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Variety of aperiodic fuzzy languages

V. P. Archana

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Abstract. In this paper we give the variety description of aperiodic fuzzy languages and also provide the Eilenberg variety theorem for the class of aperiodic fuzzy languages.

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Corresponding Author: V. P. Archana (rajesharchana09@gmail.com)

1. Intrduction

In fuzzy language theory, every monoid is the syntactic monoid of some fuzzy language [8]. By using this result, the properties of fuzzy languages can be studied by the algebraic properties of the syntactic monoids. We adapt this method to analyze different classes of fuzzy languages [1, 2, 3, 4].

The notion of aperiodic fuzzy languages was introduced by Yongming Li. Here we consider the aperiodic fuzzy languages with membership values in [0,1]. The purpose of this paper is to provide the variety description of aperiodic fuzzy languages.

2. Preliminaries

Here we recall the basic definitions and notations that will be used in the sequel. All undefined terms are as in [6, 7]. A nonempty set S with an associative binary operation is called a semigroup. If there is an element $1 \in S$ with $1s = s = s1$ for all $s \in S$, then S is called a monoid (semigroup with identity). A semigroup (monoid) S is called aperiodic if it has no proper subgroups.

Definition 2.1 (cf. [5])**.** A pseudovariety of monoids (resp. semigroups) is a class $\mathbf{V}$ of finite monoids (resp. semigroups) that is closed under finite direct product, morphic image and subobject. That is

(i) If $S, T \in \mathbf{V}$, then $S \times T \in \mathbf{V}$.
(ii) If $S, T \in \mathbf{V}$ and $\varphi : S \to T$ is a homomorphism, then $\varphi(S) \in \mathbf{V}$.
(iii) If $S \in \mathbf{V}$ and S' is a submonoid (resp. subsemigroup) of S, then $S' \in \mathbf{V}$.

Theorem 2.2 (cf. [9]). *Let S be a finite semigroup (monoid). Then S is aperiodic if and only if $x^{n+1} = x^n$ for some n and all $x \in S$.* □

The class of finite aperiodic monoids is denoted by **Ap**.

Ap is a pseudovariety of monoids.

Let A be a nonempty finite set called an alphabet. Elements of A are called letters. A word is a finite sequence of letters of A. The length of a word is the number of letters in it. A word of length zero is called the empty word, it is denoted by 1. The set of all nonempty words $A^* = A^+ \cup \{1\}$. Then $A^*(A^+)$ together with the binary operation concatenation is called a free monoid(semigroup) on A.

A fuzzy language λ in A^* is a fuzzy subset of A^*. To each fuzzy language λ we associate a congruence P_λ called syntactic congruence, as follows. For $u, v \in A^*$, $uP_\lambda v$ if and only if $\lambda(xuy) = \lambda(xvy)$ for all $x, y \in A^*$. The quotient monoid, $\mathrm{Syn}(\lambda) = A^*/P_\lambda$ is called the syntactic monoid of λ. It should be noted that if we restrict to $A^+, Syn(\lambda) = A^+/P_\lambda$ is called the syntactic semigroup of λ.

A fuzzy language λ over an alphabet A is recognizable by a monoid(semigroup) S if there is a homomorphism $\phi : A^* \to S(\phi : A^+ \to S)$ and a fuzzy subset π of S such that $\lambda = \pi\phi^{-1}$, where $\pi\phi^{-1}(u) = \pi(\phi(u)$.

For fuzzy languages $\lambda, \lambda_1, \lambda_2$ over an alphabet A, complement, union and intersection are defined respectively by $\overline{\lambda}(u) = 1 - \lambda(u)$, $(\lambda_1 \vee \lambda_2)(u) = \lambda_1(u) \vee \lambda_2(u)$, $(\lambda_1 \wedge \lambda_2)(u) = \lambda_1(u) \wedge \lambda_2(u)$.

Further left and right quotients are defined respectively by;
$(\lambda_1^{-1}\lambda_2)(u) = \bigvee_{v \in A^*} (\lambda_2(vu) \wedge \lambda_1(v))$, $(\lambda_2\lambda_1^{-1})(u) = \bigvee_{v \in A^*} (\lambda_2(uv) \wedge \lambda_1(v))$.

Let $c \in [0, 1]$ be arbitrary. Then the fuzzy language $c\lambda$ defined by $(c\lambda)(u) = c \cdot \lambda(u)$ is called multiplication by constant c.

Let A, B be finite alphabets, $\phi : A^* \to B^*(\phi : A^+ \to B^+)$ be a homomorphism and ψ a fuzzy language in $B^*(B^+)$, then the inverse image of ψ (underϕ) is a fuzzy language $\psi\phi^{-1}$ over A defined by $(\psi\phi^{-1})(u) = \psi(\phi u)$.

For a fuzzy language λ by a c-cut, $c \in [0, 1]$, we mean the crisp language λ_c defined by $\lambda_c = \{u \in A^* | \lambda(u) \geq c\}$.

The following theorem gives a characterization for regular fuzzy languages.

Theorem 2.3 ([7]). *A fuzzy language λ is regular if and only if $Im(\lambda)$ is finite and language λ_c is regular for every $c \in [0, 1]$.*

Unless otherwise specified all the fuzzy languages considered here are regular.

Definition 2.4. A family $\mathscr{F} = \mathscr{F}(A)$ of regular fuzzy languages is a variety of fuzzy languages in $A^*(A^+)$ if it is closed under unions, intersections, complements, multiplication by constants, quotients, inverse homomorphic images and cuts.

Theorem 2.5. *Let λ be a fuzzy language over an alphabet A. Then we have the following.*

(1) *Syn(λ) recognizes λ.*

(2) *If M is any other monoid (resp. semigroup) recognizing λ, then Syn(λ) is a homomorphic image of M.*

Proof. (1) Clearly $\eta_\lambda : A^* \to Syn(\lambda)$(resp. $A^+ \to Syn(\lambda)$) is an onto homomorphism. Define $\pi : Syn(\lambda) \to [0,1]$ by $\pi(\eta_\lambda(u)) = \lambda(u)$. Then $\lambda = \pi\eta_\lambda^{-1}$ and thus Syn(λ) recognizes λ.

(2) Since M recognizes λ, there is an onto homomorphism $\varphi : A^* \to M(A^+ \to M)$ and a function $\pi' : M \to [0,1]$ such that $\lambda = \pi'\varphi^{-1}$. Define $\psi : M \to Syn(\lambda)$ by $\psi(\varphi(u)) = \eta_\lambda(u)$. Then ψ is an onto homomorphism, since η_λ is an onto homomorphism. Thus Syn(λ) is a homomorphic image of M. □

For a variety of fuzzy languages $\mathscr{F}$, let $\mathscr{F}^s$ be the family of finite monoids defined by $\mathscr{F}^s = \{\text{Syn}(\lambda) | \lambda \in \mathscr{F}(A), \text{ for some } A\}$. For a variety of finite monoids $\mathscr{S}$, let $\mathscr{S}^f = \mathscr{S}^f(A)$ be the family of fuzzy languages defined by $\mathscr{S}^f(A) = \{\lambda$ is a fuzzy language over $A | \text{Syn}(\lambda) \in \mathscr{S}\}$.

Theorem 2.6 (cf.[8],Theorem 7)**.** *The mappings $\mathscr{F} \to \mathscr{F}^s$ and $\mathscr{S} \to \mathscr{S}^f$ are mutually inverse lattice isomorphisms between the lattices of all varieties of fuzzy languages and all varieties of finite monoids.*

3. Aperiodic fuzzy languages

Definition 3.1 (cf. [10], Definition 4.2)**.** A fuzzy language $\lambda : A^* \to [0,1]$ is called aperiodic, if λ is recognizable and

$$\lambda(xu^{n+1}y) = \lambda(xu^n y)$$

for all $x, y, u \in A^*$ and some non negative integer n.

Example 3.2. Let $A = \{a, b\}$ and let $\lambda' : A^* \to [0,1]$ be given by

$$\lambda'(u) = \begin{cases} \frac{1}{10} & \text{if } u \in aA^*a \cup \{a\} \\ \frac{1}{11} & \text{if } u \in bA^*b \cup \{b\} \\ 0 & \text{otherwise.} \end{cases}$$

Then λ is an aperiodic fuzzy language.

The class of aperiodic fuzzy languages in A^* is denoted by **ApFL**. The following result shows that **ApFL** is closed under the boolean operations.

3.1. Variety of aperiodic fuzzy languages.

Lemma 3.3. *Let $\lambda, \lambda_1, \lambda_2 \in$ **ApFL**. Then $\overline{\lambda} \in$ **ApFL**, $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are in **ApFL**.*

Proof. Since $\lambda \in$ **ApFL**, we have for a non negative integer n, $\lambda(xu^{n+1}y) = \lambda(xu^n y)$ for all $x, y, u \in A^*$. Then

$$\begin{aligned} \overline{\lambda}(xu^{n+1}y) &= 1 - \lambda(xu^{n+1}y) \\ &= 1 - \lambda(xu^n y) = \overline{\lambda}(xu^n y) \end{aligned}$$

for all $x, y, u \in A^*$ and for some non negative integer n. Thus $\overline{\lambda} \in$ **ApFL**.

Let $\lambda_1, \lambda_2 \in$ **ApFL**. Then $\lambda_1(xu^{n+1}y) = \lambda_1(xu^n y)$ and $\lambda_2(xu^{m+1}y) = \lambda_2(xu^m y)$ for all $x, y, u \in A^*$ and for non negative integers m and n. Then

$$\begin{aligned}(\lambda_1 \vee \lambda_2)(xu^{mn+1}y) &= \lambda_1(xu^{mn+1}y) \vee \lambda_2(xu^{mn+1}y)\\ &= \lambda_1(x(u^m)^{n+1}y) \vee \lambda_2(x(u^n)^{m+1}y)\\ &= \lambda_1(x(u^m)^n y) \vee \lambda_2(x(u^n)^m y) = (\lambda_1 \vee \lambda_2)(xu^{mn}y)\end{aligned}$$

for all $x, y, u \in A^*$. Since mn is a positive integer, $(\lambda_1 \vee \lambda_2) \in$ **ApFL**. As $\lambda_1 \wedge \lambda_2 = \overline{(\overline{\lambda_1} \vee \overline{\lambda_2})}$, it follows that $\lambda_1 \wedge \lambda_2 \in$ **ApFL**. □

The following lemma shows that **ApFL** is closed under multiplication by constants.

Lemma 3.4. *Let $\lambda \in$ **ApFL** and $c \in [0, 1]$. Then $c\lambda$ is an aperiodic fuzzy language.*

Proof. Let $\lambda(xu^{n+1}y) = \lambda(xu^n y)$ for some non negative integer n and for all $x, y, u \in A^*$. Then

$$\begin{aligned}(c\lambda)(xu^{n+1}y) &= c \cdot \lambda(xu^{n+1}y)\\ &= c \cdot \lambda(xu^n y) = (c\lambda)(xu^n y)\end{aligned}$$

for all $x, y, u \in A^*$. Thus $c\lambda \in$ **ApFL**. □

The following result shows that **ApFL** is closed under c-cuts.

Lemma 3.5. *Let $\lambda \in$ **ApFL** and $c \in [0, 1]$. Then the syntactic monoid of the language $\lambda_c = \{u \in A^* \mid \lambda(u) \geq c\}$ is an aperiodic monoid.*

Proof. Since $\lambda \in$ **ApFL**, we have for a non negative integer n and for all $x, y, u \in A^*$, $\lambda(xu^{n+1}y) = \lambda(xu^n y)$. Then

$$\begin{aligned}xu^{n+1}y \in \lambda_c &\Leftrightarrow c \leq \lambda(xu^{n+1}y) = \lambda(xu^n y)\\ &\Leftrightarrow xu^n y \in \lambda_c\end{aligned}$$

where $n > 0$ and for all $x, y, u \in A^*$. So $[u]^{n+1} = [u]^n$ for all $u \in A^*$. Thus, by Theorem 2.2, syntactic monoid of λ_c is an aperiodic monoid. □

Lemma 3.6. *Let λ be an aperiodic fuzzy language on A^*, X be a finite alphabet and $\varphi : X^* \to A^*$ be a homomorphism. Then $\lambda\varphi^{-1}$ is an aperiodic fuzzy language over A^*.*

Proof. Since $\lambda \in$ **ApFL**, we have $\lambda(xu^{n+1}y) = \lambda(xu^n y)$ for some non negative integer n and all $x, y \in A^*$. So

$$\begin{aligned}(\lambda\varphi^{-1})(xu^{n+1}y) &= \lambda(\varphi(xu^{n+1}y))\\ &= \lambda(\varphi(x)\varphi(u^{n+1})\varphi(y)) = \lambda(\varphi(x)(\varphi(u))^n\varphi(y))\\ &= \lambda(\varphi(xu^n y)) = \lambda\varphi^{-1}(xu^n y))\end{aligned}$$

for all $x, y, u \in A^*$ and $n > 0$. Thus $\lambda\varphi^{-1}$ is an aperiodic fuzzy language. □

From the above lemma it follows that **ApFL** is closed under inverse homomorphic images.

Lemma 3.7. *Let λ_1, $\lambda_2 \in$ **ApFL**. Then $\lambda_1^{-1}\lambda_2$ and $\lambda_2\lambda_1^{-1}$ are in **ApFL**.*

Proof. Let $\lambda_1, \lambda_2 \in$ **ApFL**. Then $\lambda_1(xu^{n+1}y) = \lambda_1(xu^n y)$ and $\lambda_2(xu^{m+1}y) = \lambda_2(xu^m y)$ for non negative integers m and n, and for all $x, y, u \in A^*$. So

$$\begin{aligned}(\lambda_1^{-1}\lambda_2)(xu^{mn+1}y) &= \bigvee_{v\in A^*} \{\lambda_2(vxu^{mn+1}y) \wedge \lambda_1(v)\} \\ &= \bigvee_{v\in A^*} \{\lambda_2(vx(u^n)^{m+1}y) \wedge \lambda_1(v)\} \\ &= \bigvee_{v\in A^*} \{\lambda_2((vx)(u^n)^m y) \wedge \lambda_1(v)\} \\ &= \bigvee_{v\in A^*} \{\lambda_2(vxu^{mn}y) \wedge \lambda_1(v)\} = (\lambda_1^{-1}\lambda_2)(xu^{mn}y)\end{aligned}$$

for all $x, y, u \in A^*$. Since $mn > 0$, it follows that $\lambda_1^{-1}\lambda_2 \in$ **ApFL**. Similarly if $\lambda_1, \lambda_2 \in$ **ApFL**, then $\lambda_2\lambda_1^{-1} \in$ **ApFL**. □

Theorem 3.8. ***ApFL*** *is a variety of fuzzy languages.*

Proof. By Lemma 3.3, **ApFL** is closed under the boolean operations. By Lemmas 3.4 and 3.5, **ApFL** is closed under the multiplication by constants and c-cuts. By Lemmas 3.6 and 3.7, **ApFL** is closed under the inverse homomorphic images and quotients. Hence **ApFL** is a variety of fuzzy languages. □

Theorem 3.9. *Let M be a finite aperiodic monoid recognizing the fuzzy language λ. Then $\lambda \in$* ***ApFL****.*

Proof. Let $\varphi : A^* \to M$ and a mapping $\pi : M \to [0, 1]$ such that $\lambda = \pi\varphi^{-1}$, where $\lambda(u) = \pi\varphi^{-1}(u) = \pi(\varphi(u))$ for all $u \in A^*$. Then since M is an aperiodic monoid, we have

$$\begin{aligned}\lambda(xu^{n+1}y) &= \pi\varphi^{-1}(xu^{n+1}y) \\ &= \pi(\varphi(xu^{n+1}y)) = \pi(\varphi(x)(\varphi(u))^{n+1}\varphi(y)) \\ &= \pi(\varphi(x)\varphi(u^n)\varphi(y)) = \pi\varphi^{-1}(xu^n y) = \lambda(xu^n y)\end{aligned}$$

for all $x, y, u \in A^*$. Since n is a non negative integer, $\lambda \in$ **ApFL**. □

Theorem 3.10. *Let λ be a fuzzy language. Then $\lambda \in$* ***ApFL*** *if and only if Syn(λ) is an aperiodic monoid.*

Proof. Assume that Syn(λ) is an aperiodic monoid. Since Syn(λ) recognizes λ (cf. Theorem 2.5), by Theorem 3.9, $\lambda \in$ **ApFL**.

Conversely, assume that $\lambda \in$ **ApFL**. Then $\lambda(xu^{n+1}y) = \lambda(xu^n y)$ for all $x, y, u \in A^*$ and some non negative integer n. Thus, by the definition of syntactic congruence, $[u^{n+1}]_\lambda = [u^n]_\lambda$. That is $([u]_\lambda)^{n+1} = ([u]_\lambda)^n$. So Syn($\lambda$) is an aperiodic monoid. □

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V. P. Archana (rajesharchana09@gmail.com)
Department of Mathematics, Mahatma Gandhi College, Thiruvananthapuram