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Soft $\mathcal{G}$-compactness in soft topological spaces

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Abstract. This paper aims to introduce, in terms of soft grill $\mathcal{G}$, the concepts soft $\mathcal{G}$-compactness, soft countably $\mathcal{G}$-compact, soft quasi $\mathcal{H}$-closeness (soft absolutely closeness), soft $\mathcal{H}$-closeness and soft $\mathcal{G}$-regular spaces in a soft topological space (X, τ, E). Several related results and properties of these concepts are investigated.

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1. Introduction

Several branches like medicine, economics, engineering, physics, social sciences, computer sciences etc are full of uncertainties. Before 1999, we had only four mathematical tools to deal with uncertainties, namely probability theory, fuzzy set theory [26], rough set theory and the theory of interval mathematics. To overcome the choice of degree of membership in fuzzy set theory when the facts are concerned with uncertainties, Molodtsov [17] introduced the concept of soft set theory in the year 1999 and investigated various applications in game theory, smoothness of functions, operation researches, Perron integration, probability theory, theory of measurement and so on. Later Maji et al. [16] defined various operations on soft sets to study some of the fundamental properties. At present; investigations of different properties and applications of soft set theory have attracted many researchers from various backgrounds. It has been used in fuzzy set theory too. In 2011, Shabir et al. [21] introduced the study of soft topological spaces. They defined basic notions of soft topological spaces such as soft open and soft closed sets, soft subspace, soft closure, soft neighborhood of a point, soft τ_i-spaces, for $i = 1, 2, 3, 4$, soft regular spaces, soft normal spaces and established their several properties. In the same year Cagman et al. [3] introduced soft topology in a different approach. In [5], Kandil et al. introduced some soft operations such as semi open soft, pre open soft, $\alpha-$open soft and $\beta-$open soft and investigated their properties in detail. Kandil et al. [12] introduced the notion of soft semi separation axioms. In particular they study the properties of the soft semi regular spaces and soft semi normal spaces. Hussain et al. [4] continued investigating the properties of soft open (closed), soft neighborhood and soft closure. Also, they defined and discussed the properties of soft interior, soft exterior and soft boundary. Also, Ahmad et al. [1] in 2012, explored the structures of soft topology using soft points. Aygunoglu et al. [2] considered the study of soft compactness within a soft topological space.

The notion of soft ideal was initiated for the first time by Kandil et al. [8]. They also introduced the concept of soft local function. These concepts are discussed with a view to find new soft topologies from the original one, called soft topological spaces with soft ideal $(X, \tau, E, \tilde{I})$. Applications to various fields were further investigated by Kandil et al.[6, 7, 9, 10, 11, 13]. In [20] author proposed the notion of soft grill $\mathcal{G}$ in soft topological spaces and gave a soft topology $\tau_{\mathcal{G}}$ ($\mathcal{G}$-soft topological space) over X with the same parameters E that is finer than τ which is extended in [18]. A topological space (X, τ) is absolutely closed space (which is known quasi $\mathcal{H}$-closed)[22], if for any open cover $F = \{U_i \mid i \in \Gamma\}$ of X, then there exist $i_1, i_2 ..., i_n$ such that $X = \cup_{i=1}^{n} cl(U_i)$. Also, a topological space (X, τ) is a $\mathcal{H}$-closed space [23], if it is absolutely closed and Hausdorff space.

In this paper, by using basic properties of soft topology τ and soft grill $\mathcal{G}$, the concepts soft $\mathcal{G}$-compactness, soft countably $\mathcal{G}$-compact and soft $\mathcal{G}$-regular spaces will be introduced. Furthermore, we generalize the concepts quasi $\mathcal{H}$-closeness(absolutely closeness) and $\mathcal{H}$-closeness into soft quasi $\mathcal{H}$ (soft absolutely closeness)-closeness, soft $\mathcal{H}$-closeness respectively. Some related results and properties of these concepts will be investigated.

2. Preliminaries

In this section we discuss some basic definitions and notions of soft set theory those are defined by various authors.

Definition 2.1 ([17])**.** Let X be an initial universe and E be a set of parameters. Let $\mathcal{P}(X)$ denote the power set of X. A pair (F, E) denoted by F is called a soft set over X, where F is a mapping given by $F : E \to \mathcal{P}(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X. For $e \in E$, $F(e)$ may be considered as the set of $e-$approximate elements of the soft set (F, E) i.e $(F, E) = \{F(e) : e \in E,\ F : E \to \mathcal{P}(X)\}$. If $e \notin E$, then $F(e) = \varnothing$. The set of all soft sets over X will be denoted by $S_E(X)$.

Definition 2.2 ([3])**.** Let $F, G \in S_E(X)$. Then

(i) F is said to be a null soft set, denoted by $\tilde{\varnothing}$, if $F(e) = \varnothing$ for all $e \in E$.

(ii) F is said to be absolute soft set, denoted by $\tilde{X}$, if $F(e) = X$ for all $e \in E$.

(iii) F is soft subset of G, denoted by $F \sqsubseteq G$, if $F(e) \subseteq G(e)$ for all $e \in E$.

(iv) F and G are soft equal, denoted by $F = G$, if $F \sqsubseteq G$ and $G \sqsubseteq F$.

(v) The soft union of F and G, denoted by $F \sqcup G$, is a soft set over X and defined by $F \sqcup G : E \to \mathcal{P}(X)$ such that $(F \sqcup G)(e) = F(e) \cup G(e)$ for all $e \in E$.

(vi) The soft intersection of F and G, denoted by $F \sqcap G$, is a soft set over X and defined by $F \sqcap G : E \to \mathcal{P}(X)$ such that $(F \sqcap G)(e) = F(e) \cap G(e)$ for all $e \in E$.

(vii) The soft complement $(\tilde{X} - F)$ of a soft set F is denoted by $F^{\tilde{c}}$ and defined by $F^{\tilde{c}} : E \to \mathcal{P}(X)$ such that $F^{\tilde{c}}(e) = X \setminus F(e)$ for all $e \in E$. Clearly, $(\tilde{X})^{\tilde{c}} = \tilde{\varnothing}$ and $(\tilde{\varnothing})^{\tilde{c}} = \tilde{X}$.

Definition 2.3 ([21])**.** Let F be soft set over X and $x \in X$. Then
(i) $x \tilde{\in} F$ whenever $x \in F(e)$ for all $e \in E$.
(ii) $x \tilde{\notin} F$, if $x \notin F(e)$ for some $e \in E$.

Definition 2.4 ([21])**.** A soft set F over X where $F(e) = \{x\}$, $\forall e \in E$ is called singleton soft point and denoted by x_E or (x, E).

Definition 2.5 ([27])**.** Let Γ be an arbitrary indexed set and $\Upsilon = \{F_i \mid i \in \Gamma\}$ be a subfamily of $S_E(X)$. Then

(i) The soft union of Υ is the soft set H, where $H(e) = \cup\{F_i(e) \mid i \in \Gamma\}$ for each $e \in E$. We write $H = \sqcup_{i\in\Gamma} F_i$.

(ii) The soft intersection of Υ is the soft set K, where $K(e) = \cap\{F_i(e) \mid i \in \Gamma\}$ for each $e \in E$. We write $K = \sqcap_{i\in\Gamma} F_i$.

Definition 2.6 ([21])**.** Let τ be a collection of soft sets over a universe X with a fixed set of parameters E, then $\tau \sqsubseteq S_E(X)$ is called a soft topology on X if
(i) $\tilde{X}, \tilde{\varnothing} \in \tau$, where $\tilde{\varnothing}(e) = \varnothing$ and $\tilde{X}(e) = X$, for each $e \in E$.
(ii) The soft union of any number of soft sets in τ belongs to τ.
(iii) The soft intersection of any two soft sets in τ belongs to τ. The triplet (X, τ, E) is called a soft topological space over X. A soft set F over X is said to be a soft open set in X if $F \in \tau$, and it is called a soft closed set in X, if its relative complement $F^{\tilde{c}}$ is a soft open set and we write F is a soft τ-closed [4]. A soft set is said to be soft clopen, if it is both soft open and soft closed.

Definition 2.7. [21] A soft set F of a soft topological space (X, τ, E) is called a soft neighborhood (abbreviated as a soft nbd.) of the soft set H if there exist a soft open set K such that $H \sqsubseteq K \sqsubseteq F$. If $H = x$, then F is called a soft nbd. of the soft point x. The soft neighborhood system of a soft point x, denoted by $\mathcal{N}(x)$, is the family of all its neighborhoods.

Definition 2.8 ([21])**.** Let (X, τ, E) be a soft topological space over X and $F \in S_E(X)$. Then the soft interior, soft closure and soft boundary of F, denoted by $int(F)$, $cl(F)$ and $bd(F)$ respectively, are defined as :

$$\begin{aligned} int(F) &= \sqcup\big\{G \mid G \text{ is soft open set and } G \sqsubseteq F\big\}, \\ cl(F) &= \sqcap\big\{H \mid H \text{ is soft closed set and } F \sqsubseteq H\big\}, \\ bd(F) &= cl(F) \sqcap cl(\tilde{X} - F). \end{aligned}$$

Definition 2.9 ([21])**.** Let (X, τ, E) be a soft topological space and $x, y \in X$ such that $x \neq y$. (X, τ, E) is called soft Hausdorff space or soft τ_2-space if there exist soft τ-open sets F and G such that $x \in F$, $y \in G$ and $F \sqcap G = \tilde{\varnothing}$.

Definition 2.10 ([2]). A family $\eta = \{U_i \mid i \in \Gamma\}$ of soft sets is a cover of a soft set F, if $F \sqsubseteq \sqcup_{i\in\Gamma} U_i$. It is a soft open cover, if each member of η is a soft open set. A subcover of η is a subfamily of η that is also a cover.

Definition 2.11 ([27]). A family η of soft sets has the finite intersection property, if the intersection of the members of each finite subfamily of η is not null soft set.

Definition 2.12 ([2]). A soft topological space (X, τ, E) is soft compact if each cover of $\tilde{X}$ by soft open sets has a finite subcover.

Theorem 2.1 ([24]). *Let (X, τ, E) be a soft Hausdorff space. If F is soft compact on X, then it is soft closed.*

Theorem 2.2 ([27]). *A soft topological space is soft compact if and only if each family of soft closed sets with the finite intersection property has a non null intersection.*

Definition 2.13 ([25]). A soft topological space (X, τ, E) is called soft lindelöf, if each cover of $\tilde{X}$ by soft open sets has a countable subcover.

Definition 2.14 ([14]). A soft set F is called a soft generalized closed (briefly, soft g-closed) in a soft topological space (X, τ, E), if $cl(F) \sqsubseteq H$ whenever $F \sqsubseteq H$ and H is a soft τ-open set.

Definition 2.15 ([19]). A soft set F is called a soft dense in a soft topological space (X, τ, E), if $cl(F) = \tilde{X}$.

Definition 2.16 ([21]). Let F be a soft set on soft topological space (X, τ, E) and $\tilde{Y}$ be a nonempty subset of $\tilde{X}$. Then, the sub soft set of F over $\tilde{Y}$ denoted by ${}^YF$ is defined as ${}^YF(e){=}Y \cap F(e)$, for each $e{\in}A$. In other word ${}^YF{=}\tilde{Y} \sqcap F$.

Definition 2.17 ([21]). Let (X, τ, E) be a soft topological space and $\tilde{Y}$ be a nonempty subset of $\tilde{X}$. Then, the collection of $\tau_Y = \{{}^YF \mid F \in \tau\}$ is said to be the soft relative topology on $\tilde{Y}$ and (Y, τ_Y, E) is called a soft subspace of (X, τ, E).

Proposition 2.1 ([21]). *Let (Y, τ_Y, E) be a soft subspace of a soft topological space (X, τ, E) and F be a soft open set in $\tilde{Y}$. If $\tilde{Y} \in \tau$, then $F \in \tau$.*

Theorem 2.3 ([21]). *Let (Y, τ_Y, E) be a soft subspace of a soft topological space (X, τ, E) and F be a soft set over X. Then*
(1) *F is a soft open set in $\tilde{Y}$ if and only if $F = \tilde{Y} \sqcap G$ for some $G \in \tau$.*
(2) *F is a soft closed in $\tilde{Y}$ if and only if $F = \tilde{Y} \sqcap G$ for some soft τ-closed set G.*

Definition 2.18 ([15]). Let $S_E(X)$ and $S_K(Y)$ be families of soft sets, $u : X \to Y$ and $p : E \to K$ be mappings. Then $f_{pu} : S_E(X) \to S_K(Y)$ is called a soft function.
(i) If $F \in S_E(X)$, then the image of F under f_{pu}, written as $f_{pu}(F)$, is a soft set in $S_K(Y)$ such that

$$f_{pu}(F)(k) = \begin{cases} \bigcup_{e\in p^{-1}(k)} u(F(e)), & p^{-1}(k) \neq \varnothing \\ \varnothing, & \text{otherwise.} \end{cases}$$

for each $k{\in}Y$.

(ii) If $G\tilde{\in}S_K(Y)$, then the inverse image of G under f_{pu}, written as $f_{pu}^{-1}(G)$, is a soft set in $S_E(X)$ such that

$$f_{pu}^{-1}(G)(e) = \begin{cases} u^{-1}(G(p(e))), & p(e)\in Y \\ \varnothing, & \text{otherwise.} \end{cases}$$

for each $e\in E$.
The soft function f_{pu} is called surjective if p and u are surjective, also is said to be injective if p and u are injective.

Several properties and characteristics for f_{pu} and f_{pu}^{-1} are introduced in detail in [15].

Definition 2.19 ([27])**.** Let (X, τ_1, E) and (Y, τ_2, K) be soft topological spaces and $f_{pu} : S_E(X) \to S_K(Y)$ be a function. Then, the function f_{pu} is called:
(i) Continuous soft if $f_{pu}^{-1}(G) \in \tau_1$ for each $G \in \tau_2$.
(ii) Open soft if $f_{pu}(F) \in \tau_2$ for each $F \in \tau_1$.

Definition 2.20 ([20])**.** A nonempty collection $\mathcal{G}$ of soft sets over X is called a soft grill, if the following conditions hold :
(i) $\tilde{\varnothing} \notin \mathcal{G}$.
(ii) If $F \in \mathcal{G}$ and $F \sqsubseteq H$, then $H \in \mathcal{G}$.
(iii) If $F \sqcup H \in \mathcal{G}$, then $F \in \mathcal{G}$ or $H \in \mathcal{G}$.

Definition 2.21 ([20])**.** Let $\mathcal{G}$ be a soft grill over a soft topological space (X, τ, E). Consider the self-map on $S_E(X)$ where for every soft set F, $\phi_{\mathcal{G}}(F) = \sqcup\{x \mid U \sqcap F \in \mathcal{G}$ for every soft open nbd. U of $x\}$, then $\phi_{\mathcal{S}}()$ is a soft operator. Define another soft operator $\psi_{\mathcal{G}} : S_E(X) \longrightarrow S_E(X)$ by $\psi_{\mathcal{G}}(F) = F \sqcup \phi_{\mathcal{G}}(F)$ for every $F \in S_E(X)$. Then, the soft operator $\psi_{\mathcal{G}}$ is a kuratowskiś soft closure operator and hence gives rise to a new soft topology $\tau_{\mathcal{G}} = \{H \mid \psi_{\mathcal{G}}(\tilde{X} - H) = \tilde{X} - H\}$ over X with the same parameters E, ($\mathcal{G}$-soft topological space) such that $\psi_{\mathcal{G}}(F) = \tau_{\mathcal{G}}\text{-}cl(F)$, which is a soft finer than τ in general. A soft open base $\beta(\mathcal{G}, \tau)$ for the soft topology $\tau_{\mathcal{G}}$ on $\tilde{X}$ is given by $\beta(\mathcal{G}, \tau) = \{(V - F) \mid V \in \tau, F \notin \mathcal{G}\}$ and $\tau \sqsubseteq \beta(\mathcal{G}, \tau) \sqsubseteq \tau_{\mathcal{G}}$.

Proposition 2.2 ([20])**.** *Let $\mathcal{G}$ be a soft grill over a soft topological space (X, τ, E). Then, for every $F \in S_E(X)$, the following statements hold :*

(1) *If $F \notin \mathcal{G}$, then $\phi_{\mathcal{G}}(F) = \tilde{\varnothing}$. Moreover, $\phi_{\mathcal{G}}(\tilde{\varnothing}) = \tilde{\varnothing}$.*
(2) *$\phi_{\mathcal{G}}(\phi_{\mathcal{G}}(F)) \sqsubseteq \phi_{\mathcal{G}}(F) = cl(\phi_{\mathcal{G}}(F)) \sqsubseteq cl(F)$. Moreover, $\phi_{\mathcal{G}}(F)$ is soft τ-closed.*
(3) *If F is a soft τ-closed, then $\phi_{\mathcal{G}}(F) \sqsubseteq F$. Moreover, $\psi_{\mathcal{G}}(F) = F$.*
(4) *A soft set F is $\tau_{\mathcal{G}}$-closed if and only if $\phi_{\mathcal{G}}(F) \sqsubseteq F$.*

Proposition 2.3. [20] *Let $\mathcal{G}$ be a soft grill over a soft topological space (X, τ, E). Then, for every $F \in S_E(X)$, the following statements hold :*

(1) *$F \sqsubseteq H$ implies $\phi_{\mathcal{G}}(F) \sqsubseteq \phi_{\mathcal{G}}(H)$.*
(2) *$\phi_{\mathcal{G}}(F \sqcup H) = \phi_{\mathcal{G}}(F) \sqcup \phi_{\mathcal{G}}(H), \phi_{\mathcal{G}}(F \sqcap H) \sqsubseteq \phi_{\mathcal{G}}(F) \sqcap \phi_{\mathcal{G}}(H)$.*
(3) *$\phi_{\mathcal{G}}(F) - \phi_{\mathcal{G}}(H) = \phi_{\mathcal{G}}(F - H) - \phi_{\mathcal{G}}(H)$.*
(4) *If $H \notin \mathcal{G}$, then $\phi_{\mathcal{G}}(F \sqcup H) = \phi_{\mathcal{G}}(F) = \phi_{\mathcal{G}}(F - H)$.*

Proposition 2.4 ([20])**.** *If $\mathcal{G}$ is a soft grill on a soft topological space (X, τ, E) with $(\tau - \{\tilde{\varnothing}\}) \sqsubseteq \mathcal{G}$. Then*

(1) *For any soft open set* $H, H \sqsubseteq \phi_{\mathcal{G}}(H)$.
(2) $\phi_{\mathcal{G}}(\tilde{X}) = \tilde{X}$.

Theorem 2.4 ([20]). *Let* $\mathcal{G}$ *be a soft grill on a soft topological space* (X, τ, E) *and* $F \in S_E(X)$ *such that* $F \sqsubseteq \phi_{\mathcal{G}}(F)$. *Then* $cl(F) = \psi_{\mathcal{G}}(F) = \tau_{\mathcal{G}}\text{-}cl(F) = cl(\phi_{\mathcal{G}}(F)) = \phi_{\mathcal{G}}(F)$.

Theorem 2.5 ([20]). *Let* $\mathcal{G}$ *be a soft grill on a soft topological space* (X, τ, E). *If* H *is soft* τ*-open, then* $H \sqcap \phi_{\mathcal{G}}(F) = H \sqcap \phi_{\mathcal{G}}(H \sqcap F)$, *for every soft set* F.

Proposition 2.5 ([20]). *Let* $\mathcal{G}_1, \mathcal{G}_2$ *be two soft grills over a soft topological space* (X, τ, E) *and* $F \in S_E(X)$. *If* $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$, *then* $\phi_{\mathcal{G}_1}(F) \sqsubseteq \phi_{\mathcal{G}_2}(F)$. *Moreover* $\tau_{\mathcal{G}_2} \sqsubseteq \tau_{\mathcal{G}_1}$.

Proposition 2.6 ([20]). *Let* (X, τ, E) *be a soft topological space and* $F \in S_E(X)$. *If* $\mathcal{G} = (\mathcal{P}(X) - \{\tilde{\varnothing}\})$*, then* $\phi_{\mathcal{G}}(F) = F$. *Also* $\psi_{\mathcal{G}}(F) = F$.

3. Soft $\mathcal{G}$-compactness

Definition 3.1. Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E). Then

(i) A soft set F is called soft $\mathcal{G}$-compact, if for every cover $\{U_i \in \tau \mid i \in \Gamma\}$ of F by soft open sets; there exist a finite subset Γ_0 of Γ such that $(F - \sqcup_{i \in \Gamma_0} U_i) \notin \mathcal{G}$.

(ii) A space (X, τ, E) is called soft $\mathcal{G}$-compact, if $\tilde{X}$ is a soft $\mathcal{G}$-compact as a soft subset. In other words, the space (X, τ, E) is called soft $\mathcal{G}$-compact, if every cover $\{U_i \in \tau \mid i \in \Gamma\}$ of $\tilde{X}$ by soft τ-open sets; there exist a finite subset Γ_0 of Γ such that $(\tilde{X} - \sqcup_{i \in \Gamma_0} U_i) \notin \mathcal{G}$.

Proposition 3.1. (1) *Soft compact topological space* (X, τ, E) *is a soft* $\mathcal{G}$*-compact, for any soft grill* $\mathcal{G}$ *on* $\tilde{X}$.

(2) *A soft topological space* (X, τ, E) *is a soft compact if and only if it is soft* $\mathcal{G}$*-compact, if* $\mathcal{G} = (\mathcal{P}(X) - \{\tilde{\varnothing}\})$.

Proof. (1) Let $\{U_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ-open sets. Then $\tilde{X} = \sqcup_{i \in \Gamma} U_i$. Since (X, τ, E) is a soft compact, there exist a finite subset Γ_0 of Γ such that $\tilde{X} = \sqcup_{i \in \Gamma_0} U_i$. Thus, $\tilde{X} - \sqcup_{i \in \Gamma_0} U_i = \tilde{\varnothing} \notin \mathcal{G}$. So, $\tilde{X}$ is a soft $\mathcal{G}$-compact.

(2) Let $\{U_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ-open sets. Then, $\tilde{X} = \sqcup_{i \in \Gamma} U_i$. Since (X, τ, E) is soft $\mathcal{G}$-compact, there exist a finite subset Γ_0 of Γ such that $\tilde{X} - \sqcup_{i \in \Gamma_0} U_i \notin \mathcal{G}$. Since $\mathcal{G} = (\mathcal{P}(X) - \{\tilde{\varnothing}\})$, $\tilde{X} - \sqcup_{i \in \Gamma_0} U_i = \tilde{\varnothing}$. So $\tilde{X} = \sqcup_{i \in \Gamma_0} U_i$. Hence, (X, τ, E) is a soft compact space. The other hand follows directly from (1). ☐

Theorem 3.1. *Soft* g*-closed subset of a soft* $\mathcal{G}$*-compact space* (X, τ, E) *with a soft grill* $\mathcal{G}$ *is a soft* $\mathcal{G}$*-compact.*

Proof. Let F be a soft g-closed set and let $\{U_i \mid i \in \Gamma\}$ be a cover of F by soft τ-open sets. Then $F \sqsubseteq \sqcup_{i \in \Gamma} U_i$. Since F is a soft g-closed, $cl(F) \sqsubseteq \sqcup_{i \in \Gamma} U_i$. Now $\{U_i \mid i \in \Gamma\} \sqcup \{\tilde{X} - cl(F)\}$ is a cover of $\tilde{X}$ by soft τ-open sets. Since (X, τ, E) is a soft $\mathcal{G}$-compact space, there exist a finite subset Γ_0 of Γ such that

$$\tilde{X} - [\sqcup_{i \in \Gamma_0} U_i \sqcup (\tilde{X} - cl(F))] \notin \mathcal{G}.$$

Furthermore,

$$F - \sqcup_{i \in \Gamma_0} U_i = (\tilde{X} - [\sqcup_{i \in \Gamma_0} U_i \sqcup (\tilde{X} - cl(F))]) \sqcap F \notin \mathcal{G}.$$

So, F is a soft $\mathcal{G}$-compact. □

Corollary 3.1. *Soft closed subset of a soft $\mathcal{G}$-compact space is a soft $\mathcal{G}$-compact.*

Proof. Follows directly from the fact that, every soft closed set in a soft topological space (X, τ, E) is soft g-closed. □

Theorem 3.2. *In a soft topological space (X, τ, E) with soft grill $\mathcal{G}$, if F is a soft $\mathcal{G}$-compact subset and K is a soft open set contained in F, then $(F - K)$ is a soft $\mathcal{G}$-compact set.*

Proof. Let $\{U_i \mid i \in \Gamma\}$ be a cover of $(F - K)$ by soft τ-open sets. Then, $(F - K) \sqsubseteq \sqcup_{i\in\Gamma} U_i$. Since $K \sqsubseteq F$ and K is a soft open set, $F \sqsubseteq \sqcup_{i\in\Gamma} U_i \sqcup K$. Since F is a soft $\mathcal{G}$-compact, there exist a finite subset Γ_0 of Γ such that $F - [\sqcup_{i\in\Gamma_0} U_i \sqcup K] \notin \mathcal{G}$. Thus, $[(F - K) - \sqcup_{i\in\Gamma_0} U_i] \notin \mathcal{G}$. So $(F - K)$ is a soft $\mathcal{G}$-compact. □

Corollary 3.2. *Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E). If F is a soft $\mathcal{G}$-compact subset and K is a soft closed set of (X, τ, E), then $(F \sqcap K)$ is a soft $\mathcal{G}$-compact.*

Proof. Let F be a soft $\mathcal{G}$-compact subset and K be a soft closed subsets of (X, τ, E). Then $(\tilde{X} - K)$ is a soft open set. By using Theorem 3.2, $(F \sqcap K) = (F - (\tilde{X} - K))$ is a soft $\mathcal{G}$-compact set. □

Theorem 3.3. *Let $\mathcal{G}$ be a soft grill on a soft Hausdorff space (X, τ, E). Then every soft $\mathcal{G}$-compact subset over X is a soft $\tau_{\mathcal{G}}$-closed.*

Proof. Let $x \tilde{\in} (\tilde{X} - F)$. Then, for each soft point $y \tilde{\in} F$, $x \neq y$. Since (X, τ, E) is a soft Hausdroff space, there exist soft open sets U_y and V_y such that $x \tilde{\in} U_y, y \tilde{\in} V_y$ and $U_y \sqcap V_y = \tilde{\varnothing}$. Thus, it is obvious that $x \tilde{\notin} cl(V_y)$, for $y \tilde{\in} F$. Now $\{V_y \mid y \tilde{\in} F\}$ is a cover of F by soft τ-open sets. Since F is a soft $\mathcal{G}$-compact, there exist $(i = 1, 2, ..., n)$ such that $F - \sqcup_{i=1}^n V_{yi} \notin \mathcal{G}$. So, $x \tilde{\notin} cl(V_{yi})$, $(i = 1, 2, ..., n)$ implies that $x \tilde{\notin} \sqcup_{i=1}^n cl(V_{yi}) = cl(\sqcup_{i=1}^n V_{yi})$. Let $M = \tilde{X} - cl(\sqcup_{i=1}^n V_{yi})$ and let $K = (F - cl(\sqcup_{i=1}^n V_{yi})) \sqsubseteq (F - \sqcup_{i=1}^n V_{yi}) \notin \mathcal{G}$. Hence, $(M - K) \in \beta(\mathcal{G}, \tau)$, where M is a soft τ-open and $K \notin \mathcal{G}$, and $(M - K) \sqsubseteq (\tilde{X} - F)$. Therefore, $(\tilde{X} - F)$ is $\tau_{\mathcal{G}}$-open and thus F is a soft $\tau_{\mathcal{G}}$-closed. □

Theorem 3.4. *The soft union of two soft $\mathcal{G}$-compact sets over X is a soft $\mathcal{G}$-compact.*

Proof. Let F and H be two soft $\mathcal{G}$-compact sets and let $\{U_i \mid i \in \Gamma\}$ be a cover of $F \sqcup H$ by soft τ-open sets. Then $\{U_i \mid i \in \Gamma\}$ is a cover of F and H by soft τ-open sets. Since F and H are soft $\mathcal{G}$-compact, there exist finite subsets Γ_0 and Γ_1 of Γ such that

$$F - \sqcup_{ir\in\Gamma_0} U_{ir} \notin \mathcal{G}$$

and

$$H - \sqcup_{ij\in\Gamma_1} U_{ij} \notin \mathcal{G}.$$

Thus,

$$(F - \sqcup_{ir\in\Gamma_0} U_{ir}) \sqcup (H - \sqcup_{ij\in\Gamma_1} U_{ij}) \notin \mathcal{G}.$$

So,

$$F = \sqcup_{ir\in\Gamma_0} U_{ir} \sqcup [F - \sqcup_{ir\in\Gamma_0} U_{ir}]$$

and

$$H = \sqcup_{ij\in\Gamma_1} U_{ij} \sqcup [H - \sqcup_{ij\in\Gamma_1} U_{ij}].$$

On the other hand,

$$F \sqcup H = \sqcup_{ir\in\Gamma_0} U_{ir} \sqcup [F - \sqcup_{ir\in\Gamma_0} U_{ir}] \sqcup \sqcup_{ij\in\Gamma_1} U_{ij} \sqcup [H - \sqcup_{ij\in\Gamma_1} U_{ij}].$$

Hence,

$$F \sqcup H = \sqcup_{ir\in\Gamma_0} U_{ir} \sqcup \sqcup_{ij\in\Gamma_1} U_{ij} \sqcup [F - \sqcup_{ir\in\Gamma_0} U_{ir}] \sqcup [H - \sqcup_{ij\in\Gamma_1} U_{ij}].$$

This implies,

$$F \sqcup H = \sqcup[U_{ir} \sqcup U_{ij} \mid ir \in \Gamma_0 \text{ and } ij \in \Gamma_1] \sqcup [F - \sqcup_{ir\in\Gamma_0} U_{ir}] \sqcup [H - \sqcup_{ij\in\Gamma_1} U_{ij}].$$

Consequently,

$$\begin{aligned} &(F \sqcup H) - \sqcup[U_{ir} \sqcup U_{ij} \mid ir \in \Gamma_0 \text{ and } ij \in \Gamma_1] \\ &= ([F - \sqcup_{ir\in\Gamma_0} U_{ir}] \sqcup [H - \sqcup_{ij\in\Gamma_1} U_{ij}] - \sqcup[U_{ir} \sqcup U_{ij} \mid ir \in \Gamma_0 \text{ and } ij \in \Gamma_1]) \notin \mathcal{G}. \end{aligned}$$

Which implies that $F \sqcup H$ is a soft $\mathcal{G}$-compact set. □

Corollary 3.3. *The finite soft union of soft $\mathcal{G}$-compact sets over X is a soft $\mathcal{G}$-compact.*

Proof. Let $F_1, F_2, ..., F_n$ be finite soft $\mathcal{G}$-compact sets over X and let $\{U_{\alpha i} \mid \alpha i \in \Gamma, \alpha = 1, 2, ..., n\}$ be a cover of $\sqcup_{\alpha=1}^{n} F_\alpha$ by soft τ-open sets. Then, $\{U_{\alpha i} \mid \alpha i \in \Gamma, \alpha = 1, 2, ..., n\}$ is a cover of F_α for each $\alpha = 1, 2, ..., n$ by soft τ-open sets. Thus, for each $\alpha = 1, 2, ..., n$, there exist finite subsets Γ_α of Γ such that $F_\alpha - \sqcup_{\alpha i\in\Gamma_\alpha} U_{\alpha i} \notin \mathcal{G}$. So, $\sqcup_{\alpha=1}^{n}(F_\alpha - \sqcup_{\alpha i\in\Gamma_\alpha} U_{\alpha i}) \notin \mathcal{G}$. Hence, $F_\alpha = \sqcup_{\alpha i\in\Gamma_\alpha} U_{\alpha i} \sqcup (F_\alpha - \sqcup_{\alpha i\in\Gamma_\alpha} U_{\alpha i})$, for each $\alpha = 1, 2, ..., n$ and thus

$$\sqcup_{\alpha=1}^{n} F_\alpha = (\sqcup_{\alpha=1}^{n} \sqcup_{\alpha i\in\Gamma_\alpha} U_{\alpha i}) \sqcup (\sqcup_{\alpha=1}^{n}(F_\alpha - \sqcup_{\alpha i\in\Gamma_\alpha} U_{\alpha i})).$$

Therefore,

$$\sqcup_{\alpha=1}^{n} F_\alpha - (\sqcup_{\alpha=1}^{n} \sqcup_{\alpha i\in\Gamma_\alpha} U_{\alpha i}) = (\sqcup_{\alpha=1}^{n}(F_\alpha - \sqcup_{\alpha i\in\Gamma_\alpha} U_{\alpha i})) - (\sqcup_{\alpha=1}^{n} \sqcup_{\alpha i\in\Gamma_\alpha} U_{\alpha i}) \notin \mathcal{G}.$$

□

Theorem 3.5. *Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E), then the following statements are equivalent :*

(1) *(X, τ, E) is soft $\mathcal{G}$-compact.*

(2) *$(X, \tau_{\mathcal{G}}, E)$ is soft $\mathcal{G}$-compact.*

(3) *For any family $\{M_i \mid i \in \Gamma\}$ of soft τ-closed sets of X such that $\sqcap\{M_i \mid i \in \Gamma_0\} = \tilde{\varnothing}$, there exist a finite subset Γ_0 of Γ such that $\sqcap\{M_i \mid i \in \Gamma_0\} \notin \mathcal{G}$.*

Proof. (1) $\Longrightarrow$ (2): Let $\{U_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft $\tau_{\mathcal{G}}$-open sets. Then, there exist soft τ-open sets V_i and $F_i \notin \mathcal{G}$ such that $U_i = [V_i - F_i]$, for each $i \in \Gamma$. Thus, $\{V_i \mid i \in \Gamma\}$ is a cover of $\tilde{X}$ by soft τ-open sets. Since (X, τ, E) is a soft $\mathcal{G}$-compact, then there exist a finite subset Γ_0 of Γ such that $(\tilde{X} - \sqcup_{i\in\Gamma_0} V_i) \notin \mathcal{G}$. Since $F_i \notin \mathcal{G}$ for every $i \in \Gamma_0$,

$$\tilde{X} - \sqcup_{i\in\Gamma_0} U_i = [\tilde{X} - \sqcup_{i\in\Gamma_0}(V_i - F_i)] \sqsubseteq [\tilde{X} - \sqcup_{i\in\Gamma_0} V_i] \sqcup [\sqcup_{i\in\Gamma_0} F_i] \notin \mathcal{G}.$$

So, $(X, \tau_{\mathcal{G}}, E)$ is soft $\mathcal{G}$-compact.

(2) $\Longrightarrow$ (1): Follows directly from $\tau \sqsubseteq \tau_{\mathcal{G}}$.

(1) $\Longrightarrow$ (3): Let $\{M_i \mid i \in \Gamma\}$ be a family of soft τ-closed sets over X such that $\sqcap\{M_i \mid i \in \Gamma\} = \tilde{\varnothing}$. Then, $\{\tilde{X} - M_i \mid i \in \Gamma\}$ is a cover of $\tilde{X}$ by soft τ-open sets. Since (X, τ, E) is a soft $\mathcal{G}$-compact, then there exist a finite subset Γ_0 of Γ such that $\tilde{X} - \sqcup\{\tilde{X} - M_i \mid i \in \Gamma_0\} \notin \mathcal{G}$. This implies that, $\sqcap\{M_i \mid i \in \Gamma_0\} \notin \mathcal{G}$.

(3) $\Longrightarrow$ (1) Let $\{U_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ-open sets. Then, $\{\tilde{X} - U_i \mid i \in \Gamma\}$ is a collection of soft τ-closed sets and $\sqcap\{\tilde{X} - U_i \mid i \in \Gamma\} = \tilde{\varnothing}$. Thus, there exist a finite subset Γ_0 of Γ such that $\sqcap\{\tilde{X} - U_i \mid i \in \Gamma_0\} \notin \mathcal{G}$. This implies that, $\tilde{X} - \sqcup\{U_i \mid i \in \Gamma_0\} \notin \mathcal{G}$. This shows (X, τ, E) is a soft $\mathcal{G}$-compact. □

Theorem 3.6. *Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E) and $(X, \tau_{\mathcal{G}}, E)$ be its $\mathcal{G}$-soft topological space. Then, the following implications hold:*

$(X, \tau_{\mathcal{G}}, E)$ is a soft compact space	$\Longrightarrow$	*(X, τ, E) is soft compact space*
$\Downarrow$		$\Downarrow$
$(X, \tau_{\mathcal{G}}, E)$ is a soft $\mathcal{G}$-compact space	$\Longrightarrow$	*(X, τ, E) is soft $\mathcal{G}$-compact space.*

Proof. Follows immediately from Definitions 2.12, 2.21 and 3.1. □

Theorem 3.7. *Let $\mathcal{G}$ be a soft grill over soft topological spaces $(X, \tau_1, E), (X, \tau_2, E)$. If (X, τ_2, E) is a soft $\mathcal{G}$-compact space and $\tau_1 \sqsubseteq \tau_2$, then (X, τ_1, E) is a soft $\mathcal{G}$-compact.*

Proof. Let $\{U_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ_1-open sets and let $\tau_1 \sqsubseteq \tau_2$. Then, $\{U_i \mid i \in \Gamma\}$ is a cover of $\tilde{X}$ by soft τ_2-open sets. Since (X, τ_2, E) is a soft $\mathcal{G}$-compact space, there exist a finite subset Γ_0 of Γ such that $(F - \sqcup_{i\in\Gamma_0} U_i) \notin \mathcal{G}$. Thus, (X, τ_1, E) is a soft $\mathcal{G}$-compact. □

Theorem 3.8. *Let (X, τ, E) be a soft topological space with two soft grills $\mathcal{G}_1, \mathcal{G}_2$ over $\tilde{X}$. If (X, τ, E) is a soft $\mathcal{G}_2$-compact and $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$, then (X, τ, E) is soft $\mathcal{G}_1$-compact space.*

Proof. Let $\{U_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ-open sets and let (X, τ, E) be a soft $\mathcal{G}_2$-compact. Then, there exist a finite subset Γ_0 of Γ such that $(F - \sqcup_{i\in\Gamma_0} U_i) \notin \mathcal{G}_2$. Since $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$, there exist a finite subset Γ_0 of Γ such that $(F - \sqcup_{i\in\Gamma_0} U_i) \notin \mathcal{G}_1$. Thus, (X, τ, E) is soft $\mathcal{G}_1$-compact space. □

Remark 3.1. If there are two soft grills not comparable $\mathcal{G}_1$ and $\mathcal{G}_2$ over $\tilde{X}$ with (X, τ, E) is a soft $\mathcal{G}_1$-compact, then (X, τ, E) it is not necessary to be not soft $\mathcal{G}_2$-compact.

Example 3.1. Let X be an uncountable soft set endowed with soft co-countable topology τ. Let $G_x = \{V \mid x\tilde{\in}V\}$ and let $G_y = \{V \mid y\tilde{\in}V\}$ be soft grills over $\tilde{X}$, where x, y are distinct points. Clearly, (X, τ, E) is soft G_x-compact and also it is soft G_y-compact, but G_x and G_y are obviously not comparable.

4. Soft quasi $\mathcal{H}$-closeness

Definition 4.1. (i) A soft topological space (X, τ, E) is called soft quasi $\mathcal{H}$-closed (SQ$\mathcal{H}$C, in short) (or soft absolutely closed), if for every cover $\eta = \{V_i \mid i \in \Gamma\}$ of $\tilde{X}$ by soft τ-open sets, has a finite subcollection Γ_0 of Γ such that

$$\tilde{X} = \sqcup\{cl(V_i) \mid i \in \Gamma_0\}.$$

(ii) A soft set F of a soft topological space (X, τ, E) is called soft quasi $\mathcal{H}$-closed (SQ$\mathcal{H}$C, in short) (or soft absolutely closed), if for every cover $\eta = \{V_i \mid i \in \Gamma\}$ of F by soft τ-open sets, has a finite subcollection Γ_0 of Γ such that

$$F \sqsubseteq \sqcup\{cl(V_i) \mid i \in \Gamma_0\}.$$

(iii) A soft topological space (X, τ, E) is soft $\mathcal{H}$-closed (or soft absolutely closed) space, if it is soft Hausdorff and soft quasi $\mathcal{H}$-closed space.

Proposition 4.1. *A soft closure of a soft quasi $\mathcal{H}$-closed set of a soft topological space (X, τ, E) is a soft quasi $\mathcal{H}$-closed.*

Proof. Let F be a soft quasi $\mathcal{H}$-closed set and let $\eta = \{V_i \mid i \in \Gamma\}$ be a cover of clF by soft τ-open sets. Then, η be a cover of F by soft τ-open sets. Thus, there exist finite subset Γ_0 of Γ such that $F \sqsubseteq \sqcup_{i\in\Gamma_0} cl(V_i)$. So $cl(F) \sqsubseteq \sqcup_{i\in\Gamma_0} cl(V_i)$. □

Proposition 4.2. (1) *If there exist a soft dense set F of a soft topological space (X, τ, E), which is soft quasi $\mathcal{H}$-closed, then $\tilde{X}$ is soft quasi $\mathcal{H}$-closed.*

(2) *A soft topological space (X, τ, E) is soft quasi $\mathcal{H}$-closed if and only if every cover of $\tilde{X}$ by soft clopen sets has a finite sucover.*

Proof. (1) Let $\eta = \{V_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ-open sets and F be a soft set over X. Then, η is cover of F by soft τ-open sets. Since F is soft quasi $\mathcal{H}$-closed set, then there exist finite subfamily Γ_0 of Γ such that $F \sqsubseteq \sqcup_{i\in\Gamma_0} cl(V_i)$ and so $cl(F) \sqsubseteq \sqcup_{i\in\Gamma_0} cl(V_i)$. Since F is soft dense set, then $\tilde{X} = \sqcup_{i\in\Gamma_0} cl(V_i)$. Consequently, $\tilde{X}$ is soft quasi $\mathcal{H}$-closed.

(2) Let $\eta = \{V_i \mid i \in \Gamma\}$ be any cover of $\tilde{X}$ by soft τ-clopen sets. Then, $\tilde{X} = \sqcup_{i\in\Gamma}(V_i)$. Since $\tilde{X}$ is soft quasi $\mathcal{H}$-closed, then there exist finite subfamily Γ_0 of Γ such that $\tilde{X} = \sqcup_{i\in\Gamma_0} cl(V_i) = \sqcup_{i\in\Gamma_0}(V_i)$. Thus, every cover of $\tilde{X}$ by soft clopen sets has a finite sucover. On the other hand, let $\eta = \{V_i \mid i \in \Gamma\}$ be any cover of $\tilde{X}$ by soft τ-open sets. Since every cover of $\tilde{X}$ by soft clopen sets has a finite sucover, then there exist finite subfamily Γ_0 of Γ such that $\tilde{X} = \sqcup_{i\in\Gamma_0}(V_i) = \sqcup_{i\in\Gamma_0} cl(V_i)$. Hence, $\tilde{X}$ is soft quasi $\mathcal{H}$-closed. □

Proposition 4.3. *The finite soft union of soft quasi $\mathcal{H}$-closed sets of a soft topological space (X, τ, E) is so soft quasi $\mathcal{H}$-closed set.*

Proof. Let F, K be soft quasi $\mathcal{H}$-closed sets and $\eta = \{V_i \mid i \in \Gamma\}$ be a cover of $F \sqcup K$ by soft τ-open sets. Then, η is cover of both F and K by soft τ-open sets. Thus, there exist finite subsets Γ_0, Γ_1 of Γ such that $F \sqsubseteq \sqcup_{i\in\Gamma_0} cl(V_i)$ and $K \sqsubseteq \sqcup_{i\in\Gamma_1} cl(V_i)$. So, $(F \sqcup K) \sqsubseteq \sqcup_{i\in\Gamma_0} cl(V_i) \sqcup \sqcup_{i\in\Gamma_1} cl(V_i)$. Suppose $\Gamma_2 = max\{\Gamma_0, \Gamma_1\}$. Then $(F \sqcup K) \sqsubseteq \sqcup_{i\in\Gamma_2} cl(V_i)$. Hence, $(F \sqcup K)$ is soft quasi $\mathcal{H}$-closed set. □

Definition 4.2. A soft subspace (Y, τ_Y, E) of a soft topological space (X, τ, E) is a soft quasi $\mathcal{H}$-closed relative to $\tilde{X}$ if each soft τ-open family which covers $\tilde{Y}$ has a finite subfamily whose union is τ-dense in $\tilde{Y}$.

Remark 4.1. The next example shows that, the soft quasi $\mathcal{H}$-closeness is not a hereditary property.

Example 4.1. Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $\tau = \{\tilde{\varnothing}, \tilde{X}, F_1, F_2, F_3, F_4, F_5, F_6\}$ where the soft sets $F_1, F_2, F_3, F_4, F_5, F_6$ over X are given as follow:

$F_1 = \{\{a\}, \{a\}\}, F_2 = \{\{b\}, \{c\}\}, F_3 = \{\{a, b\}, \{a, c\}\}$,

$F_4 = \{\{c\}, \{b\}\}, F_5 = \{\{a, c\}, \{a, b\}\}, F_6 = \{\{b, c\}, \{b, c\}\}$.

It is clear that $\tilde{X}$ is soft quasi $\mathcal{H}$-closed space, but $\tilde{Y} = \{\{c\}, \{c\}\}$ is not soft quasi $\mathcal{H}$-closed.

Theorem 4.1. *Let F be a soft closed set of a soft quasi $\mathcal{H}$-closed topological space (X, τ, E). If the soft boundary of F is soft quasi $\mathcal{H}$-closed, then So is F.*

Proof. Let $\eta = \{V_i \mid i \in \Gamma\}$ be a cover of F by soft τ-open sets. Then, $F \sqsubseteq \sqcup_{i\in\Gamma}(V_i)$. Thus, $\tilde{X} = F \sqcup (\tilde{X} - F) = \sqcup_{i\in\Gamma}(V_i) \sqcup (\tilde{X} - F)$. Since (X, τ, E) is soft quasi $\mathcal{H}$-closed space, then there exist finite sub collection $V_{i1}, V_{i2}, ..., V_{in}$ of η such that

$$\tilde{X} = cl(V_{i1}) \sqcup cl(V_{i2}) \sqcup ... \sqcup cl(V_{in}) \sqcup cl(\tilde{X} - F).$$

So,

$$F \sqsubseteq cl(V_{i1}) \sqcup cl(V_{i2}) \sqcup ... \sqcup cl(V_{in}) \sqcup (F \sqcap cl(\tilde{X} - F))$$

and thus

$$F \sqsubseteq cl(V_{i1}) \sqcup cl(V_{i2}) \sqcup ... \sqcup cl(V_{in}) \sqcup (bd(F)).$$

Since $bd(F)$ is soft quasi $\mathcal{H}$-closed and $\xi = \{U_i \mid i \in \Gamma\}$ is a cover of $bd(F)$ by soft τ-open sets, then there exist finite sub collection $U_{i1}, U_{i2}, ..., U_{in}$ of ξ such that

$$bd(F) = cl(U_{i1}) \sqcup cl(U_{i2}) \sqcup ... \sqcup cl(U_{in}).$$

Hence,

$$F \sqsubseteq cl(V_{i1}) \sqcup cl(V_{i2}) \sqcup ... \sqcup cl(V_{in}) \sqcup cl(U_{i1}) \sqcup cl(U_{i2}) \sqcup ... \sqcup cl(U_{in})$$

and so F is soft quasi $\mathcal{H}$-closed. □

Theorem 4.2. *If every closed soft subspace of a soft topological space (X, τ, E) is a soft quasi $\mathcal{H}$-closed, then $\tilde{X}$ is a soft quasi $\mathcal{H}$-closed.*

Proof. Let $\eta = \{V_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ-open sets and $V_0 \in \eta$. Let $\tilde{Y} = (\tilde{X} - V_0)$. Then, $\{(\tilde{Y} \sqcap V_i) \mid i \in \Gamma\}$ is cover of $\tilde{Y}$ by soft τ_Y-open sets. Since $\tilde{Y}$ is soft τ-closed, then $\tilde{Y}$ is a soft quasi $\mathcal{H}$-closed. So there exist finite subsets Γ_0 of Γ such that $\tilde{Y} = \sqcup_{i\in\Gamma_0} cl_Y((\tilde{Y} \sqcap V_i))$. Therefore, there exist finite subsets Γ_0 of Γ such that $\tilde{X} = \sqcup_{i\in\Gamma_0} cl(V_i)$ and so $\tilde{X}$ is a soft quasi $\mathcal{H}$-closed space. □

Theorem 4.3. *Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E) and $(\tau - \{\tilde{\varnothing}\}) \sqsubseteq \mathcal{G}$. If (X, τ, E) is a soft $\mathcal{G}$-compact space, then it is a SQ$\mathcal{H}$C.*

Proof. Let $\{U_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ-open sets and (X, τ, E) be a soft $\mathcal{G}$-compact space. Then, there exist a finite subset Γ_0 of Γ such that $(\tilde{X} - \sqcup_{i\in\Gamma_0} U_i) \notin \mathcal{G}$. Thus, $\phi_{\mathcal{G}}(\tilde{X} - \sqcup_{i\in\Gamma_0} U_i) = \tilde{\varnothing}$ and so $\tilde{X} - \phi_{\mathcal{G}}(\sqcup_{i\in\Gamma_0} U_i) = \tilde{\varnothing}$, follows from Propositions 2.1, 2.2 and 2.3. Also, $\tilde{X} - cl(\sqcup_{i\in\Gamma_0} U_i) = \tilde{\varnothing}$. Hence, $int(\tilde{X} - \sqcup_{i\in\Gamma_0} U_i) = \tilde{\varnothing}$. Otherwise, $int(\tilde{X} - \sqcup_{i\in\Gamma_0} U_i) \in (\tau - \{\tilde{\varnothing}\})$. Since $(\tau - \{\tilde{\varnothing}\}) \sqsubseteq \mathcal{G}$, then $(\tilde{X} - \sqcup_{i\in\Gamma_0} U_i) \in \mathcal{G}$, a contradiction. Therefore, $\tilde{X} = \sqcup_{i\in\Gamma_0} cl(U_i)$ and so $\tilde{X}$ is SQ$\mathcal{H}$C. □

Theorem 4.4. *Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E) with $(\tau - \{\tilde{\varnothing}\}) \sqsubseteq \mathcal{G}$. A space (X, τ, E) is SQ$\mathcal{H}$C if and only if a space $(X, \tau_{\mathcal{G}}, E)$ is SQ$\mathcal{H}$C.*

Proof. Let $(X, \tau_{\mathcal{G}}, E)$ be a SQ$\mathcal{H}$C and $\tau \sqsubseteq \tau_{\mathcal{G}}$. Then, the space (X, τ, E) is SQ$\mathcal{H}$C. On the other hand, let $\eta = \{(V_i - F_i) \mid V_i \in \tau, F_i \notin \mathcal{G}, i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft $\tau_{\mathcal{G}}$-open sets. Then, $\{V_i \mid i \in \Gamma\}$ is a cover of $\tilde{X}$ by soft τ-open sets. Since (X, τ, E) is SQ$\mathcal{H}$C, then there exist finite subset Γ_0 of Γ such that $\tilde{X} = \sqcup_{i\in\Gamma_0} cl(V_i)$. Since $(\tau - \{\tilde{\varnothing}\}) \sqsubseteq \mathcal{G}$, then $\psi_{\mathcal{G}}(V_i) = \tau_{\mathcal{G}}\text{-}cl(V_i) = \phi_{\mathcal{G}}(V_i) = cl(V_i)$, follows from Theorem 2.3. Since $F_i \notin \mathcal{G}$ for each i, then by using Proposition 2.3, $\phi_{\mathcal{G}}(V_i) = \phi_{\mathcal{G}}(V_i - F_i)$. Hence, $\sqcup_{i\in\Gamma_0}\ \phi_{\mathcal{G}}(V_i - F_i) = \sqcup_{i\in\Gamma_0} \phi_{\mathcal{G}}(V_i) = \sqcup_{i\in\Gamma_0} cl(V_i) = \tilde{X}$. Therefore, $\sqcup_{i\in\Gamma_0} \psi_{\mathcal{G}}(V_i - F_i) = \sqcup_{i\in\Gamma_0} [\phi_{\mathcal{G}}(V_i - F_i) \sqcup (V_i - F_i)] = \tilde{X}$ and so $(X, \tau_{\mathcal{G}}, E)$ is SQ$\mathcal{H}$C. □

Corollary 4.1. *Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E) such that $(\tau - \{\tilde{\varnothing}\}) \sqsubseteq \mathcal{G}$. If (X, τ, E) is a soft $\mathcal{G}$-compact, then $(X, \tau_{\mathcal{G}}, E)$ is SQ$\mathcal{H}$C.*

Proof. Let (X, τ, E) is a soft $\mathcal{G}$-compact and $(\tau - \{\tilde{\varnothing}\}) \sqsubseteq \mathcal{G}$. Then, (X, τ, E) is SQ$\mathcal{H}$C, by using Theorem 4.3. Thus, $(X, \tau_{\mathcal{G}}, E)$ is SQ$\mathcal{H}$C, in view of Theorem 4.4. □

It is clear that the collection $\{F \mid int(cl(F)) \neq \tilde{\varnothing}\}$ is a soft grill on $\tilde{X}$, which is denoted by $\mathcal{G}_\delta$.

Theorem 4.5. *A space (X, τ, E) is a soft $\mathcal{G}_\delta$-compact, if it is SQ$\mathcal{H}$C.*

Proof. Let $\{U_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ-open sets and (X, τ, E) be a soft quasi $\mathcal{H}$-closed. Then there exist a finite subset Γ_0 of Γ such that $\sqcup_{i\in\Gamma_0} cl(U_i) = \tilde{X}$. Suppose $(\tilde{X} - \sqcup_{i\in\Gamma_0} U_i) \in \mathcal{G}_\delta$ and U_i is a soft τ-open set for every i. Then $\tilde{\varnothing} \neq int\ cl(\tilde{X} - \sqcup_{i\in\Gamma_0} U_i) \sqsubseteq (\tilde{X} - \sqcup_{i\in\Gamma_0} cl(U_i))$, a contradiction. Thus, (X, τ, E) is soft $\mathcal{G}_\delta$-compact. □

Definition 4.3. Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E). Then, the space (X, τ, E) is called soft $\mathcal{G}$-regular if for any soft closed set F in $\tilde{X}$ with soft point $x \tilde{\notin} F$, there exist disjoint soft open sets U and V such that $x \tilde{\in} U$ and $(F - V) \notin \mathcal{G}$.

Theorem 4.6. *A space (X, τ, E) with a soft grill $\mathcal{G}$ is soft $\mathcal{G}$-regular, if it is a soft $\mathcal{G}$-compact Hausdorff.*

Proof. Let F be a soft τ-closed set over X and $x \in X$ with $x \tilde{\notin} F$. By soft Hausdorfness of $\tilde{X}$, for each $y \tilde{\in} F$ there exist disjoint soft τ-open sets U_y and V_y such that $x \tilde{\in} U_y$ and $y \tilde{\in} V_y$. Now, $\{V_y \mid y \tilde{\in} F\} \sqcup (\tilde{X} - F)$ is a cover of $\tilde{X}$ by soft τ-open sets. Thus, by soft $\mathcal{G}$-compactness of $\tilde{X}$, there exist finitely many points $y_1, y_2, ..., y_n$ in F such that $\tilde{X} - [(\sqcup_{i=1}^{n} V_{yi}) \sqcup (\tilde{X} - F)] \notin \mathcal{G}$. Let $K = \tilde{X} - (\sqcup_{i=1}^{n} cl(V_{yi}))$ and $H = \sqcup_{i=1}^{n} V_{yi}$. Then, K and H are two disjoint nonempty soft τ-open sets in $\tilde{X}$ such that $x \tilde{\in} K, (F - H) = F \sqcap [\tilde{X} - \sqcup_{i=1}^{n} V_{yi}] = \tilde{X} - [(\sqcup_{i=1}^{n} V_{yi}) \sqcup (\tilde{X} - F)] \notin \mathcal{G}$. So, (X, τ, E) is a soft $\mathcal{G}$-regular. □

Corollary 4.2. *Let $\mathcal{G}$ be a soft grill on a soft Hausdorff space (X, τ, E) such that $(\tau - \{\tilde{\varnothing}\}) \sqsubseteq \mathcal{G}$. If (X, τ, E) is soft $\mathcal{G}$-compact, then it is soft $\mathcal{H}$-closed and soft $\mathcal{G}$-regular.*

Proof. Since soft $\mathcal{G}$-compact Hausdorff space is soft $\mathcal{G}$-regular, in view of Theorem 4.6. Since (X, τ, E) is a soft $\mathcal{G}$-compact space and $(\tau - \{\tilde{\varnothing}\}) \sqsubseteq \mathcal{G}$, then (X, τ, E) is a SQ$\mathcal{H}$C space, by using Theorem 4.3. Since (X, τ, E) is soft Hausdorff space, then it is soft $\mathcal{H}$-closed. □

Theorem 4.7. *A space (X, τ, E) with a soft grill $\mathcal{G}$ is soft $\mathcal{G}$-compact, if it is soft $\mathcal{H}$-closed $\mathcal{G}$-regular.*

Proof. Let η be a cover of $\tilde{X}$ by soft τ-open sets. Then, for each soft point x, there exist $U_x \in \eta$ such that $x \tilde{\in} F U_x$. Thus, $x \tilde{\notin} (\tilde{X} - U_x)$ where $(\tilde{X} - U_x)$ is a soft τ-closed set. By soft $\mathcal{G}$-regularity of $\tilde{X}$, there exist two disjoint soft τ-open sets K_x and H_x such that $M_x = [(\tilde{X} - U_x) - H_x] \notin \mathcal{G}$ and $x \tilde{\in} K_x$. Now, $cl(K_x) \sqcap H_x = \tilde{\varnothing}$. Thus, $cl(K_x) \sqsubseteq (\tilde{X} - H_x) \sqsubseteq (\tilde{X} - H_x) \sqcup U_x = [\tilde{X} - (H_x \sqcup U_x)] \sqcup U_x = M_x \sqcup U_x$. Since $\{K_x \mid x \tilde{\in} F \tilde{X}\}$ is a cover of $\tilde{X}$ by soft τ-open sets and (X, τ, E) is soft $\mathcal{H}$-closed space, then there are finitely many points $x_1, x_2, ..., x_n$ such that $\tilde{X} = \sqcup_{i=1}^{n} cl(K_{xi})$. So, $\tilde{X} = \sqcup_{i=1}^{n} cl(K_{xi}) \sqsubseteq \sqcup_{i=1}^{n} [M_{xi} \sqcup U_{xi}]$. Hence, $\tilde{X} - \sqcup_{i=1}^{n} U_{xi} \sqsubseteq \sqcup_{i=1}^{n} M_{xi} \notin \mathcal{G}$. Therefore, (X, τ, E) is a soft $\mathcal{G}$-compact. □

Theorem 4.8. *The soft quasi $\mathcal{H}$-closeness is a soft topological property.*

Proof. Let (X, τ_1, E), (Y, τ_2, K) be soft topological spaces and $\eta = \{V_i \mid i \in \Gamma\}$ be a cover of $\tilde{Y}$ by soft τ_2-open sets. Since f_{pu} is soft continuous, then $\{f_{pu}^{-1}(V_i) \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ_1-open sets. Since (X, τ_1, E) is soft quasi $\mathcal{H}$-closed space, then there exist a finite subset Γ_0 of Γ such that $\tilde{X} = \sqcup_{i \in \Gamma_0} cl(f_{pu}^{-1}(V_i))$. Since f_{pu} is soft open bijective function, then $\tilde{Y} = \sqcup_{i \in \Gamma_0} cl(V_i)$ and so (Y, τ_2, K) is soft quasi $\mathcal{H}$-closed space. □

Definition 4.4. Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E). Then,

(i) A soft set F is called soft countably $\mathcal{G}$-compact, if for every countable cover $\{U_i \mid i \in \Gamma\}$ of F by soft τ-open sets, there exist a finite subset Γ_0 of Γ such that $(F - \sqcup_{i \in \Gamma_0} U_i) \notin \mathcal{G}$.

(ii) The space (X, τ, E) is called soft countably $\mathcal{G}$-compact, if $\tilde{X}$ is soft countably $\mathcal{G}$-compact as a soft subset.

Theorem 4.9. *Let (X, τ, E) be a soft topological space with two soft grills $\mathcal{G}_1, \mathcal{G}_2$ over $\tilde{X}$. If (X, τ, E) is a soft countably $\mathcal{G}_2$-compact and $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$. Then, (X, τ, E) is soft countably $\mathcal{G}_1$-compact space.*

Proof. Let $\{U_i \mid i \in \Gamma\}$ be a countable cover of $\tilde{X}$ by soft τ-open sets and (X, τ, E) be a soft countably $\mathcal{G}_2$-compact, then there exist a finite subset Γ_0 of Γ such that $(F - \sqcup_{i \in \Gamma_0} U_i) \notin \mathcal{G}_2$. Since $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$, then there exist a finite subset Γ_0 of Γ such that $(F - \sqcup_{i \in \Gamma_0} U_i) \notin \mathcal{G}_1$. Hence, (X, τ, E) is soft countably $\mathcal{G}_1$-compact space. □

Proposition 4.4. *Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E). Then,*

(1) *Every soft $\mathcal{G}$-compact space is soft countably $\mathcal{G}$-compact.*

(2) *The space (X, τ, E) is soft countably $\mathcal{G}$-compact if and only if it is soft lindelöf, whenever $\mathcal{G} = (\mathcal{P}(X) - \{\tilde{\varnothing}\})$.*

Proof. Follows immediately from Definition 4.4. □

Theorem 4.10. *Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E), then the following statements are equivalent :*

(1) *(X, τ, E) is soft countably $\mathcal{G}$-compact.*

(2) *For any countable family $\{M_i \mid i \in \Gamma\}$ of soft τ-closed sets of $\tilde{X}$ such that $\sqcap\{M_i \mid i \in \Gamma\} = \tilde{\varnothing}$, there exist a finite subset Γ_0 of Γ such that $\sqcap\{M_i \mid i \in \Gamma_0\} \notin \mathcal{G}$.*

Proof. It is similar to the proof of Theorem 3.5. □

Theorem 4.11. *Let $\mathcal{G}$ be a soft grill on a soft topological space (X, τ, E). If the space (X, τ, E) is soft countably $\mathcal{G}$-compact and soft lindelöf, then it is a soft $\mathcal{G}$-compact.*

Proof. Let $\{U_i \mid i \in \Gamma\}$ be a cover of $\tilde{X}$ by soft τ-open sets and (X, τ, E) be a soft lindelöf space. Then, there exist a countable subcover $\{U_i \mid i \in \Gamma_1\}$ such that $\tilde{X} = \sqcup\{U_i \mid i \in \Gamma_1\}$. Since (X, τ, E) be a soft countably $\mathcal{G}$-compact space, then there exist a finite subset Γ_0 of Γ_1 such that $(\tilde{X} - \sqcup_{i \in \Gamma_0} U_i) \notin \mathcal{G}$. Consequently, (X, τ, E) is soft $\mathcal{G}$-compact. □

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