Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 6, (June 2016), pp. 989–1001
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

Fuzzy convex sub l-groups

G. S. V. Satya Saibaba

Received 16 November 2015; Revised 19 December 2016; Accepted 9 January 2016

Abstract. The aim of this paper is to introduce the notions of L-fuzzy convex sub l-groups, L-fuzzy cosets, L-fuzzy prime convex sub l-groups of l-group G and to introduce ordering between two L-fuzzy Cosets of a L-fuzzy convex sub l-group.

2010 AMS Classification: 03E72, 08A72, 06D72, 18B35

Keywords: L-fuzzy sub l-group, L-fuzzy convex sub l-groups, L-fuzzy prime convex sub l-groups, L-fuzzy cosets.

Corresponding Author: G. S. V. S. Saibaba (saibabagannavarapu65@yahoo.com)

1. Introduction

The notion of fuzzy subset of a set was introduced by Zadeh[22]. Rosenfeld [19] applied this concept to the theory of groups and groupoids. Since then these ideas so many have been applied to various algebraic structures. Swamy and Swamy studied Fuzzy Prime ideals of rings[21] and Naseem Ajmal studied Fuzzy lattices[18].For further study see [3, 10, 11, 12, 13, 14, 15, 17, 23]. Goguen [8]initiated a more abstract study of fuzzy sets by replacing the values set $[0, 1]$, by a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L-fuzzy sets. Most of the authors considered fuzzy subsets by taking values in a complete lattice. Fuzzy algebra is now a well developed part of algebra. Partially ordered algebraic systems play an important role in algebra. Especially l-groups, l-rings, Vector lattices, and f-rings are important concepts in algebra [5, 1, 2, 4, 6, 7, 9, 16]. We introduced the concepts of L-fuzzy sub l-groups and L-fuzzy l-ideals of l-group in [20]. The objective of this paper is to study L-fuzzy convex sub l-groups which assume values in a complete lattice that satisfies the infinite meet distributive law.

In this paper, we introduce the concepts of L-fuzzy convex sub l-groups, L-fuzzy prime convex sub l-groups and L-fuzzy maximal convex sub l-groups and ordering between two L-fuzzy cosets of a L-fuzzy convex sub l-group, of l-group G. With this ordering, we prove that the set of L-fuzzy cosets $\frac{G}{\lambda}$ form a distributive lattice, if λ is a L-fuzzy convex sub l-group. We obtain one to one correspondence between

L-fuzzy prime convex sub l-groups of G and pairs (P, α), where P is a prime convex l-subgroup and α is an irreducible element in L. We prove that $\frac{G}{\lambda}$ is totally ordered, if λ is a L-fuzzy prime convex sub l-group and a particular L-fuzzy convex sub l-group can be expressed as the intersection of L-fuzzy prime convex sub l-groups under certain conditions.

Throughout this paper, let $G \neq 0$ be an l-group and L stands for a nontrivial complete lattice in which the infinite meet distributive law $a \wedge (\vee_{s\in S} s) = \vee_{s\in S}(a \wedge s)$ for any $S \subseteq L$ and $a \in L$, holds and we consider meet irreducible elements of L only.

2. Preliminaries

Definition 2.1 ([5])**.** A lattice ordered group is a system $G = (G, +, \leq)$, where
(i) $(G, +)$ is a group,
(ii) $(G, \leq)$ is a lattice and
(iii) the inclusion is invariant under all translations $x \mapsto a + x + b$ i.e, $x \leq y \Rightarrow a + x + b \leq a + y + b$, for all $a, b \in G$.

Definition 2.2 ([5])**.** If a is an element of l-group G, then $a \vee (-a)$ is called the absolute value of a and is denoted by $|a|$. Any element a of an l-group G can be written as $a = a \vee 0 + a \wedge 0$, i.e., $a = a^+ + a^-$, where a^+ is called positive part of a and a^- is called negative part of a.

Theorem 2.3 ([5])**.** *In any l-group G, for all $a \in G$,*
(1) $|a| \geq 0$, *moreover* $|a| > 0$ *unless* $a = 0$,
(2) $a^+ \wedge (-a^+) = 0$,
(3) $|a| = a^+ - a^-$.

Theorem 2.4 ([5])**.** *For all x, y in any l-group G, the absolute value satisfies :*
(1) $|nx| = |n||x|$ *for any integer* n.
(2) $|x - y| = x \vee y - x \wedge y$.
(3) $|x \vee y - x' \vee y| \leq |x - x'|$ *and dually.*
(4) $|(x \vee z) - (y \vee z)| + |(x \wedge z) - (y \wedge z)| = (x \vee y) - (x \wedge y)$.
(5) $|x + y| \leq |x| + |y| + |x|$.

Theorem 2.5 ([5])**.** *For all x, y, a in any l-group G, addition is distributive on meets and joins, i.e.,*
(1) $a + (x \vee y) = (a + x) \vee (a + y)$, $\quad$ $(x \vee y) + b = (x + b) \vee (y + b)$,
(2) $a + (x \wedge y) = (a + x) \wedge (a + y)$, $\quad$ $(x \wedge y) + b = (x + b) \wedge (y + b)$.

Theorem 2.6 ([5])**.** *In any l-group G, $a - (a \wedge b) + b = b \vee a$, for all $a, b \in G$.*

Definition 2.7 ([2])**.** A l-subgroup C of G is called convex if $0 \leq g \leq c \in C$ and $g \in G$ imply $g \in C$.

The set of all convex l-subgroups is denoted by $\mathcal{C}(G)$. It is closed with respect to arbitrary intersections and it is a distributive sublattice of the lattice of all l-subgroups of G.

Definition 2.8 ([2])**.** Let C be a convex l-subgroup of G. For $x, y \in G$, define $C + x \leq C + y$ if and only if $c + x \leq y$ for some $c \in C$.

Theorem 2.9 ([2])**.** *The set $R(C)$ of right cosets in G is a lattice with the above ordering.*

Definition 2.10 ([2])**.** A convex l-subgroup M of G is called regular if M is maximal without some $g \in G$ and in this case M is called a value of g.

Theorem 2.11 ([2])**.** *Let $P \in \mathcal{C}(G)$. Then, P is meet irreducible in $\mathcal{C}(G)$ if and only if P is maximal in $\mathcal{C}(G)$ with respect to not containing some $g \in G$.*

Theorem 2.12 ([2])**.** *Every non zero element of an l-group G has at least one value. Consequently, each convex l-subgroup can be obtained as the intersection of regular l-subgroups.*

Theorem 2.13 ([2])**.** *If G is an l-group and $M \in \mathcal{C}(G)$ then the following are equivalent :*
(1) *M is regular.*
(2) *$M \subset M^* = \cap\{C \mid M \subseteq C \in \mathcal{C}(G)\}$.*
(3) *M is meet irreducible in $\mathcal{C}(G)$.*

Corollary 2.14 ([2])**.** *If M is a regular convex l-subgroups of G and $a, b \in G^+ - M$, then $a \wedge b \in G^+ - M$.*

Theorem 2.15 ([2])**.** *If G is an l-group and $M \in \mathcal{C}(G)$, then the following are equivalent :*
(1) *If $A, B \in \mathcal{C}(G)$ and $M \supseteq A \cap B$, then $M \supseteq A$ or $M \supseteq B$.*
(2) *If $A, B \in \mathcal{C}(G)$, $A \supset M$ and $B \supset M$, then $A \cap B \supset M$.*
(3) *If $a, b \in G^+ - M$, then $a \wedge b \in G^+ - M$.*
(4) *If $a, b \in G^+ - M$, then $a \wedge b > 0$.*
(5) *The lattice of right cosets of M is totally ordered.*
(6) *$\{C \in \mathcal{C}(G) \mid C \supseteq M\}$ is chain.*
(7) *M is the intersection of a chain of regular convex l-subgroups.*

Definition 2.16 ([2])**.** A convex l-subgroup that satisfies any one of the conditions of above theorem will be called prime.

Each regular l-subgroup is a prime convex l-subgroup. Each maximal convex l-subgroup of G is a value of some element of G, so maximal convex l-subgroup is prime. In special case that $R(C)$ becomes linearly ordered with the above ordering, we call C is a prime convex l-subgroup of G.

Theorem 2.17 ([2])**.** *Let $S \neq \varnothing$ be a subset of an l-group of G. Then*
$$\langle S \rangle = \{x \in G \mid |x| \leq |s_1| + |s_2| + \cdots + |s_n|,\ s_i \in S, i = 1, 2, \cdots n\}$$
is the convex l-subgroup of G generated by S.

Let G be an l-group and $g \in G$. Let $\mathcal{C}_g$ be the set of all convex l-subgroups of G that contain g. Then $\cap\{c \mid c \in \mathcal{C}_g\}$ contains g and is contained in every element of $\mathcal{C}_g$. It is the smallest convex l-subgroup of G containing g, and will be denoted by $\langle g \rangle$.

Lemma 2.18 ([7])**.** *If G is a l-group and $g \in G$, then*
$$\langle g \rangle = \{f \in G \mid |f| \leq n|g|, n \in \mathbb{Z}^+\}.$$

Corollary 2.19. [7] *If G is a l-group and $f, g \in G^+$, then*
$$\langle f \wedge g \rangle = \langle f \rangle \wedge \langle g \rangle \text{ and } \langle f \vee g \rangle = \langle f \rangle \vee \langle g \rangle.$$

Corollary 2.20. [7] *Let $g \in G$. Then, $\langle |g| \rangle = \langle g \rangle$.*

Definition 2.21. [7] Let $e \in G$ be called a strong unit if for any $a \in G$, $|a| < n|e|$, for some $n \in \mathbb{Z}^+$.

Clearly $\langle e \rangle = G$.

Definition 2.22 ([17])**.** Let X be a non empty set. A L-Fuzzy subset λ of X is a mapping from X into L, where L is a complete lattice satisfying the infinite meet distributive law.

Definition 2.23 ([17])**.** Let $\lambda : X \to L$ be a L-fuzzy sub set of X. Then the set $\{\lambda(x) \mid x \in X\}$ is called the image of λ and is denoted by $\lambda(x)$ or $Im(\lambda)$. The set $\{x \mid x \in X, \lambda(x) > 0\}$ is called the support of λ and is denoted by $Supp(\lambda)$. The set $X_\lambda = \{x \in X \mid \lambda(x) = \lambda(0)\}$. For $t \in L$, $\lambda_t = \{x \in X \mid \lambda(x) \geq t\}$ is called a t-cut or t-level set of λ.

Definition 2.24 ([17])**.** Let λ, μ be two L-fuzzy subsets of X. If $\lambda(x) \leq \mu(x)$ for all $x \in X$, then we say that λ is contained in μ and we write $\lambda \subseteq \mu$. Define $\lambda \cup \mu$ and $\lambda \cap \mu$ are L-fuzzy subsets of X by for all $x \in X$, $(\lambda \cup \mu)(x) = \lambda(x) \vee \mu(x)$, $(\lambda \cap \mu)(x) = \lambda(x) \wedge \mu(x)$. Then $\lambda \cup \mu$ and $\lambda \cap \mu$ are called the union and intersection of λ and μ, respectively.

Definition 2.25 ([17])**.** Let f be a mapping from X into Y, and let λ and μ be L-fuzzy subsets of X and Y respectively. The L-fuzzy subsets $f(\lambda)$ of Y and $f^{-1}(\mu)$ of X, defined by
$$f(\lambda)(y) = \begin{cases} \vee\{\lambda(x) \mid x \in X, f(x) = y\} & \text{if } f^{-1}(y) \neq \varnothing; \\ 0 & \text{otherwise,} \end{cases}$$
where $y \in Y$, and $f^{-1}(\mu)(x) = \mu(f(x))$, for all $x \in X$, are called the image of λ under f and the pre-image of μ under f, respectively.

Definition 2.26 ([17])**.** A L-fuzzy subset λ of X is said to have *sup* property if, for any subset A of X, there exists $a_0 \in A$ such that $\lambda(a_0) = \vee_{a \in A} \lambda(a)$.

Definition 2.27 ([17])**.** Let f be any function from a set X to a set Y, and let λ be any L-fuzzy subset of X. Then λ is called f-invariant if $f(x) = f(y)$ implies $\lambda(x) = \lambda(y)$, where $x, y \in X$.

Definition 2.28 ([17])**.** Let X be nonempty set. Let $Y \subseteq X$ and $a \in Y$. We define, a L-fuzzy set a_Y is defined as follows:
$$a_Y(x) = \begin{cases} a & \text{if } x \in Y \\ 0 & \text{if } x \in X - Y. \end{cases}$$
In particular, if Y is a singleton, say, $\{y\}$, then a_y is called as L-fuzzy point.

Definition 2.29 ([20])**.** Let $G = (G, +, \vee, \wedge)$ be an l-group with 0 as the additive identity in G. A L-fuzzy subset λ of G is said to be a L-fuzzy sub l-group of G, if

(i) $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$,

(ii) $\lambda(-x) = \lambda(x)$,
(iii) $\lambda(x \vee y) \geq \lambda(x) \wedge \lambda(y)$,
(iv) $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$,
for all $x, y \in G$.

Theorem 2.30 ([20]). *Let λ be a L-fuzzy subset of an l-group G. λ is a L-fuzzy sub l-group of G if and only if for all $x, y \in G$,*

$$\lambda(x - y) \geq \lambda(x) \wedge \lambda(y) \text{ and } \lambda(x \wedge y) \wedge \lambda(x \vee y) \geq \lambda(x) \wedge \lambda(y).$$

Theorem 2.31 ([20]). *If λ is a L-fuzzy sub l-group of G, then*
(1) *$\lambda(0) \geq \lambda(x)$ for all $x \in G$,*
(2) *$\lambda(x^+) \geq \lambda(x), \lambda(x^-) \geq \lambda(x)$ and $\lambda(|x|) \geq \lambda(x)$ for all $x \in G$,*
(3) *$Supp(\lambda)$ is a l-subgroup of G, if $Supp(\lambda) \neq \varnothing$ and L is regular, i.e., if $a \neq 0, b \neq 0$, then $a \wedge b \neq 0$ where $a, b \in L$.*

Definition 2.32 ([20]). A L-fuzzy sub l-group λ of a l-group G is said to be a L-fuzzy l-ideal of G if
(i) $\lambda(x + y) = \lambda(y + x)$ for all $x, y \in G$ and
(ii) $x, a \in G, |x| \leq |a| \Rightarrow \lambda(x) \geq \lambda(a)$.

3. L-fuzzy convex sub l-groups

Definition 3.1. A L-fuzzy sub l-group λ of G is said to be a L-fuzzy convex sub l-group of G if $x, a \in G$, $0 \leq x \leq a \Rightarrow \lambda(x) \geq \lambda(a)$ (*Convexity condition*).

Theorem 3.2. *Let λ be a L-fuzzy sub l-group of G. Then λ is a L-fuzzy convex sub l-group of G if and only if $0 \leq x \leq a$ implies $\lambda(0) \geq \lambda(x) \geq \lambda(a)$, for all $x, a \in G$.*

Lemma 3.3. *Let λ be a L-fuzzy convex sub l-group. Then $|x| \leq |a|$ implies $\lambda(x) \geq \lambda(a)$, for $x,\ a \in G$.*

Theorem 3.4. *A L-fuzzy sub l-group λ of a l-group G is a L-fuzzy convex sub l-group of G if and only if for each l-subgroup λ_t, $t \in \lambda(G) \cup \{t \in L \mid \lambda(0) \geq t\}$ is a convex l-subgroup of G.(In fact, for each $t \in L$, λ_t is empty or a convex l-subgroup of G).*

Example 3.5. Let $L = [0, 1]$. Let G=$\mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}$ be the set all integers. By ordering lexicographically $(a, b) \geq (0, 0)$ if and only if $a > 0$ or $a = 0$ and $b \geq 0$. Let $+$ be usual addition. $(G, +, \vee, \wedge)$ is an l-group with above ordering. Define a L-fuzzy subset $\mu : G \to L$, by

$$\mu(x) = \begin{cases} 1, & \text{if } (x, y) \in \{(0, 0)\} \\ 0.5 & \text{if } (x, y) \in (\{(0, 0)\} \times \mathbb{Z}) - \{(0, 0)\} \\ 0.25, & \text{otherwise.} \end{cases}$$

Clearly the level sets $\mu_t = \{(0, 0)\}$, if $0.5 < t \leq 1$, μ_t=$\{(0, 0)\} \times \mathbb{Z}$, if $0.25 < t \leq 0.5$ and μ_t=G for $0 \leq t \leq 0.25$, are convex l-subgroups of G. Therefore, μ is a L-fuzzy convex sub l-group of l-group G.

Example 3.6. Let L=$\{1, a, b, 0\}$, where $1 > a > 0$, $1 > b > 0$ and $a \parallel b$ be the lattice.

Let $G=\mathbb{Z}\times\mathbb{Z}$, where $\mathbb{Z}$ be the set all integers. Clearly $(G,+,\vee,\wedge)$ is an l-group with point wise ordering. Clearly $I_1=\mathbb{Z}\times\{0\}$, $I_2=\{0\}\times\mathbb{Z}$ are l-subgroups of G. Define a L-fuzzy subset $\mu : G \to L$, by

$$\mu(x,y)=\begin{cases} 1, & \text{if } (x,y)=(0,0) \\ a, & \text{if } (x,y)\in I_1-\{(0,0)\} \\ b, & \text{if } (x,y)\in I_2-\{(0,0)\} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the level sets $\mu_1=\{(0,0)\}$, $\mu_a=I_1$, $\mu_b=I_2$, and $\mu_0=G$ are convex l-subgroups of G. Therefore, μ is a L-fuzzy convex sub l-group of l-group G.

Example 3.7. Let $L=[0,1]$. Let $G=\mathbf{C(X)}$ be the set of all continuous real valued functions on X, where X is a Hausdroff space. If all the operations defined pointwise i.e, for $f,g\in\mathbf{C(X)}$ and for each $x\in X$, $(f+g)(x)=f(x)+g(x)$, $(f\vee g)(x)=f(x)\vee g(x)$, $(f\wedge g)(x)=f(x)\wedge g(x)$, then $\mathbf{C(X)}$ becomes an l-group. With each point x of X and a neighborhood U of x there are two l-subgroups defined by, $O_x=\{f\in\mathbf{C(X)}\mid f(U)=\{0\}\}$ and $M_x=\{f\in\mathbf{C(X)}\mid f(x)=0\}$. Let $x\in X$ and a neighborhood U of x. Define $\mu:\mathbf{C(X)}\to L$ as follows:

$$\mu(f)=\begin{cases} 1, & \text{if } f=\widehat{0} \\ .75, & \text{if } f\in O_x-\{\widehat{0}\} \\ .5, & \text{if } f\in M_x-O_x \\ 0, & \text{otherwise.} \end{cases}$$

Clearly the level sets $\mu_t=\{\widehat{0}\}$ for $0.75<t\leq 1$, $\mu_t=O_x$ for $0.5<t\leq 0.75$, $\mu_t=M_x$ for $0<t\leq 0.5$ and $\mu_0=G$ are all convex l-subgroups of G. Then μ is a L-fuzzy convex sub l-group of $\mathbf{C(X)}$.

Theorem 3.8. *If λ is a L-fuzzy convex sub l-group of G, then $Supp(\lambda)=\{x\in G\mid \lambda(x)>0\}$ is a convex l-subgroup of G if $Supp(\lambda)\neq\varnothing$ and L is regular.*

Theorem 3.9. *The intersection of any non empty family of L-fuzzy convex sub l-groups of G is a L-fuzzy convex sub l-group.*

Theorem 3.10. *If λ is a L-fuzzy convex sub l-group of G, then $G_\lambda=\{x\in G\mid \lambda(x)=\lambda(0)\}$ is a convex l-subgroup of G.*

Theorem 3.11. *If A is any convex l-subgroup of G, then the L-fuzzy subset λ of G defined by*

$$\lambda(x)=\begin{cases} s & \text{if } x\in A \\ t & \text{if } x\notin A, \end{cases}$$

where $s,t\in L$ and $t<s$, is a L-fuzzy convex sub l-group of G.

Theorem 3.12. *A nonempty subset A of an l-group of G is a convex l-subgroup of G if and only if χ_A is a L-fuzzy convex sub l-group of G.*

Theorem 3.13. *Let G and G' be two l-groups. Let λ and μ be L-fuzzy convex sub l-groups of G and G' respectively. If $f:G\to G'$ be a epimorphism, then*

(1) *$f(\lambda)$ is a L-fuzzy convex sub l-group of G', provided that λ is f-invariant.*
(2) *$f^{-1}(\mu)$ is a L-fuzzy convex sub l-group of G.*

Theorem 3.14. *Let f be a homomorphism of G onto G'. If λ and μ are L-fuzzy convex sub l-groups of G, then $f(\lambda \cap \mu) = f(\lambda) \cap f(\mu)$, provided that if at least one of λ or μ is f-invariant.*

Theorem 3.15. *For any, L-fuzzy subset λ of G, there exists smallest L-fuzzy convex sub l-group of G containing λ.*

Definition 3.16. Let λ be a L-fuzzy subset of an l-group G. The smallest L-fuzzy convex sub l-group of G which contains λ is called the L-fuzzy convex sub l-group of G, generated by λ and is denoted by $\langle\lambda\rangle$.

Theorem 3.17. *Let μ be a L-fuzzy subset of an l-group G. Define $\nu : G \to L$ be a L-fuzzy subset as follows:*

$$\nu(x) = \vee\{\wedge_{y\in A}\mu(y) \mid A \subseteq G, 1 \leq |A| < \infty,\ x \in \langle A\rangle\}(x \in G),$$

where $\langle A\rangle$ denotes convex l-subgroup generated by A. Then $\nu = \langle\mu\rangle$, L-fuzzy convex sub l-group generated by μ.

Theorem 3.18. *The set $C(G)$ of all L-fuzzy convex sub l-groups of an l-group G is a complete lattice under the relation $\subseteq$. The Sup and Inf of any family $\{\lambda_i \mid i \in \Delta\}$ of L-fuzzy convex sub l-groups are $\langle\cup\{\lambda_i \mid i \in \Delta\}\rangle$ and $\cap\{\lambda_i \mid i \in \Delta\}$ respectively. The greatest and the least elements are χ_G and $\chi_\varnothing$ where, $\chi_G(x) = 1$, for all $x \in G$, $\chi_\varnothing(x) = 0$, for all $x \in G$.*

We note that $C(G)$ is a sublattice of the lattice of all L-fuzzy sub l-groups. Now, we close this section with some results of L-fuzzy points. Here $\langle x_a\rangle$ is a L-fuzzy convex sub l-group generated by L-fuzzy point x_a and $\langle x\rangle$ denotes the convex l-subgroup of G generated by x.

Lemma 3.19. *Let $x \in G$ and $a \in L - \{0\}$. Then $\langle x_a\rangle = a_{\langle x\rangle}$.*

Theorem 3.20. *Let $x, y \in G$ and $a, b \in L - \{0\}$. Then*

$$a_{\langle x\rangle} \cap b_{\langle y\rangle} = (a \wedge b)_{\langle x\rangle\cap\langle y\rangle},\ i.e.,\ \langle x_a\rangle \cap \langle y_b\rangle = (\langle x\rangle \cap \langle y\rangle)_{a\wedge b}.$$

Theorem 3.21. *If $x,\ y \in G^+$ and $a,\ b \in L - \{0\}$, then $\langle x_a\rangle \cap \langle y_b\rangle = \langle x \wedge y\rangle_{a\wedge b}$.*

Lemma 3.22. *Let $x, y \in G$ and $a, b \in L - \{0\}$. Then*
(1) $a \leq b \Rightarrow a_{\langle x\rangle} \subseteq b_{\langle x\rangle}$.
(2) $\langle x\rangle \subseteq \langle y\rangle \Rightarrow a_{\langle x\rangle} \subseteq a_{\langle y\rangle}$.
(3) $a_{\langle x\rangle} \vee b_{\langle y\rangle} \subseteq (a \vee b)_{\langle x\rangle\vee\langle y\rangle}$.

Proof. (i) and (ii) are clear.
(iii) Clearly, $a \leq a \vee b$. Then $a_{\langle x\rangle} \subseteq (a \vee b)_{\langle x\rangle} \subseteq (a \vee b)_{\langle x\rangle\vee\langle y\rangle}$.
Similarly, $b_{\langle y\rangle} \subseteq (a \vee b)_{\langle y\rangle} \subseteq (a \vee b)_{\langle x\rangle\vee\langle y\rangle}$. Thus $a_{\langle x\rangle} \vee b_{\langle y\rangle} \subseteq (a \vee b)_{\langle x\rangle\vee\langle y\rangle}$. □

Theorem 3.23. *If $x,\ y \in G^+$ and $a,\ b \in L - \{0\}$, then $\langle x_a\rangle \cup \langle y_b\rangle \leq \langle x \vee y\rangle_{a\vee b}$.*

4. L-fuzzy cosets

Definition 4.1. Let $\lambda : G \to L$ be a L-fuzzy convex sub l-group of G and $x \in G$. The L-fuzzy subset $x + \lambda : G \to L$ defined by $(x + \lambda)(y) = \lambda(-x + y)$,for all $y \in G$ is called a L-fuzzy left coset of the L-fuzzy convex sub l-group λ corresponding to x.

Definition 4.2. Let $\lambda : G \to L$ be a L-fuzzy convex sub l-group of G and $x \in G$. The L-fuzzy subset $\lambda + x : G \to L$ defined by $(\lambda + x)(y) = \lambda(y - x)$, for all $y \in G$ is called a L-fuzzy right coset of the L-fuzzy convex sub l-group λ corresponding to x.

Theorem 4.3. *If $\lambda : G \to L$ is a L-fuzzy convex sub l-group of G, then $x+\lambda = y+\lambda$ if and only if $\lambda(-x + y) = \lambda(0) = \lambda(-y + x)$.*

Theorem 4.4. *If $\lambda : G \to L$ is a L-fuzzy convex sub l-group of G, then $\lambda+x = \lambda+y$ if and only if $\lambda(y - x) = \lambda(0) = \lambda(x - y)$.*

Theorem 4.5. *Let λ be a L-fuzzy convex sub l-group of G. If $x + \lambda = y + \lambda$ i.e, $\lambda(-x + y) = \lambda(0)$, then $\lambda(x) = \lambda(y)$.*

Theorem 4.6. *Let λ be a L-fuzzy convex sub l-group of G. If $\lambda + x = \lambda + y$ i.e, $\lambda(y - x) = \lambda(0)$, then $\lambda(x) = \lambda(y)$.*

Theorem 4.7. *Let λ be a L-fuzzy convex sub l-group of G. Then, for any $x, y \in G$, $x + \lambda = y + \lambda$ if and only if $x + G_\lambda = y + G_\lambda$.*

Theorem 4.8. *Let λ be a L-fuzzy convex sub l-group of G. Then, for any $x, y \in G$, $\lambda + x = \lambda + y$ if and only if $G_\lambda + x = G_\lambda + y$.*

Definition 4.9. Let λ be a L-fuzzy convex sub l-group of G. For $x, y \in G$, define $x + \lambda \leq y + \lambda$ if and only if $x + c \leq y$ for some $c \in G_\lambda$.

Remark 4.10. (1) We note that the above definition $x + \lambda \leq y + \lambda$ is equivalent to $x \leq y + c$ for some $c \in G_\lambda$.

(2) If $x \leq y$ in G, then $x + \lambda \leq y + \lambda$ (as $x + 0 \leq y$, $0 \in G_\lambda$).

Definition 4.11. Let λ be a L-fuzzy convex sub l-group of G. For $x, y \in G$, define $\lambda + x \leq \lambda + y$ if and only if $c + x \leq y$ for some $c \in G_\lambda$.

Remark 4.12. (1) We note that the above definition $\lambda + x \leq \lambda + y$ is equivalent to $x \leq c + y$ for some $c \in G_\lambda$.

(2) If $x \leq y$ in G, then $\lambda + x \leq \lambda + y$ (as $0 + x \leq y$, $0 \in G_\lambda$).

Theorem 4.13. *The set of all L-fuzzy left cosets of λ i.e, $\frac{G}{\lambda} = \{x + \lambda \mid x \in G\}$, is a distributive lattice with the ordering mentioned in Definition 4.9.*

Proof. (I) ordering is well defined : Let $x, y, x', y' \in G$ such that

$$x + \lambda = x' + \lambda,\ y + \lambda = y' + \lambda \text{ and } x \leq y + c \text{ for some } c \in G_\lambda.$$

Then $-x + x' \in G_\lambda$, $-y + y' \in G_\lambda$. On the one hand,

$$\begin{aligned} x' &= x - x + x' \\ &\leq y + c - x + x' \\ &= y' - y' + y + c - x + x' \\ &= y' + (-(-y + y')) + c + (-x + x') \\ &= y' + s, \end{aligned}$$

where $s = (-(-y + y')) + c + (-x + x') \in G_\lambda$. Thus $x' + \lambda \leq y' + \lambda$. So the ordering $\leq$ is well defined.

(i) $\leq$ is reflexive : Let $x+\lambda \in \frac{G}{\lambda}$. Clearly, $x+0=x$ and $0 \in G_\lambda$. By definition, $x+\lambda \leq x+\lambda$, for all $x \in G$.

(ii) $\leq$ is anti-symmetric : Let $x, y \in G$ such that $x+\lambda \leq y+\lambda$ and $y+\lambda \leq x+\lambda$. Then

$$x+\lambda \leq y+\lambda \Rightarrow x+s_1 \leq y \text{ for some } s_1 \in G_\lambda$$

and

$$y+\lambda \leq x+\lambda \Rightarrow y+s_2 \leq x \text{ for some } s_2 \in G_\lambda.$$

Thus $y+s_2+s_1 \leq x+s_1 \leq y$. So $s_2+s_1 \leq -y+x+s_1 \leq 0$. Hence

$$s_2+s_1 \leq -y+x+s_1 \leq 0 \leq -(-y+x+s_1) \leq -(s_2+s_1).$$

Since λ is a L-fuzzy convex sub l-group of G, we have

$$\begin{aligned}
\lambda(-s_1-x+y) &\geq \lambda(-(s_2+s_1)) \\
&= \lambda(s_2+s_1) \\
&\geq \lambda(s_2) \wedge \lambda(s_1) \\
&= \lambda(0) \\
&\geq \lambda(-s_1-x+y).
\end{aligned}$$

Since $s_1 \in G_\lambda$, $(-s_1-x+y) \in G_\lambda$, i.e., $-x+y \in G_\lambda$. Therefore $x+\lambda = y+\lambda$.

(iii) $\leq$ Transitive : Let $x, y, z \in G$ such that $x+\lambda \leq y+\lambda$ and $y+\lambda \leq z+\lambda$. Then

$$x+s_1 \leq y \text{ and } y+s_2 \leq z \text{ for some } s_1 \in G_\lambda,\ s_2 \in G_\lambda.$$

Thus $x+s_1+s_2 \leq y+s_2 \leq z$. So $x+s \leq z$, where $s = s_1+s_2 \in G_\lambda$. Hence $x+\lambda \leq z+\lambda$ and thus $\leq$ is transitive.

Therefore, by (i), (ii) and (iii), $\leq$ is a partial order on $\frac{G}{\lambda}$.

(II) $\frac{G}{\lambda}$ is Lattice : Let $x+\lambda,\ y+\lambda \in \frac{G}{\lambda}$. We prove that $(x+\lambda)\vee(y+\lambda) = (x\vee y)+\lambda$, i.e., l.u.b of $\{x+\lambda, y+\lambda\}$.

Clearly, $x \leq x\vee y$ and $x \leq x \vee y$. Then $x+\lambda \leq x\vee y+\lambda$ and $y+\lambda \leq x\vee y+\lambda$. Thus $x\vee y+\lambda$ is an upper bound of $x+\lambda$ and $y+\lambda$. Let $z+\lambda$ be any upper bound of $x+\lambda$ and $y+\lambda$, i.e., $x+\lambda \leq z+\lambda$ and $y+\lambda \leq z+\lambda$. Then

$$x \leq z+s_1 \quad \text{and} \quad y \leq z+s_2 \text{ for some } s_1, s_2 \in G_\lambda.$$

Thus

$$x\vee y \leq (z+s_1)\vee(z+s_2) = z+(s_1\vee s_2).$$

Clearly, $s_1 \vee s_2 \in G_\lambda$ and G_λ is l-subgroup of G. So $x\vee y+\lambda \leq z+\lambda$. Hence $(x+\lambda)\vee(y+\lambda) = (x\vee y)+\lambda$.

Similarly, we can show that $(x+\lambda)\wedge(y+\lambda) = (x\wedge y)+\lambda$. Therefore $\frac{G}{\lambda}$ is a lattice.

(III) $\frac{G}{\lambda}$ distributive lattice : $\frac{G}{\lambda}$ is a distributive lattice, since l-group G is a distributive lattice. $\square$

Theorem 4.14. *The set of all L-fuzzy right cosets of λ i.e, $\frac{G}{\lambda} = \{\lambda + x \mid x \in G\}$, is a distributive lattice with the ordering mentioned in Definition 4.11.*

Theorem 4.15. *Let λ be a L-fuzzy convex sub l- group of G. Then the following are equivalent :*

(1) $x, y \in G, x \vee y = 0$ *implies* $x \in G_\lambda$ *or* $y \in G_\lambda$.
(2) $x,\ y \in G,\ x \wedge y = 0$ *implies* $x \in G_\lambda$ *or* $y \in G_\lambda$.
(3) $\frac{G}{\lambda}$ *is totally ordered.*

If λ is a L-fuzzy convex sub l-group, then G_λ is a convex l-subgroup of G. Thus the set $\frac{G}{G_\lambda}$ of left cosets of λ, is a distributive lattice.

Theorem 4.16. *Let λ be a L-fuzzy convex sub l-group of G. Then there is an order isomorphism between $\frac{G}{\lambda}$ and $\frac{G}{G_\lambda}$.*

Now we can easily observe that, the above theorem can holds for right cosets also.

Definition 4.17. Let λ be a L-fuzzy convex sub l-group of G. Then λ is called a L-fuzzy maximal convex sub l-group of G, if λ is a maximal element in the set of all non constant L-fuzzy convex sub l-groups of G under point wise partial ordering.

Now we close this section with the characterization of L-fuzzy maximal convex sub l-groups of G.

Theorem 4.18. *Let λ be a L-fuzzy subset of an l-group G. Then λ is a L-fuzzy maximal convex sub l-group of G if and only if there exist, a maximal convex l-subgroup M of G and maximal element α in L such that*

$$\lambda(x) = \begin{cases} 1, & \text{if } x \in M \\ \alpha, & \text{otherwise.} \end{cases}$$

5. L-fuzzy prime convex sub l-group

Definition 5.1. A non constant L-fuzzy convex sub l-group of an l-group G is called L-fuzzy prime convex sub l-group if and only if for any L-fuzzy convex sub l-groups μ and ν, $\mu \cap \nu \subseteq \lambda \Rightarrow$ either $\mu \subseteq \lambda$ or $\nu \subseteq \lambda$.

Clearly, L-fuzzy prime convex sub l-group of G are precisely the prime elements in the complete lattice $C(G)$ of all L-fuzzy convex sub l-group of G.

Theorem 5.2. *If λ is a L-fuzzy prime convex sub l-group of G, then $G_\lambda = \{x \in G \mid \lambda(x) = \lambda(0)\}$ is a prime convex l-subgroup of G.*

Lemma 5.3. *If λ is a L-fuzzy prime convex sub l-group of G, then $\lambda(0) = 1$.*

Now the following theorem establishes one to one correspondence between L-fuzzy prime convex sub l-groups of G and pairs (P, α), where P is a prime convex l-subgroup and α is an irreducible element in L.

Theorem 5.4. *Let λ be a L-fuzzy subset of G. Then, λ is a L-fuzzy prime convex sub l-group of G if and only if there exists a pair (P, α), where P is a prime convex l-subgroup and α is an irreducible element of L, such that*

$$\lambda(x) = \begin{cases} 1, & \text{if } x \in P \\ \alpha, & \text{otherwise.} \end{cases}$$

Corollary 5.5. *χ_P is a L-fuzzy prime convex sub l-group of G if and only if P is a prime convex l-subgroup and 0 is irreducible element in L.*

Theorem 5.6. *Let λ be a L-fuzzy prime convex sub l-group of G. If $\mu \supset \lambda$ and $\nu \supset \lambda$, where μ and ν are L-fuzzy convex sub l-groups of G, then $\mu \cap \nu \supset \lambda$.*

Theorem 5.7. *If λ is a L-fuzzy prime convex sub l-group of G, then $\frac{G}{\lambda}$ is totally ordered.*

Proof. Let λ be a L-fuzzy prime convex sub l-group of G. Then G_λ is a prime convex l-subgroup of G. Let $x+\lambda, y+\lambda \in \frac{G}{\lambda}$. We know that, in l-groups, $\frac{G}{G_\lambda}$ is totally ordered. Thus $x+G_\lambda \leq y+G_\lambda$ or $y+G_\lambda \leq x+G_\lambda$, i.e., $x+s_1 \leq y$ or $y+s_2 \leq x$, for some $s_1, s_2 \in G_\lambda$, i.e., $x+\lambda \leq y+\lambda$ or $y+\lambda \leq x+\lambda$. So $\frac{G}{\lambda}$ is totally ordered. □

Theorem 5.8. *Let λ be a L-fuzzy prime convex sub l-group of G. If $x, y \in G$ such that $x \wedge y = 0$, then $x \in G_\lambda$ or $y \in G_\lambda$.*

Theorem 5.9. *Let λ be a L-fuzzy prime convex sub l-group of G. If $x, y \in G$ such that $x \vee y = 0$, then $x \in G_\lambda$ or $y \in G_\lambda$.*

Theorem 5.10. *Let λ be a L-fuzzy prime convex sub l-group of G. Then for any $x, y \in G$, either $\lambda(x \wedge y) = \lambda(x)$ or $\lambda(x \wedge y) = \lambda(y)$.*

Proof. Let $x, y \in G$. By the above theorem, $\frac{G}{\lambda}$ is totally ordered. Then either $x+\lambda \leq y+\lambda$ or $y+\lambda \leq x+\lambda$. Thus $x \wedge y+\lambda = x+\lambda$ or $x \wedge y+\lambda = y+\lambda$. So $\lambda(-x+x \wedge y) = \lambda(0)$ or $\lambda(-y+x \wedge y) = \lambda(0)$. Hence $\lambda(x) = \lambda(x \wedge y)$ or $\lambda(y) = \lambda(x \wedge y)$. Therefore $\lambda(x \wedge y) = \lambda(x)$ or $\lambda(x \wedge y) = \lambda(y)$. □

Theorem 5.11. *Let λ be a L-fuzzy prime convex sub l-group of G. Then for any $x, y \in G$, either $\lambda(x \vee y) = \lambda(x)$ or $\lambda(x \vee y) = \lambda(y)$.*

Proof. $\lambda(x \vee y) = \lambda(-(x \vee y)) = \lambda((-x) \wedge (-y)) = \lambda(-x)$ or $\lambda(-y) = \lambda(x)$ or $\lambda(y)$. □

Theorem 5.12. *Let f be a homomorphism of G onto G'. If λ is a L-fuzzy prime convex sub l-group of G, then $f(\lambda)$ is a L-fuzzy prime convex sub l-group of G, provided that λ is f-invariant.*

Theorem 5.13. *Let f be a homomorphism of G onto G'. If μ is a L-fuzzy prime convex sub l-group of G', then $f^{-1}(\mu)$ is a L-fuzzy prime convex sub l-group of G, provided that every L-fuzzy convex sub l-group of G is f-invariant.*

From [2], we have each non zero element x of an l-group has at least one value i.e, there exists a convex l-subgroup of G which is maximal with respect to the property of not containing x (such a convex l-subgroup is called regular). Consequently each convex l-subgroup can be obtained as the intersection of regular l-subgroups. Also each regular convex l-subgroup is prime convex l-subgroup. So, each convex l-subgroup can be written as intersection of prime convex l-subgroups. Now, we have the following theorem in fuzzy setting.

Theorem 5.14. *Let G be a l-group. Let λ be a L-fuzzy convex sub l-group with $Im\lambda = \{1, c\}$, where c is irreducible element in L. Then λ can be expressed as the intersection of L-fuzzy prime convex sub l-groups of G.*

Proof. Since c is irreducible in L and $c \neq 1$. Since $Im\lambda = \{1, c\}$ we have $G_\lambda \neq G$, G_λ is a convex l-subgroup of G. G_λ is the intersection of regular convex l-subgroups of G and each regular convex l-subgroup is a prime convex l-subgroup. So, G_λ is the intersection of prime convex l-subgroups of G. So there exists a family $\{G_i\}_{i \in I}$ of prime convex l-subgroups of G such that $G_\lambda = \cap_{i \in I} G_i$.
Let

$$(\lambda_i)(x) = \begin{cases} 1, & \text{if } x \in G_i \\ c, & \text{otherwise,} \end{cases}$$

for $i \in I$. Each $\lambda_i, i \in I$ is a L-fuzzy prime convex sub l-group of G. Clearly, $\lambda = \cap_{i \in I} \lambda_i$. □

Corollary 5.15. *Each L-fuzzy maximal convex sub l-group is L-fuzzy prime convex sub l-group.*

Acknowledgements. The author is grateful to Prof. K. L. N. Swamy and Prof. P. Ranga Rao for their help and guidence in this work.

References

[1] A. Bigard, K. Keimel and S. Wolfenstein, Groupes et Anneaux Recticules, Springer-Verlag, Berlin 1977.
[2] Paul F. Conrad, Lattice ordered groups, Tulane Lecture notes, New Orleans 1970.
[3] V. N. Dixit, R. Kumar and N. Ajmal, Fuzzy ideals and fuzzy prime ideals of ring, Fuzzy sets and systems 44 (1991) 127–138.
[4] L. Fuchs, Partially ordered algebraic systems, Pergamon Press 1963.
[5] Garrett Birkhoff, Lattice Theory, American Mathematical Society colloquium publications, Volume XXV 1940.
[6] Andrew M. W. Glass and W. Charles Holland, Lattice ordered groups, D. Reidal, Dordrecht 1987.
[7] Andrew M. W. Glass , Partially ordered groups, World Scientific, London 1999.
[8] J. A. Goguen, *L*-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145–174.
[9] K. Keimel, The representation of lattice ordered groups and rings, by sections in sheaves, Lecture notes in Maht., Vol. 248, Springer, Berlin 1971.
[10] R. Kumar, Fuzzy irreducible ideals in rings, Fuzzy sets and systems 42 (1991) 360–379.
[11] R. Kumar, Fuzzy cosets and some radicals, Fuzzy sets and systems 46 (1992) 261–265.
[12] R. Kumar, Fuzzy subgroups, fuzzy ideals and fuzzy cosets; some properties, Fuzzy sets and systems 48 (1992) 267–274.
[13] R. Kumar, Fuzzy Algebra, Volume 1, University of Delhi Pulication Division 1993.
[14] D. S. Malik and J. N. Mordeson, Fuzzy prime ideals of rings, Fuzzy sets and systems 37 (1990) 93–98.
[15] D. S. Malik and J. N. Mordeson, Fuzzy maximal radical and primary ideals of a ring, Inform Sci. 53 (1991) 237–250.
[16] Marlow Anderson and Todd Feil, Lattice-ordered groups An introduction, D. Reidel Publishing Company, Kulwer Academic, Boston 1988.
[17] J. N. Mordeson and D. S. Malik , Fuzzy commutative algebra, World Scientific publishing. co. pvt. Ltd 1998.
[18] Naseem Ajmal, Fuzzy lattices, Inform Sci. 79 (1994) 271–291.
[19] A. Rosenfeld, fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[20] G. S. V. S. Saibaba, Fuzzy lattice ordered groups, Southeast Asian Bulletin of Mathematics 32 (2008) 749–766.
[21] K. L. N. Swamy and U. M. Swamy, Fuzzy Prime ideals of rings, J. Math. Anal. Appl. 134 (1988) 94–103.
[22] L. A. Zadeh, Fuzzy sets, Information and control 8 (1965) 338–353.

[23] Yue Zhang, Prime L-fuzzy ideals and Primary L-fuzzy ideals, Fuzzy sets and systems 27 (1988) 345–350.

G. S. V. Satya Saibaba (saibabagannavarapu65@yahoo.com)
Department of Mathematics, Sri. Y. N. College(A), Narsapur -534 275,W. G. Dt, A.P, India