of X to be the size of a basis B of $M \mid X$. Evidently r maps $P(U)$ into the set of non-negative integers, and is called the rank function of M and written as r_M. When there is no confusion, the subscript M can be omitted.

Proposition 2.15 ([9])**.** *Let* $M = (U, \mathcal{I})$ *be a matroid. Then the rank function* r_M *of* M *satisfies the following properties :*

(1) *For all* $X \subseteq U$, $0 \leq r_M(X) \leq |X|$.

(2) *If* $X \subseteq Y \subseteq U$, *then* $r_M(X) \leq r_M(Y)$.

(3) *If* $X, Y \subseteq U$, *then* $r_M(X) + r_M(Y) \geq r_M(X \cup Y) + r_M(X \cap Y)$.

Definition 2.16 ([9])**.** (Closure). Let $M = (U, \mathcal{I})$ be a matroid. The closure operator cl_M of M is defined as $cl_M(X) = \{u \in U \mid r_M(X) = r_M(X \cup \{u\})\}$ for all $X \subseteq U$. $cl_M(X)$ is called the closure of X in M.

Definition 2.17 ([9])**.** (Closed set). Let $M = (U, \mathcal{I})$ be a matroid. For all $X \subseteq U$, X is called a closed set of M if $cl_M(X) = X$.

The following closure axiom shows that a matroid uniquely determines a closure operator, and vice versa.

Proposition 2.18 ([9])**.** *(Closure axiom). Let* $cl : P(U) \to P(U)$ *be an operator. Then there exists a matroid* $M = (U, \mathcal{I})$ *such that* $cl = cl_M$ *if and only if* cl *satisfies the following conditions :*

(CL1) *For all* $X \subseteq U$, $X \subseteq cl(X)$.

(CL2) *If* $X \subseteq Y \subseteq U$, *then* $cl(X) \subseteq cl(Y)$.

(CL3) *For all* $X \subseteq U$, $cl(cl(X)) = cl(X)$.

(CL4) *For all* $x, y \in U$, *if* $y \in cl(X \cup \{x\}) - cl(X)$, *then* $x \in cl(X \cup \{y\})$.

Definition 2.19 ([21])**.** ((Upper approximation number). Let R be an equivalence relation on U. For all $X \subseteq U$, we can define $f_R(X) = |\{K \in U/R \mid K \cap X \neq \varnothing\}|$. $f_R(X)$ is called the upper approximation number of X, f_R the upper approximation number function with respect to R.

3. The relationships between classical degree rough sets and matroids

Wang et al. induced a matroid by an equivalence relation in Ref. [20]:

Let R be an equivalence relation on U. we define a family $\mathcal{C}(R)$ of subsets of U as follows:

$$\forall\ x, y \in U \text{ and } x \neq y, (x, y) \in R \Leftrightarrow \{x, y\} \in \mathcal{C}(R),$$

where $\mathcal{C}(R)$ satisfies the circuit axiom. Therefore, there is a unique matroid $M(R)$ which determined by $\mathcal{C}(R)$. We can propose another equivalent description of $\mathcal{C}(R)$ if we characterize it from the viewpoint of upper approximation number, that is,

$$\mathcal{C}(R) = \{X \subseteq U \mid f_R(X) = 1 \textit{ and } |X| = 2\}.$$

We can replace 2 with an arbitrary positive integer k, and get the family of subsets of U as follows :

$$\mathcal{C}_k(R) = \{X \subseteq U \mid f_R(X) = 1 \textit{ and } |X| = k\}. \tag{3.1}$$

Remark 3.1. In this paper, all of k are positive integers.

In the following theorem, we show that $\mathcal{C}_k(R)$ is the family of circuits of a matroid.

Theorem 3.2. *Let R be an equivalence relation on U, and $\mathcal{C}_k(R) = \{X \subseteq U \mid f_R(X) = 1$ and $|X| = k\}$. Then $\mathcal{C}_k(R)$ satisfies the circuit axiom.*

Proof. (1) Since $f_R(\varnothing) = 0$, $\varnothing \notin \mathcal{C}_k(R)$. Then $\mathcal{C}_k(R)$ satisfies (C1).

(2) For all $C_1, C_2 \in \mathcal{C}_k(R)$, by formula (3.1), we have that $|C_1| = |C_2| = k$. Then $C_1 \subseteq C_2$ implies $C_1 = C_2$. We have proved that $\mathcal{C}_k(R)$ satisfies (C2).

(3) For all $C_1, C_2 \in \mathcal{C}_k(R)$, $C_1 \neq C_2$ and $z \in C_1 \cap C_2$. Then

$$\begin{aligned}
&|C_1| = |C_2| = k,\\
&|(C_1 \cup C_2) - \{z\}| = |C_1 \cup C_2| - 1\\
&\qquad = |C_1| + |C_2| - |C_1 \cap C_2| - 1\\
&\qquad = k + (k - |C_1 \cap C_2|) - 1\\
&\qquad > k - 1 \geq k.
\end{aligned}$$

Thus we can take $C_3 \subseteq (C_1 \cup C_2) - \{z\}$ and $|C_3| = k$. It is straightforward that $f_R(C_1 \cup C_2) = 1$, which implies $f_R(C_3) = 1$. Thus $C_3 \in \mathcal{C}_k(R)$. We have proved that there exists $C_3 \in \mathcal{C}_k(R)$ such that $C_3 \subseteq (C_1 \cup C_2) - \{z\}$. So $\mathcal{C}_k(R)$ satisfies (C3).

In summary, by Proposition 2.11, $\mathcal{C}_k(R)$ satisfies the circuit axiom. □

Remark 3.3. According to Theorem 3.2 and Proposition 2.11, there exists a matroid on the universe such that $\mathcal{C}_k(R)$ is the family of its circuits, denote this matroid as $M_k(R)$. Especially, if $k = 2$, then the matroid $M_2(R)$ is coincided with the matroid induced by equivalence relation in Ref. [20].

Example 3.4. Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, R be an equivalence relation on U, and $U/R = \{\{1, 2, 3\}, \{4, 5, 6, 7\}\}$. If $k = 3$, then $\mathcal{C}_3(R) = \{\{1, 2, 3\}, \{4, 5, 6\}, \{4, 5, 7\}, \{4, 6, 7\}, \{5, 6, 7\}\}$.

Next, we discuss the relationship between degree rough sets and the dependent sets, independent sets of the matroid $M_k(R)$.

For each $X \subseteq U$: $\overline{R}_{k-1}(X) \neq \varnothing$ if and only if there exists $K \in U/R$ such that $|K \bigcap X| > k - 1 \geq k$, i.e., $|X| \geq k$ and $f_R(X) \geq 1$ if and only if X is a dependent set of $M_k(R)$. Thus the following two propositions are obvious.

Proposition 3.5. *Let R be an equivalence relation on U. For all $X \subseteq U$, X is a dependent set in the matroid $M_k(R)$ if and only if $\overline{R}_{k-1}(X) \neq \varnothing$.*

Proposition 3.6. *Let R be an equivalence relation on U. For all $X \subseteq U$, X is an independent set in the matroid $M_k(R)$ if and only if $\overline{R}_{k-1}(X) = \varnothing$.*

The above two conclusions link the dependent sets and the independent sets of matroids to degree approximation operators closely. Then we can characterize the circuit, rank function, base, closure and so on of the matroid $M_k(R)$ by degree approximation operators. In the following proposition, we present the circuit of the matroid $M_k(R)$ by degree approximation operators.

Proposition 3.7. *Let R be an equivalence relation on U. Denote $\mathcal{C}_k(R) = \{X \subseteq U \mid f_R(X) = 1, \overline{R}_{k-1}(X) \neq \varnothing$ and $\overline{R}_k(X) = \varnothing\}$. Then $\mathcal{C}_k(R)$ is the family of all circuits of the matroid $M_k(R)$.*

This proposition can be directly proved by Proposition 3.5, Proposition 3.6 and Theorem 3.2.

In the follows, we will discuss the independent sets of the matroid $M_k(R)$.

Remark 3.8. $M_k(R) = (U, \mathcal{I}_k(R))$ are a series of matroids, which induced by upper approximation number functions. So it is necessary for us to discuss the different situations in which the positive integer k are different.

Proposition 3.9. *Let R be an equivalence relation on U and $\mathcal{C}_k(R) = \{X \subseteq U \mid f_R(X) = 1$ and $|X| = k\}$ be the family of all circuits. Then the family of independent sets $\mathcal{I}_k(R)$ of the matroid $M_k(R)$ has following properties :*

(1) *If $k = 1$, then $\mathcal{I}_k(R) = \{\varnothing\}$.*

(2) *If $k = 2$, then $\mathcal{I}_k(R) = \{X \subseteq U \mid \forall\ x, y \in X, x \neq y \Rightarrow (x, y) \notin R\}$.*

(3) *If $k > \lambda$, then $\mathcal{I}_k(R) = P(U)$, where $\lambda = max\{|K| \mid K \in U/R\}$.*

Proof. (1) If $k = 1$, $|X| = 1$, then X is a single point set, that is, all single point sets are dependent sets, thus any nonempty set is dependent set. It is straightforward that there is only $\varnothing$ is independent set, that is, $\mathcal{I}_k(R) = \{\varnothing\}$.

(2) According to Ref. [20], it is straightforward.

(3) By formula (3.1) , $\lambda = max\{|K| \mid K \in U/R\}$ and $k > \lambda$, we have that there not exists $X \in \mathcal{C}_k(R)$ such that $f_R(X) = 1$ and $|X| = k$, that is, $\mathcal{C}_k(R) = \varnothing$. In other words, there has no dependent set in the family of subsets of universe U. Hence $\mathcal{I}_k(R) = P(U)$. $\square$

Now we discuss the rank function of the matroid $M_k(R)$.

Lemma 3.10. *Let R be an equivalence relation on U, $X \subseteq U$ and $K \in U/R$. Then the following properties hold :*

(1) *If $|K \cap X| > k - 2$, then $r_{M_k(R)}(K \cap X) = k - 1$.*

(2) *If $|K \cap X| \leq k - 2$, then $r_{M_k(R)}(K \cap X) = |K \cap X|$.*

Proof. (1) Since $|K \cap X| > k - 2$, $|K \cap X| \geq k - 1$. Let $K \cap X = \{x_1, x_2, \cdots, x_m\}$, where m is a positive integer and $m \geq k - 1$. Thus $\{x_1, x_2, \cdots, x_{k-1}\} \subseteq K \cap X$. According to Definition 2.4, $\overline{R}_{k-1}(\{x_1, x_2, \cdots, x_{k-1}\}) = \varnothing$, according to Proposition 3.5, $\{x_1, x_2, \cdots, x_{k-1}\}$ is an independent set. According to formula (3.1), every subset of k elements in $K \cap X$ is dependent set. So $\{x_1, x_2, \cdots, x_{k-1}\}$ is a maximum independent set in $K \cap X$. Hence by Definition 2.14, we can get that

$$r_{M_k(R)}(K \cap X) = |\{x_1, x_2, \cdots, x_{k-1}\}| = k - 1.$$

(2) Since $|K \cap X| \leq k - 2$, according to Definition 2.4, $\overline{R}_{k-1}(K \cap X) = \varnothing$. Thus $K \cap X$ is an independent set in $M_k(R)$. So, according to Definition 2.14,

$$r_{M_k(R)}(K \cap X) = |K \cap X|.$$

□

Proposition 3.11. *Let R be an equivalence relation on U. Then for all $X \subseteq U$,*

$$r_{M_k(R)}(X) = (k-1) f_R(\overline{R}_{k-2}(X)) + |X - \overline{R}_{k-2}(X)|.$$

Proof. Let $f_R(\overline{R}_{k-2}(X)) = s$, where s is a positive integer. By Definition 2.19, we may assume that

$$\{K \in U/R \mid |K \cap X| > k-2\} = \{K_1, K_2, \cdots, K_s\}$$

and

$$U/R = \{K_1, K_2, \cdots, K_s,\ K_{s+1}, \cdots, K_n\},$$

where n is a positive integer. Then $|K_j \cap X| \leq k-2, j = s+1, s+2, \cdots, n$. By Lemma 3.10, we can get that

$$r_{M_k(R)}(K_i \cap X) = k-1,\ i = 1, 2, \cdots, s$$

and

$$r_{M_k(R)}(K_j \cap X) = |K_j \cap X|,\ j = s+1, s+2, \cdots, n.$$

Since $X = (K_1 \cap X) \cup (K_2 \cap X) \cup \cdots \cup (K_n \cap X)$ and $(K_i \cap X) \cap (K_j \cap X) = \varnothing$, where $i \neq j$ and $i, j = 1, 2, \cdots, n$, it follows from Proposition 2.15 that

$$\begin{aligned} & r_{M_k(R)}(X) \\ &= r_{M_k(R)}((K_1 \cap X) \cup (K_2 \cap X) \cup \cdots \cup (K_n \cap X)) \\ &\leq r_{M_k(R)}(K_1 \cap X) + r_{M_k(R)}(K_2 \cap X) + \cdots + r_{M_k(R)}(K_n \cap X) \\ &= (k-1) + \cdots + (k-1) + |K_{s+1} \cap X| + |K_{s+2} \cap X| + \cdots + |K_n \cap X| \\ &= (k-1) \cdot s + |X - \overline{R}_{k-2}(X)| \\ &= (k-1) f_R(\overline{R}_{k-2}(X)) + |X - \overline{R}_{k-2}(X)|. \end{aligned}$$

On the other hand, let

$$X_i = \{x_{i1}, x_{i2}, \cdots, x_{i,k-1}\} \subseteq K_i \cap X, i = 1, 2, \cdots, s$$

and

$$X_j = K_j \cap X, j = s+1, s+2, \cdots, n.$$

Then $Y = X_1 \cup X_2 \cup \cdots \cup X_s \cup X_{s+1} \cup \cdots \cup X_n \subseteq X$. According to Definition 2.4, $\overline{R}_{k-1}(Y) = \varnothing$. Thus, by Proposition 3.5, Y is an independent set in $M_k(R)$. It follows from Definition 2.14 that

$$\begin{aligned} & r_{M_k(R)}(X) \geq |Y| \\ &= (k-1) \cdot s + |K_{s+1} \cap X| + |K_{s+2} \cap X| + \cdots + |K_n \cap X| \\ &= (k-1) f_R(\overline{R}_{k-2}(X)) + |X - \overline{R}_{k-2}(X)|. \end{aligned}$$

Hence $r_{M_k(R)}(X) = (k-1) f_R(\overline{R}_{k-2}(X)) + |X - \overline{R}_{k-2}(X)|$. □

According to Proposition 3.5 and Proposition 3.6, we can present two formulations of the base of the matroid $M_k(R)$:

(1) $\mathcal{B}(M_k(R)) = \{X \subseteq U \mid \overline{R}_{k-1}(X) = \varnothing \textit{ and } \forall\, Y \subseteq U,\ X \subset Y \Rightarrow \overline{R}_{k-1}(Y) \neq \varnothing\}$,

(2) $\mathcal{B}(M_k(R)) = \{X \cup \underline{R}_{k-2}(\varnothing) \mid X \subseteq U, \overline{R}_{k-2}(X) = \overline{R}_{k-2}(U) \textit{ and } \overline{R}_{k-1}(X) = \varnothing\}$,
$k \geq 2$;
where R be an equivalence relation on U.

Example 3.12. Let $U = \{a, b, c, d, e\}$, R be an equivalence relation on U, and $U/R = \{\{a, b, c\}, \{d, e\}\}$. If $k = 3$, $X = \{a, d\}$, $Y = \{a, c, d, e\}$, then $r_{M_3(R)}(X) = 2$, $r_{M_3(R)}(Y) = 4$, $r_{M_3(R)}(U) = 4$; $\mathcal{B}(M_3(R)) = \{\{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}$.

Proposition 3.13. [9] *Let M be a matroid, $X \subseteq U$, and $x \in U$. Then $x \in cl_M(X) \Leftrightarrow x \in X$ or there exists a circuit C of M such that $C - \{x\} \subseteq X$.*

We know that a closure operator uniquely determines a matroid. In the following proposition, we characterize the closure operator of the matroid $M_k(R)$ from the viewpoint of degree rough sets.

Lemma 3.14. *Let R be an equivalence relation on U, $X \subseteq U$, $K \in U/R$, and $|K| > k - 2$. Then $|K \cap X| > k - 2$ if and only if $K \subseteq cl_{M_k(R)}(X)$.*

Proof. (⇒): Suppose $|K \cap X| > k - 2$. Then $|K \cap X| \geq k - 1$. Let $K \cap X = \{x_1, x_2, \cdots, x_m\}$, where m is a positive integer and $m \geq k - 1$. For all $u \in K$, if $u \in X$, then, by Definition 2.16, $u \in cl_{M_k(R)}(X)$. If $u \notin X$, then, according to formula (3.1), $\{x_1, x_2, \cdots, x_{k-1}, u\} \in \mathcal{C}_k(R)$, that is, $\{x_1, x_2, \cdots, x_{k-1}, u\}$ is a circuit of $M_k(R)$. Thus, according to Proposition 3.13, $u \in cl_{M_k(R)}(X)$. So $K \subseteq cl_{M_k(R)}(X)$.

(⇐): If $K \subseteq X$, then it is straightforward. If $K \nsubseteq X$, we may assume that $\exists\ u \in K$ and $u \notin X$. Since $K \subseteq cl_{M_k(R)}(X)$, we have that $u \in cl_{M_k(R)}(X)$. Then, according to Proposition 3.13, there exists $C \in \mathcal{C}_k(R)$ such that $u \in C$ and $C - \{u\} \subseteq X$. Thus, according to formula (3.1), $C - \{u\} \subseteq K$ and $|C| = k$. So $C - \{u\} \subseteq K \cap X$. Hence $|K \cap X| \geq |C - \{u\}| = k - 1 > k - 2$. □

Proposition 3.15. *Let R be an equivalence relation on U and $X \subseteq U$. Then*

$$cl_{M_k(R)}(X) = \overline{R}_{k-2}(X) \cup X.$$

Proof. Let $x \in \overline{R}_{k-2}(X) \cup X$. Then $x \in X$ or $x \in \overline{R}_{k-2}(X)$. If $x \in X$, then, according to Definition 2.16, $x \in cl_{M_k(R)}(X)$. If $x \in \overline{R}_{k-2}(X)$, then, according to Definition 2.4, there exists $K \in U/R$ such that $|K \cap X| > k - 2$ and $x \in K$. Thus, according to Lemma 3.14, $x \in K \subseteq cl_{M_k(R)}(X)$. So $\overline{R}_{k-2}(X) \cup X \subseteq cl_{M_k(R)}(X)$.

On the other hand, let $u \notin \overline{R}_{k-2}(X) \cup X$. Then $u \notin \overline{R}_{k-2}(X)$ and $u \notin X$. By $u \notin \overline{R}_{k-2}(X)$ and Definition 2.4, $|[u]_R \cap X| \leq k - 2$.

(i) If $|[u]_R \cap X| = k - 2$, by $u \notin X$, we have that $|[u]_R \cap (X \cup \{u\})| = k - 1 > k - 2$. Thus, by Definition 2.4, $[u]_R \in \{K \in U/R \mid |K \cap (X \cup \{u\})| > k - 2\}$. So, it is easy to prove that $\overline{R}_{k-2}(X \cup \{u\}) = \bigcup\{K \cup [u]_R \mid K \in U/R \text{ and } K \subseteq \overline{R}_{k-2}(X)\}$. Hence

$$(X \cup \{u\}) - \overline{R}_{k-2}(X \cup \{u\}) = (X - \overline{R}_{k-2}(X)) - (X \cap [u]_R) \tag{3.2}$$

and

$$f_R(\overline{R}_{k-2}(X \cup \{u\})) = f_R(\overline{R}_{k-2}(X)) + 1. \tag{3.3}$$

Since R is an equivalence relation, $X \cap [u]_R \subseteq X - \overline{R}_{k-2}(X)$. Thus

$$|(X \cup \{u\}) - \overline{R}_{k-2}(X \cup \{u\})| = |X - \overline{R}_{k-2}(X)| - |X \cap [u]_R|. \tag{3.4}$$

Therefore, according to Proposition 3.11, formula (3.3) and formula (3.4),

$$\begin{aligned} r_{M_k(R)}(X \cup \{u\}) &= (k-1)f_R(\overline{R}_{k-2}(X \cup \{u\})) + |(X \cup \{u\}) - \overline{R}_{k-2}(X \cup \{u\})| \\ &= (k-1)(f_R(\overline{R}_{k-2}(X)) + 1) + |X - \overline{R}_{k-2}(X)| - |X \cap [u]_R| \\ &= (k-1)f_R(\overline{R}_{k-2}(X)) + (k-1) + |X - \overline{R}_{k-2}(X)| - (k-2) \\ &= (k-1)f_R(\overline{R}_{k-2}(X)) + |X - \overline{R}_{k-2}(X)| + 1 \\ &= r_{M_k(R)}(X) + 1 \neq r_{M_k(R)}(X). \end{aligned}$$

(ii) If $|[u]_R \cap X| < k-2$, then $|[u]_R \cap (X \cup \{u\})| \leq k-2$. Thus $[u]_R \notin \{K \in U/R \mid |K \cap (X \cup \{u\})| > k-2\}$. So

$$\overline{R}_{k-2}(X \cup \{u\}) = \overline{R}_{k-2}(X) \tag{3.5}$$

and

$$f_R(\overline{R}_{k-2}(X \cup \{u\})) = f_R(\overline{R}_{k-2}(X)). \tag{3.6}$$

Because $u \notin X$,

$$|(X \cup \{u\}) - \overline{R}_{k-2}(X \cup \{u\})| = |(X \cup \{u\}) - \overline{R}_{k-2}(X)| = |X - \overline{R}_{k-2}(X)| + 1. \tag{3.7}$$

By Proposition 3.11, formula (3.6) and formula(3.7),

$$\begin{aligned} r_{M_k(R)}(X \cup \{u\}) &= (k-1)f_R(\overline{R}_{k-2}(X \cup \{u\})) + |(X \cup \{u\}) - \overline{R}_{k-2}(X \cup \{u\})| \\ &= (k-1)f_R(\overline{R}_{k-2}(X)) + |X - \overline{R}_{k-2}(X)| + 1 \\ &= r_{M_k(R)}(X) + 1 \neq r_{M_k(R)}(X). \end{aligned}$$

According to (i) and (ii), if $u \notin \overline{R}_{k-2}(X) \cup X$, then

$$u \notin \{u \in U \mid r_{M_k(R)}(X) = r_{M_k(R)}(X \cup \{u\})\}.$$

Thus, according to Definition 2.16, $u \notin cl_{M_k(R)}(X)$. So,

$$cl_{M_k(R)}(X) \subseteq \overline{R}_{k-2}(X) \cup X.$$

Hence $cl_{M_k(R)}(X) = \overline{R}_{k-2}(X) \cup X$. □

Example 3.16. In Example 3.12, $k = 3$, $\overline{R}_{3-2}(X) = \varnothing$. Then

$$cl_{M_3(R)}(X) = X = \{a, d\}; \overline{R}_{3-2}(Y) = \{a, b, c, d, e\}.$$

Thus $cl_{M_3(R)}(Y) = \overline{R}_{3-2}(Y) \cup Y = \{a, b, c, d, e\} \cup \{a, c, d, e\} = \{a, b, c, d, e\} = U$.

4. The relationships between generalized degree rough setsaAnd matroids

We have discussed the relationships between classical rough sets and matroids in Section 3. However, it is difficult to ensure the relation R is equivalence relation always. Then we will discuss the relationships between generalized degree rough sets which based on generalized binary relations and matroids. According to Proposition 3.7, if replace the equivalence relation by binary relation, that is, R is a binary relation, denote

$$\mathcal{C}'_k(R) = Min\{X \subseteq U | \overline{apr}_{k-1}(X) \neq \varnothing\}. \tag{4.1}$$

Whether $\mathcal{C}'_k(R)$ can induce an matroid structure?

Definition 4.1. Let R be a binary relation on U. For all $x, y \in U$, if $R_s(x) \cap R_s(y) \neq \varnothing$ implies $\exists\, z \in U$, such that $R_s(x) \cup R_s(y) \subseteq R_s(z)$, then R is called an union closed relation.

Lemma 4.2. *Let R be a binary relation on U and $\mathcal{C}'_k(R) = Min\{X \subseteq U \mid \overline{apr}_{k-1}(X) \neq \varnothing\}$. Then for all $C \in \mathcal{C}'_k(R)$, there exists $x \in U$ such that $C \subseteq R_s(x)$ and $|C| = k$.*

Proof. $\forall\, C \in \mathcal{C}'_k(R)$, according to formula (4.1), $\overline{apr}_{k-1}(C) \neq \varnothing$. By Definition 2.6, there exists $x \in U$ such that $|R_s(x) \cap C| > k-1$. That is, $|R_s(x) \cap C| \geq k$. Let $K \subseteq R_s(x) \cap C$ and $|K| = k$. Then $k = |K| = |K \cap R_s(x)| > k-1$. Thus $x \in \overline{apr}_{k-1}(K)$. So $\overline{apr}_{k-1}(K) \neq \varnothing$. That is to say, $K \in \{X \subseteq U \mid \overline{apr}_{k-1}(X) \neq \varnothing\}$. By $C \in \mathcal{C}'_k(R)$, $K \subseteq C$ and formula (4.1), we can get $K = C$. Hence $|C| = |K| = k$ and $C = K \subseteq R_s(x)$. □

In the following theorem, we present the condition of a matroid induced by a generalized binary relation.

Theorem 4.3. *Let R be a union closed relation on U. Then $\mathcal{C}'_k(R)$ satisfies the circuit axiom.*

Proof. (1) By Definition 2.6, it is easy to verify $\overline{apr}_{k-1}(\varnothing) = \varnothing$. Thus, by formula (4.1), $\varnothing \notin \mathcal{C}'_k(R)$.

(2) For all $C_1, C_2 \in \mathcal{C}'_k(R)$, by formula (4.1), C_1 and C_2 are minimum sets of $\{X \subseteq U \mid \overline{apr}_{k-1}(X) \neq \varnothing\}$. Then, if $C_1 \subseteq C_2$, we can get that $C_1 = C_2$.

(3) Let $C_1, C_2 \in \mathcal{C}'_k(R)$, $C_1 \neq C_2$ and $z \in C_1 \cap C_2$. By Lemma 4.2, there exists $x, y \in U$ such that $C_1 \subseteq R_s(x)$, $C_2 \subseteq R_s(y)$ and $|C_1| = |C_2| = k$. By $C_1 \cap C_2 \neq \varnothing$, we have that $R_s(x) \cap R_s(y) \neq \varnothing$. Since R is a union closed relation, there exists $a \in U$ such that $R_s(x) \cup R_s(y) \subseteq R_s(a)$. Thus $(C_1 \cup C_2) - \{z\} \subseteq R_s(a)$. It is easy to prove that $|(C_1 \cup C_2) - \{z\}| > k-1$. So $|R_s(a) \cap ((C_1 \cup C_2) - \{z\})| > k-1$. It follows from Definition 2.6 that $\overline{apr}_{k-1}((C_1 \cup C_2) - \{z\}) \neq \varnothing$. This implies $(C_1 \cup C_2) - \{z\} \in \{X \subseteq U \mid \overline{apr}_{k-1}(X) \neq \varnothing\}$. Hence there exists $C_3 \in Min\{X \subseteq U \mid \overline{apr}_{k-1}(X) \neq \varnothing\}$ such that $C_3 \subseteq (C_1 \cup C_2) - \{z\}$, that is, $C_3 \in \mathcal{C}'_k(R)$.

In summary, by Proposition 2.11, $\mathcal{C}'_k(R)$ satisfies the three conditions of circuit axiom. □

Corollary 4.4. *Let R be a binary relation on U. For all $x, y \in U$, if $R_s(x) \cap R_s(y) \neq \varnothing$ implies $R_s(x) \subseteq R_s(y)$ or $R_s(y) \subseteq R_s(x)$, then $\mathcal{C}'_k(R) = Min\{X \subseteq U \mid \overline{apr}_{k-1}(X) \neq \varnothing\}$ satisfies the circuit axiom.*

We can find some binary relations satisfying the condition in Corollary 4.4. In the following conclusions, we show that the generalized degree approximation operators with respect to R can also induce matroids when R is a symmetric and transitive relation or transitive and Euclid relation on U.

Lemma 4.5. *Let R be a symmetric and transitive relation on U. Then for all $x, y \in U$, $x \in R_s(y)$ implies $R_s(x) = R_s(y)$.*

Proof. Since R is a symmetric relation and $x \in R_s(y)$, according to Proposition 2.7, $y \in R_s(x)$. Since R is a transitive relation, according to Proposition 2.7, $R_s(x) \subseteq R_s(y)$, $R_s(y) \subseteq R_s(x)$. Then $R_s(x) = R_s(y)$. □

According to Corollary 4.4, Lemma 4.5 and Proposition 2.11, the generalized degree approximation operators with respect to R can induce matroids when R is a symmetric and transitive relation on U.

Proposition 4.6. *Let R be a symmetric and transitive relation on U. Then $\mathcal{C}'_k(R)$ satisfies the circuit axiom.*

Lemma 4.7. *Let R be a transitive and Euclid relation on U. Then for all $x, y \in U$, $x \in R_s(y)$ implies $R_s(x) = R_s(y)$.*

Proof. Since R is a transitive relation and $x \in R_s(y)$, according to Proposition 2.7, $R_s(x) \subseteq R_s(y)$. Since R is an Euclid relation, according to Proposition 2.7, $R_s(y) \subseteq R_s(x)$. Then $R_s(x) = R_s(y)$. □

According to Corollary 4.4, Lemma 4.7 and Proposition 2.11, the generalized degree approximation operators with respect to R can also induce matroids when R is a transitive and Euclid relation on U.

Proposition 4.8. *Let R be a transitive and Euclid relation on U. Then $\mathcal{C}'_k(R)$ satisfies the circuit axiom.*

However, the generalized degree approximation operators with respect to R can induce matroids not only when R is a symmetric and transitive relation or transitive and Euclid relation on U. In Example 4.9, R isn't a reflexive, symmetric, transitive or Euclid relation, its generalized degree approximation operators can also induce a matroid.

Example 4.9. Let $U = \{1, 2, 3, 4, 5\}$, $R_s(1) = \{1, 2, 3\}$, $R_s(2) = \{3\}$, $R_s(3) = \{4, 5\}$, $R_s(4) = \{4\}$, $R_s(5) = \{5\}$. If $k = 2$, then $\mathcal{C}'_2(R) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}\}$.

5. The same matroid induced by different relations

According to the above sections, there are many ways to induce matroids by binary relations, and these matroid structures induced by same relation coincide with each other. However, whether different relations can induce the same matroid? In the following section, we discuss the conditions of inducing the same matroid.

Symbol description. In this section, for all $X \subseteq U$, we denote the upper approximation numbers of X with respect to R_1, R_2 as $f_{R_1}(X)$, $f_{R_2}(X)$ respectively. Denote the upper degree approximations of X with respect to equivalence relations R_1, R_2 as $\overline{R}_{1,(k-1)}(X)$, $\overline{R}_{2,(k-1)}(X)$ respectively. Denote the upper generalized degree approximations of X with respect to binary relations R_1, R_2 as $\overline{apr}_{1,(k-1)}(X)$, $\overline{apr}_{2,(k-1)}(X)$ respectively.

Theorem 5.1. *Let R_1, R_2 be equivalence relations on U, k is a positive integer. Then the following three conditions are equivalent :*
(1) $\mathcal{C}_k(R_1) = \mathcal{C}_k(R_2)$.
(2) *For all $x \in U$, if $|[x]_{R_1}| > k-1$ or $|[x]_{R_2}| > k-1$, then $[x]_{R_1} = [x]_{R_2}$.*
(3) *For all $X \subseteq U$, $\overline{R}_{1,(k-1)}(X) = \overline{R}_{2,(k-1)}(X)$.*

Proof. (1)⇒(2): Let $x \in U$ and assume that $|[x]_{R_1}| > k-1$. Then $|[x]_{R_1}| \geq k$. Take $X \subseteq [x]_{R_1}$ and $|X| = k$. Then, according to formula (3.1), $X \in \mathcal{C}_k(R_1)$. Since $\mathcal{C}_k(R_1) = \mathcal{C}_k(R_2)$, $X \in \mathcal{C}_k(R_2)$. By formula (3.1), $f_{R_2}(X) = 1$. According to $f_{R_2}(X) = 1$, $|X| = k$, and Definition 2.4, we can get that $X \subseteq \overline{R}_{2,(k-1)}(X)$ and $\overline{R}_{2,(k-1)}(X) = [x]_{R_2}$, that is, $X \subseteq [x]_{R_2}$. We have proved that for all $X \subseteq [x]_{R_1}$, if $|X| = k$, then $X \subseteq [x]_{R_2}$. Thus $[x]_{R_1} \subseteq [x]_{R_2}$.

Similarly, we can prove that $[x]_{R_2} \subseteq [x]_{R_1}$. So $[x]_{R_1} = [x]_{R_2}$.

(2)⇒(3): For all $x \in U$, if $|[x]_{R_1}| > k-1$ or $|[x]_{R_2}| > k-1 \Rightarrow [x]_{R_1} = [x]_{R_2}$, then, by Definition 2.4, $\overline{R}_{1,(k-1)}([x]_{R_1}) = \overline{R}_{2,(k-1)}([x]_{R_2})$. In consideration of the arbitrariness of x, we have that $\forall\ X \subseteq U$, $\overline{R}_{1,(k-1)}(X) = \overline{R}_{2,(k-1)}(X)$.

(3)⇒(1): For all $X \subseteq U$, if $\overline{R}_{1,(k-1)}(X) = \overline{R}_{2,(k-1)}(X)$, then

$$Min\{X \subseteq U \mid \overline{R}_{1,(k-1)}(X) \neq \varnothing\} = Min\{X \subseteq U \mid \overline{R}_{2,(k-1)}(X) \neq \varnothing\}.$$

Since $Min\{X \subseteq U \mid \overline{R}_{k-1}(X) \neq \varnothing\} = \{X \subseteq U \mid f_R(X) = 1 \text{ and } |X| = k\}$, (1) holds. □

In the above theorem, we present the necessary and sufficient condition of the same matroid induced by different equivalence relations. Next, we discuss the situation of binary relations.

Proposition 5.2. *Let R_1, R_2 be binary relations on U, and $X \subseteq U$. If $\overline{apr}_{1,(k-1)}(X)$ $= \overline{apr}_{2,(k-1)}(X)$, then $\mathcal{C}'_k(R_1) = \mathcal{C}'_k(R_2)$.*

Proof. According to formula (4.1), it is straightforward. □

However, in the following example, we demonstrate that Proposition 5.2 is not sufficient.

Example 5.3. Let $U = \{1, 2, 3, 4, 5\}$, $R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 4), (3, 5), (4, 4), (5, 5)\}$, $R_2 = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 4), (3, 5), (4, 4), (5, 5)\}$, $k = 2$. Then $\mathcal{C}'_2(R_1) = \mathcal{C}'_2(R_2) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}\}$. If we let that $X = \{1, 2, 3\}$, then $\overline{apr}_{1,(2-1)}(X) = \{1\}$, $\overline{apr}_{2,(2-1)}(X) = \{1, 2\}$.

6. Conclusions And Future Works

This paper established a bridge between degree rough sets and matroids. Specifically, upper approximation number functions based on equivalence relations were used to induce the kind of matroids, then we found many connections between degree rough sets and these matroids. Therefore, we used degree rough sets to study matroids in detail. We also extended degree rough sets to generalized degree rough sets, and studied the relationships between generalized degree rough sets and matroids. Finally, we discussed the conditions of the same matroid induced by different binary relations. In a word, this paper provided a new perspective, that is, we studied matroids by degree rough sets. Though much research has been conducted in this paper, there are still many interesting issues worth studying in future work.

1. Applications of the matroids induced by upper approximation number functions(or degree approximation operators) in attribute reduction.
2. The methods of matroids induced by lower approximation number functions.
3. The more relationships with generalized degree rough sets and matroids.

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