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R_0 concepts in fuzzy bitopological spaces

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Abstract. In this paper, we introduce four notions of R_0–property in fuzzy bitopological spaces by using quasi-coincident sense. We show that all these notions satisfy good extension property. Also hereditary and productive properties are satisfied by these concepts. We observe that all these concepts are preserved under one-one, onto and continuous mappings. Finally, we show that initial and final fuzzy bitopological spaces are R_0.

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1. Introduction

Fuzzy topology, as an important research field in fuzzy set theory, has been established by Chang [6] in 1968 based on Zadeh's [33] concept of fuzzy sets. Since then much attention [24, 28, 29, 10] has been paid to generalize the basic concepts of general topology in fuzzy settings. In classical topology the concept of R_0-property first defined in 1943 by Shanin [30]. After then Dude [8], Naimpally [23], Dorsett [7], Caldas et al. [5], Ekici [9], Keskin and Nori [19] and Roy and Mukherjee [26] defined and studied many characterizations of R_0-properties. The concepts of fuzzy R_0- type axioms for fuzzy topology was first introduced by Hutton and Reilly [13] in 1980. In 1990, Ali et al. [4] introduced some other definitions of fuzzy R_0-axioms. Srivastava et al. [31], Hossain and Ali [11], Khedr et al. [20], Jun and Lee [14], Ali and Azam [3], Zhou and Zhao [12] and Zhang et al. [34] also gave some new concepts of R_0-property in fuzzy topology. In 1989, Kandil and El-Shafee [15] introduced the concept of fuzzy bitopological spaces (fbts, in short) as an extension of fuzzy topological space and as a generalization of bitopological spaces which was introduced by Kelly [18]. After then in 1991, Kandil and El-Shafee [16] first defined R_0-property in fuzzy bitopological spaces. Then after Abu Safiya et al. [1], Kandil et al. [17] and Nouh [25] defined several types of R_0-properties. The objective of this paper is to introduce a set of new notions of R_0-properties of fbts each of which is shown as the good extension of bitopological R_0-property. We have also shown that these notions are preserved under one-one, onto and continuous mappings. Finally we have established that initial and final properties of fuzzy bitopological spaces hold the R_0-properties.

2. Preliminaries

In this section, we give some elementary concepts and results which will be used in the sequel. Throughout this paper, X will be a non-empty set, $I = [0,1]$, $I_0 = (0,1]$, $I_1 = [0,1)$ and FP(resp P) stands for fuzzy pairwise (resp pairwise). The class of all fuzzy sets on a universe X will be denoted by I^X and fuzzy sets on X will be denoted by u, v, w, etc. Crisp subset of X will be denoted by capital letters U, V, W etc. In this paper (X, t) and (X, s, t) will be denoted fuzzy topological space and fuzzy bitopological space respectively. $x_r q u$ denotes x_r is quasi-coincident with u and $x_r \bar{q} u$ denotes that x_r is not quasi-coincident with u throughout this paper.

Definition 2.1 ([33])**.** A fuzzy set μ in a set X is a function from X into the closed unit interval $I = [0,1]$. For every $x \in X$, $\mu(x) \in I$ is called the grade of membership of x. A member of I^X may also be called fuzzy subset of X.

Definition 2.2 ([33])**.** Let f be a mapping from a set X into a set Y and u be a fuzzy set in X. Then the image of u, written as $f(u)$, is a fuzzy set in Y whose membership function is given by

$$f(u)(y) = \begin{cases} \sup\{u(x)\}, & \text{if } f^{-1}[\{y\}] \neq \Phi,\ x \in X, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3 ([33])**.** Let f be a mapping from a set X into a set Y and v be a fuzzy set in Y. Then the inverse of v denoted as $f^{-1}(v)$ is a fuzzy set in X defined by

$$f^{-1}(v)(x) = v(f(x)), \text{ for all } x \in X.$$

Definition 2.4 ([32])**.** A fuzzy set μ in X is called a fuzzy singleton if and only if $\mu(x) = r$, $(0 < r \leq 1)$ for a certain $x \in X$ and $\mu(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. We call x_r is a fuzzy point if $0 < r < 1$. The class of all fuzzy singletons in X will be denoted by $S(X)$.

Definition 2.5 ([6])**.** A fuzzy topology t on X is a collection of members of I^X which is closed under arbitrary suprema and finite infima and which contains constant fuzzy sets 1 and 0. The pair (X, t) is called a fuzzy topological space (fts, in short) and members of t are called t-open (or simply open) fuzzy sets. A fuzzy set μ is called t- closed (or simply closed) fuzzy set if $1 - \mu \in t$.

Definition 2.6 ([27])**.** Let f be a real valued function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real $\alpha \in I_1$, then f is called lower semi continuous function.

Definition 2.7 ([6])**.** A function f from a fuzzy topological space (X, t) into a fuzzy topological space (Y, s) is called fuzzy continuous if and only if for every $u \in s$, $f^{-1}(u) \in t$.

Definition 2.8 ([4])**.** Let X be a nonempty set and T be a topology on X. Let $t = w(T)$ be the set of all lower semi continuous (lsc, in short) functions from (X, T) to I (with usual topology). Thus $w(T) = \{\mu \in I^X : \mu^{-1}(\alpha, 1] \in T\}$ for each $\alpha \in I_1$. It can be shown that $w(T)$ is a fuzzy topology on X.

Let P be a property of topological spaces and FP be its fuzzy topological analogue. Then FP is called a 'good extension' of P if and only if "the statement (X, T) has P if and only if $(X, w(T))$ has FP" holds good for every topological space (X, T).

Definition 2.9 ([16])**.** A fuzzy singleton x_r is said to be quasi-coincident with a fuzzy set μ, denoted by $x_r q\mu$ iff $r + \mu(x) > 1$. If x_r is not quasi-coincident with μ, we write $x_r \bar{q}\mu$.

Definition 2.10 ([11])**.** A fuzzy topological space (X, t) is called

(i) $FR_0(i)$ iff for any pair x_r, y_s in $S(X)$ with $x \neq y$, whenever $\exists u \in t$ with $x_r qu$, $y_s \bar{q}u$, then $\exists v \in t$ such that $y_s qv$, $x_r \bar{q}v$.

(ii) $FR_0(ii)$ iff for any pair x_r, y_s in $S(X)$ with $x \neq y$, whenever $\exists u \in t$ with $x_r \in u$, $y_s \bar{q}u$, then $\exists v \in t$ such that $y_s \in v$, $x_r \bar{q}v$.

(iii) $FR_0(iii)$ iff for any pair x_r, y_s in $S(X)$ with $x \neq y$, whenever $\exists u \in t$ with $x_r qu$, $y_s \cap u = 0$ then $\exists v \in t$ such that $y_s qv$, $x_r \cap v = 0$.

(iv) $FR_0(iv)$ iff for any pair x_r, y_s in $S(X)$ with $x \neq y$, whenever $\exists u \in t$ with $x_r \in u$, $y_s \subseteq u^c$ then $\exists v \in t$ such that $y_s \in v$, $x_r \subseteq v^c$.

(v) $FR_0(v)$ iff for any pair x_r, y_s in $S(X)$ with $x \neq y$, whenever $\exists u \in t$ with $x_r \in u$, $y_s \cap u = 0$ then $\exists v \in t$ such that $y_s \in v$, $x_r \cap v = 0$.

Definition 2.11 ([18])**.** Let X be any non empty set and S and T be any two general topologies on X. Then the triple (X, S, T) is called a bitopological space.

Definition 2.12 ([16])**.** A fuzzy bitopological space (fbts, in short) is a triple (X, s, t), where s and t are arbitrary fuzzy topologies on X.

Definition 2.13 ([16])**.** A fuzzy bitopological space (X, t_1, t_2) is called FPR_0 if and only if $x_t \bar{q} t_i.cl(y_r)$ implies $y_s \bar{q} t_j.cl(x_t)\,(i, j \in \{1, 2\}, i \neq j)$.

Definition 2.14 ([2])**.** A fbts (X, t_1, t_2) is said to be PFR_0 if and only if for any distinct fuzzy points p and q in X, whenever there exists $\mu \in t_i$ such that $p \in \mu$ and $q \cap \mu = 0$, then there exists $\gamma \in t_j$ such that $p \cap \gamma = 0$ and $q \in \gamma\,(i,\, j = 1, 2,\, i \neq j)$.

Definition 2.15 ([2])**.** A fbts (X, t_1, t_2) is said to be PFR_{0w} if for any $x, y \in X$, $x \neq y$, whenever there exists $\mu \in t_i$ such that $\mu(x) = 1$, $\mu(y) = 0$, then there exists $\gamma \in t_j$ such that$\gamma(x) = 0$, $\gamma(y) = 1$. $(i, j = 1, 2, i \neq j)$.

Further, let $A \subseteq X$ and $s_A = \{u/A : u \in s\}$, $t_A = \{v/A : v \in t\}$ denoted the subspace topology on A induced by s_A, t_A respectively. Then (A, s_A, t_A) is called subspace of (X, s, t) with the underlying set A.

A fuzzy bitopological property P is called hereditary if each subspace of a fbts with property P, also has property P.

Definition 2.16 ([22])**.** A function f from a fuzzy bitopological space (X, s, t) into a fuzzy bitopological space (Y, s_1, t_1) is called FP-continuous if and only if $f : (X, s) \to (Y, s_1)$ and $f : (X, t) \to (Y, t_1)$ are both fuzzy continuous.

Definition 2.17 ([32])**.** Let $\{(X_i, s_i, t_i), i \in \bigwedge\}$ be a family of fuzzy bitopological space. Then the space $(\prod X_i, \prod s_i, \prod t_i)$ is called product fuzzy bitopological space of the family $\{(X_i, s_i, t_i), i \in \bigwedge\}$, where $\prod s_i$, $\prod t_i$ respectively denote the usual product fuzzy topologies of the families $\{\prod s_i : i \in \bigwedge\}$ and $\{\prod t_i : i \in \bigwedge\}$ of the fuzzy topologies on X_i, $i \in \bigwedge$.

A fuzzy topological property P is called productive if the product of a family of fbts, each having property P, has property P.

Definition 2.18 ([18])**.** A bitopological space (X, S, T) is called pairwise-R_0 (PR_0, in short) if for all $x, y \in X$, $x \neq y$, whenever $\exists U \in S$ with $x \in U$, $y \notin U$, then $\exists V \in T$ such that $y \in V$, $x \notin V$.

Definition 2.19 ([21])**.** The initial fuzzy topology on a set X for the family of fts $\{(X_i, t_i)\}_{i \in J}$ and the family of functions $\{f_i : X \longrightarrow (X_i, t_i)\}_{i \in J}$ is smallest fuzzy topology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(u_i) : u_i \in t_i\}_{i \in J}$.

Definition 2.20 ([21])**.** The final fuzzy topology on a set X for the family of fts $\{(X_i, t_i)\}_{i \in J}$ and the family of functions $\{f_i : (X_i, t_i) \longrightarrow X\}_{i \in J}$ is finest fuzzy topology on X making each f_i fuzzy continuous.

3. Fuzzy pairwise R_0-spaces

The goal of this section is to introduce a set of new notions of fuzzy pairwise R_0-type axioms in fuzzy bitopological spaces. These new concepts are generated by using quasi-coincidence sense.

Definition 3.1. A fuzzy bitopological space (X, s, t) is called

(i) $FPR_0(i)$ iff for every pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever $\exists \mu \in s$ with $x_r q \mu$, $y_s \bar{q} \mu$, then $\exists \lambda \in t$ such that $y_s q \lambda$, $x_r \bar{q} \lambda$.

(ii) $FPR_0(ii)$ iff for every pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever $\exists \mu \in s$ with $x_r \in \mu$, $y_s \bar{q} \mu$, then $\exists \lambda \in t$ such that $y_s \in \lambda$, $x_r \bar{q} \lambda$.

(iii) $FPR_0(iii)$ iff for every pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever $\exists \mu \in s$ with $x_r q \mu$, $y_s \cap \mu = 0$ then $\exists \lambda \in t$ such that $y_s q \lambda$, $x_r \cap \lambda = 0$.

(iv) $FPR_0(iv)$ iff for every pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever $\exists \mu \in s$ with $x_r \in \mu$, $y_s \subseteq \mu^c$ then $\exists \lambda \in t$ such that $y_s \in \lambda$, $x_r \subseteq \lambda^c$.

(v) ([1]). $FPR_0(v)$ iff for every pair of fuzzy singletons x_r, y_s in X with $x \neq y$, whenever $\exists \mu \in s$ with $x_r \in \mu$, $y_s \cap \mu = 0$ then $\exists \lambda \in t$ such that $y_s \in \lambda$, $x_r \cap \lambda = 0$.

In general, it is true that union of fuzzy topologies is not a fuzzy topology. But if union of two fuzzy topologies is again a fuzzy topology, then we have the following theorem.

Theorem 3.2. *Let (X, s, t) be a fuzzy bitopological space and $(X, s \cup t)$ be a fuzzy topological space. Then the followings hold :*

(1) (X, s, t) *is* $FPR_0(i) \Rightarrow (X, s \cup t)$ *is* $FR_0(i)$.
(2) (X, s, t) *is* $FPR_0(ii) \Rightarrow (X, s \cup t)$ *is* $FR_0(ii)$.
(3) (X, s, t) *is* $FPR_0(iii) \Rightarrow (X, s \cup t)$ *is* $FR_0(iii)$.
(4) (X, s, t) *is* $FPR_0(iv) \Rightarrow (X, s \cup t)$ *is* $FR_0(iv)$.

Proof. First suppose that (X, s, t) is $FPR_0(i)$. We have to prove that $(X, s \cup t)$ is $FR_0(i)$. Let x_r, y_s be two distinct fuzzy singletons in X and $\mu \in s \cup t$ such that $x_r q\mu$, $y_s\bar{q}\mu$, that is, either $\mu \in s$ or $\mu \in t$. Suppose $\mu \in s$ with $x_r q\mu$, $y_s\bar{q}\mu$. Since (X, s, t) is $FPR_0(i)$, there exists $\lambda \in t$ such that $y_s q\lambda, x_r\bar{q}\lambda$. But if $\lambda \in t$, then $\lambda \in s \cup t$ with $y_s q\lambda$ and $x_r\bar{q}\lambda$. Thus the topological space $(X, s \cup t)$ is $FR_0(i)$.
Similarly, (2), (3) and (4) can be proved. □

For non-implications, the following examples will serve the purpose.

Example 3.3. Let $X = \{x, y\}$ and s be a fuzzy topology on X generated by $\{x_1, y_1\} \cup \{constant\}$. Again t be the indiscrete fuzzy topology on X. Then $(X, s \cup t)$ is $FR_0(i)$, $FR_0(ii)$, $FR_0(iii)$, $FR_0(iv)$ and $FR_0(v)$. On the other hand, (X, s, t) is none of the $FPR_0(i)$, $FPR_0(ii)$, $FPR_0(iii)$, $FPR_0(iv)$ and $FPR_0(v)$.

Remark: Let (X, s) and (X, t) be two fuzzy topological spaces and (X, s, t) be its corresponding fuzzy bitopological space. Then "(X, s, t) is $FPR_0(j)$" does not imply (X, s) and (X, t) are $FR_0(j)$ in general, where $j = i, ii, iii, iv, v$.

Example 3.4. Let $X = \{x, y\}$ and s be a fuzzy topology on X generated by $\{x_1\} \cup \{constant\}$. Again t be a fuzzy topology on X generated by $\{y_1\} \cup \{constant\}$. Then (X, s, t) is $FPR_0(i)$, $FPR_0(ii)$, $FPR_0(iii)$, $FPR_0(iv)$ and $FPR_0(v)$. On the other hand, (X, s) and (X, t) are none of the $FR_0(i)$, $FR_0(ii)$, $FR_0(iii)$, $FR_0(iv)$ and $FR_0(v)$.

Remark: Let (X, s) and (X, t) be two fuzzy topological spaces and (X, s, t) be its corresponding fuzzy bitopological space. Then "(X, s) and (X, t) are both $FR_0(j)$" does not imply (X, s, t) is $FPR_0(j)$ in general, where $j = i, ii, iii, iv, v$.

Example 3.5. Let $X = \{x, y\}$ and s be a fuzzy topology on X generated by $\{x_1, y_1\} \cup \{constant\}$. Again t be a fuzzy topology on X generated by $\{constant\}$. Then it is clear that (X, s) and (X, t) are both $FR_0(i)$, $FR_0(ii)$, $FR_0(iii)$, $FR_0(iv)$ and $FR_0(v)$. But on the other hand, the fuzzy bitopological space (X, s, t) is none of $FPR_0(i)$, $FPR_0(ii)$, $FPR_0(iii)$, $FPR_0(iv)$ and $FPR_0(v)$.

Now we discuss hereditary property of $FPR_0(j)$ concepts, where $(j = i, ii, iii, iv, v)$ in the following theorem.

Theorem 3.6. *Let (X, s, t) be a fuzzy bitopological space, $A \subseteq X$ and $s_A = \{u/A : u \in s\}$, $t_A = \{v/A : v \in t\}$. Then the followings hold :*
(1) *(X, s, t) is $FPR_0(i) \Rightarrow (A, s_A, t_A)$ is $FPR_0(i)$.*
(2) *(X, s, t) is $FPR_0(ii) \Rightarrow (A, s_A, t_A)$ is $FPR_0(ii)$.*
(3) *(X, s, t) is $FPR_0(iii) \Rightarrow (A, s_A, t_A)$ is $FPR_0(iii)$.*
(4) *(X, s, t) is $FPR_0(iv) \Rightarrow (A, s_A, t_A)$ is $FPR_0(iv)$.*
(5) *(X, s, t) is $FPR_0(v) \Rightarrow (A, s_A, t_A)$ is $FPR_0(v)$.*

Proof. First suppose that (X, s, t) is $FPR_0(i)$. We have to prove that (A, s_A, t_A) is $FPR_0(i)$. Let x_r *and* y_s be two distinct fuzzy singletons in A and $\mu \in s_A$ such that $x_r q\mu$ *and* $y_s\bar{q}\mu$. Clearly x_r *and* y_s are two distinct fuzzy singletons in X. Let $\mu = \gamma/A$, where $\gamma \in s$ with $x_r q\gamma$ *and* $y_s\bar{q}\gamma$. Since (X, s, t) is $FPR_0(i)$, there exists $\lambda \in t$ such that $y_s q\lambda$ *and* $x_r\bar{q}\lambda$. But by the definition, $\lambda/A \in t_A$ as $\lambda \in t$ and thus $y_s q(\lambda/A)$ *and* $x_r\bar{q}(\lambda/A)$. So the topological subspace (A, s_A, t_A) is $FPR_0(i)$.
Similarly, (2), (3), (4) and (5) can be proved. □

In the following theorem, we observe that all the $FPR_0(j)$ properties are good extension of their bitopological counter parts, where $j = i, ii, iii, iv, v$.

Theorem 3.7. *Let (X, S, T) be a bitopological space. Then (X, S, T) is PR_0 iff $(X, w(S), w(T))$ is $FPR_0(j)$, where $j = i, ii, iii, iv, v$.*

Proof. Suppose that (X, S, T) is PR_0. We have to show that $(X, w(S), w(T))$ is $FPR_0(i)$. Let $x_r, y_r \in S(X)$ with $x \neq y$ and $\mu \in w(S)$ such that $x_r q\mu$ *and* $y_r \bar{q}\mu$. Then $\mu(x) + r > 1$ and $\mu(y) + r \leq 1$, i.e., $\mu(x) > 1 - r$ and $\mu(y) \leq 1 - r$. Thus $x \in \mu^{-1}(1-r, 1]$ and $y \notin \mu^{-1}(1-r, 1]$. Since (X, S, T) is PR_0, there exists $U \in T$ such that $y \in U$ *and* $x \notin U$. So $1_U \in w(T)$ and $y_r q 1_U$, $x_r \bar{q} 1_U$. Hence $(X, w(S), w(T))$ is $FPR_0(i)$.

Conversely, suppose that $(X, w(S), w(T))$ is $FPR_0(i)$. We have to show that (X, S, T) is PR_0. Let $x, y \in X$ with $x \neq y$ and let $V \in S$ with $x \in V$ *and* $y \notin V$. Then $x_r q 1_V$ *and* $y_r \bar{q} 1_V$. Since x_r *and* $y_r \in S(X)$ with $x \neq y$ and $1_V \in w(S)$ with $x_r q 1_V$ *and* $y_r \bar{q} 1_V$, there exists $\lambda \in w(T)$ such that $y_r q\lambda$ *and* $x_r \bar{q}\lambda$. That is, $\lambda(y) + r > 1$ and $\lambda(x) + r \leq 1$, that is, $\lambda(y) > 1 - r$ and $\lambda(x) \leq 1 - r$. Thus $y \in \lambda^{-1}(1 - r, 1]$ and $x \notin \lambda^{-1}(1 - r, 1]$. So (X, S, T) is PR_0.

We can prove similar way for $j = ii, iii, iv, v$. □

In the following theorem, we show that all the FPR_0-type axioms are productive.

Theorem 3.8. *Given that $\{(X_i, s_i, t_i) : i \in \bigwedge\}$ is a family of fuzzy bitopological spaces. Then the product fbts $(\prod X_i, \prod s_i, \prod t_i)$ is $FPR_0(j)$ if each coordinate space (X_i, s_i, t_i) is $FPR_0(j)$,where $j = i, ii, iii, iv, v$.*

Proof. Suppose $(X_i, s_i, t_i) : i \in \bigwedge$ be $FPR_0(i)$. We claim that $(\prod X_i, \prod s_i, \prod t_i)$ is $FPR_0(i)$. Let $x_p, y_r \in S(\prod X_i)$ with $x \neq y$ and $u \in \prod s_i$ such that $x_p qu$ *and* $y_r \bar{q}u$. That is, $u(x) + p > 1$ and $u(x) + r \leq 1$, that is, $u(x) > 1 - p$ and $u(x) \leq 1 - r$.
But $u(x) = \min\{u_i(x_i) : i \in \bigwedge\}$ and $u(y) = \min\{u_i(y_i) : i \in \bigwedge\}$. Thus
we can find an $u_i \in s_i$ and $x_i \neq y_i$ such that $u_i(x_i) > 1 - p$ and $u_i(y_i) \leq 1 - r$. Since (X_i, s_i, t_i) is $FPR_0(i)$, there exists $v_i \in t_i$ such that $v_i(x_i) \leq 1 - p$ and $v_i(y_i) > 1 - r$. But we have $\prod_i(x) = x_i$ and $\prod_i(y) = y_i$ and thus

$$v_i(\textstyle\prod_i(x)) \leq 1 - p,\ v_i(\prod_i(y)) > 1 - r$$

and

$$v_i(\textstyle\prod_i(x)) + p \leq 1,\ v_i(\prod_i(y)) + r > 1.$$

So $x_p \bar{q}(v_i \circ \prod_i)$, $y_r q(v_i \circ \prod_i)$. Hence $(\prod X_i, \prod s_i, \prod t_i)$ is $FPR_0(i)$.

We can prove similar way for $j = ii, iii, iv, v$. □

In following two theorems, we observe here that $FPR_0(j), (j = i, ii, iii, iv, v)$ concepts are preserved under continuous, one-one, onto and open mappings.

Theorem 3.9. *Let (X, s, t) and (Y, s_1, t_1) be two fuzzy bitopological spaces and let $f : X \to Y$ be bijective, FP-continuous and FP-open. If (X, s, t) is $FPR_0(j)$, then (Y, s_1, t_1) is $FPR_0(j)$, where $j = i, ii, iii, iv, v$.*

Proof. Suppose that (X, s, t) is $FPR_0(i)$. We have to show that (Y, s_1, t_1) is $FPR_0(i)$.

Let $a_r, b_p \in S(Y)$ with $a \neq b$ and let $w \in s_1$ with $a_r q w$ *and* $b_p \bar{q} w$. That is, $w(a) + r > 1$ and $w(b) + p \leq 1$. Since f is bijective, there exist c_r, $d_p \in S(X)$ such that $f(c) = a$, $f(d) = b$ and $c \neq d$. Again since f is continuous and $w \in s_1$, $f^{-1}(w) \in s$. On the one hand,

$$f^{-1}(w)(c) + r = w(f(c) + r = w(a) + r > 1.$$

Thus $c_r q f^{-1}(w)$. Similarly, $d_r \bar{q} f^{-1}(w)$. Since c_r, d_p are two distinct fuzzy singletons in X and $f^{-1}(w) \in s$ with $c_r q f^{-1}(w)$ *and* $d_r \bar{q} f^{-1}(w)$, there exists $u \in t$ such that $d_p q u$ *and* $c_r q u$, that is, $u(d)+p > 1$ and $u(c)+r \leq 1$. Now $f(u)(b)+p = \sup u(d)+p = u(d) + p > 1$ as $f(d) = b$ and f is bijective. So $b_p q f(u)$. Similarly, $a_r \bar{q} f(u)$. Since f is FP-open, $f(u) \in t_1$. Now it is clear that, there exists $f(u) \in t_1$ such that $b_p q f(u)$ and $a_r \bar{q} f(u)$. Hence (Y, s_1, t_1) is $FPR_0(i)$.

The proofs are similar for $j = ii, iii, iv, v$. □

Theorem 3.10. *Let (X, s, t) and (Y, s_1, t_1) be two fuzzy bitopological spaces and $f : X \to Y$ be bijective, FP-continuous and FP-open. If (Y, s_1, t_1) is $FPR_0(j)$, then (X, s, t) is $FPR_0(j)$, where $j = i, ii, iii, iv, v$.*

Proof. Suppose that (Y, s_1, t_1) is $FPR_0(i)$. We have to show that (X, s, t) is $FPR_0(i)$. Let c_r, $d_p \in S(X)$ with $c \neq d$ and let $u \in s$ with $c_r q u$, $d_p \bar{q} u$. That is, $u(c) + r > 1$ and $u(d) + p \leq 1$. Now put $f(c) = a$ and $f(d) = b$. Then $a \neq b$ as f is one-one. Since f is FP-open, $f(u) \in s_1$. Since f is bijective and $f^{-1}(a) = \{c\}$, $f(u)(a) = \sup u(c) = u(c)$. Thus $f(u)(a) + r = u(c) + r > 1$. So $a_r q f(u)$. Similarly, $b_p \bar{q} f(u)$. Since (Y, s_1, t_1) is $FPR_0(i)$ and $f(u) \in s_1$ with $a_r q f(u)$ and $b_p \bar{q} f(u)$, there exists a fuzzy set $v \in t_1$ such that $b_p q v$ and $a_r \bar{q} v$. Thus $v(b) + p > 1$ and $v(a) + r \leq 1$. Now $f^{-1}(v)(c) = v(f(c)) = v(a)$. Similarly, $f^{-1}(v)(d) = v(b)$. So $f^{-1}(v)(c) + r = v(a) + r \leq 1$ and $f^{-1}(v)(d) + p = v(b) + p > 1$. Hence $d_p q f^{-1}(v)$ and $c_r \bar{q} f^{-1}(v)$. Since f is FP-continuous, $f^{-1}(v) \in t$. Therefore there exists a fuzzy set $f^{-1}(v) \in t$ such that $d_p q f^{-1}(v)$ and $c_r \bar{q} f^{-1}(v)$. Hence (X, s, t) is $FPR_0(i)$.

The proofs are similar for $j = ii, iii, iv, v$. □

Definition 3.11. The initial fuzzy bitopology on a set X for the family of fbts $\{(X_i, s_i, t_i)\}_{i \in J}$ and the family of functions $\{f_i : X \to (X_i, s_i, t_i)\}_{i \in J}$ is smallest fuzzy bitopology on X making each f_i FP-continuous.

Definition 3.12. The final fuzzy bitopology on a set X for the family of fbts $\{(X_i, s_i, t_i)\}_{i \in J}$ and the family of functions $\{f_i : (X_i, s_i, t_i) \to X\}_{i \in J}$ is finest fuzzy bitopology on X making each f_i FP-continuous.

Theorem 3.13. *If $\{(X_i, s_i, t_i)\}_{i \in J}$ is family of $FPR_0(j)$ fbts and $\{f_i : X \to (X_i, s_i, t_i)\}_{i \in J}$, a family of functions, then the initial fuzzy bitopology on X for the family $\{f_i\}_{i \in J}$ is $FPR_0(j)$, where $j = i, ii, iii, iv, v$.*

Proof. We shall prove the theorem for $j = iii$ only. Let s and t be the initial topologies on X. Let x_r, $y_s \in S(X)$ with $x \neq y$ and let a fuzzy set $u \in s$ with $x_r q u$, $u \cap y_s = 0$. That is, $u(x)+r > 1$ and $u(y) = 0$. For any $\alpha \in (0, 1-r)$, consider the fuzzy point x_α. Then $x_\alpha \in u$ and thus it is possible to find a basic fuzzy s-open set, say

$$f_{i_1}^{-1}(u_{i_1}^{\alpha}) \cap f_{i_2}^{-1}(u_{i_2}^{\alpha}) \cap \ldots \cap f_{i_n}^{-1}(u_{i_n}^{\alpha}),\ u_{i_k}^{\alpha}\ (1 \leq k \leq n)$$

being s_{i_k}-open fuzzy set such that

$$x_\alpha \in \inf f_{i_k}^{-1}(u_{i_k}^{\alpha}) \subset u. \tag{1}$$

Thus for all $\alpha \in (0, 1-r)$,

$$\alpha < \inf f_{i_k}^{-1}(u_{i_k}^{\alpha})(x) < u(x)$$

or

$$\alpha < \inf u_{i_k}^{\alpha}(f_{i_k}(x))\ \text{(for all } \alpha \in (0,1-r)).$$

SoThus

$$1 - r = \sup \inf u_{i_k}^{\alpha}(f_{i_k}(x)).$$

Now as for all $\alpha \in (0, 1-r)$,

$$u_{i_k}^{\alpha}(f_{i_k}(x)) \leq \sup u_{i_k}^{\alpha}(f_{i_k}(x)),$$

$$\inf u_{i_k}^{\alpha}(f_{i_k}(x)) \leq \inf \sup u_{i_k}^{\alpha}(f_{i_k}(x)).$$

Hence

$$1 - r = \sup \inf u_{i_k}^{\alpha}(f_{i_k}(x)) \leq \inf \sup u_{i_k}^{\alpha}(f_{i_k}(x)).$$

This implies that

$$\sup u_{i_k}^{\alpha}(f_{i_k}(x)) > 1 - r$$

for all k, $1 \leq k \leq n$. In particular,

$$\sup u_{i_1}^{\alpha}(f_{i_1}(x)) > 1 - r.$$

Now let $u_1 = \sup u_{i_1}^{\alpha}$. Then $u_1 \in s_{i_1}$ and $u_1(f_{i_1}(x)) > 1 - r$. Also as $u(y) = 0$, from (1), $u_{i_1}^{\alpha}(f_{i_1}(y)) = 0\, \forall \alpha \in (0, 1-r)$. Thus $u_1(f_{i_1}(x)) = 0$. Since $(X_{i_1}, s_{i_1}, t_{i_1})$ is $FPR_0(iii)$, then for every two fuzzy points $(f_{i_1}(x))_r$, $(f_{i_1}(y))_s$ of X_{i_1}, there exists fuzzy set $v_1 \in t_{i_1}$ such that

$$(f_{i_1}(y))_s q v_1 \text{ and } (f_{i_1}(x))_r \cap v_1 = 0.$$

Now, let $v_r = f_{i_1}^{-1}(v_1)$. Then $y_s q v_r$, for this, since $(f_{i_1}(y))_s q v_1$ we have $v_1(f_{i_1}(y)) + s > 1$, that is $f_{i_1}^{-1}(v_1)(y) + s > 1$, that is, $v_r(y) + s > 1$. Hence $y_s q v_r$. Now, we have to show that $x_r \cap v_r = 0$. Suppose $x_r \cap v_r \neq 0$. Then $v_r(x) > 0$.
But $v_r(x) = f_{i_1}^{-1}(v_1)(x) = v_1(f_{i_1}(x)) > 0$ which contradicts that $(f_{i_1}(x))_r \cap v_1 = 0$. Therefore (X, s, t) is must $FPR_0(iii)$.

Similarly, we can prove for $j = i, ii, iv, v$. □

Theorem 3.14. *If $\{(X_i, s_i, t_i)\}_{i \in J}$ is family of $FPR_0(j)$ fbts and $\{f_i : (X_i, s_i, t_i) \to X\}_{i \in J}$, a family of FP-open and bijective, then the final fuzzy bitopology on X for the family $\{f_i\}_{i \in J}$ is $FPR_0(j)$, where $j = i, ii, iii, iv, v$.*

Proof. We shall prove the above theorem for j=iii only. Let s and t be the final topologies on X for the family $\{f_i\}_{i \in J}$. Let x_r, $y_s \in S(X)$ with $x \neq y$ and let a fuzzy set $u \in s$ with $x_r q u$, $u \cap y_s = 0$. That is, $u(x) + r > 1$ and $u(y) = 0$. That is, $u(x) > 1 - r$ and $u(y) = 0$.

For any $\alpha \in (0, 1-r)$, consider the fuzzy point x_α. Then $x_\alpha \in u$ and so it is possible to find a basic fuzzy s-open set, say

$$f_{i_1}(u^\alpha_{i_1}) \cap f_{i_2}(u^\alpha_{i_2}) \cap \cdots \cap f_{i_n}(u^\alpha_{i_n}),\ u^\alpha_{i_k},\ (1 \leq k \leq n)$$

being s_{i_k}-open fuzzy set such that

$$x_\alpha \in \inf f_{i_k}(u^\alpha_{i_k}) \subset u.$$

But for all $\alpha \in (0, 1-r)$,

$$\alpha < \inf f_{i_k}(u^\alpha_{i_k})(x) < u(x)$$

or

$$1 - r = \sup \inf f_{i_k}(u^\alpha_{i_k})(x).$$

On one hand, as $\forall \alpha \in (0, 1-r)$,

$$f_{i_k}(u^\alpha_{i_k})(x) \leq \sup f_{i_k}(u^\alpha_{i_k})(x),$$

we have $\forall \alpha \in (0, 1-r)$,

$$\inf f_{i_k}(u^\alpha_{i_k})(x) \leq \inf \sup f_{i_k}(u^\alpha_{i_k})(x).$$

Thus

$$1 - r = \sup \inf f_{i_k}(u^\alpha_{i_k})(x) \leq \inf \sup f_{i_k}(u^\alpha_{i_k})(x).$$

This implies that

$$\sup f_{i_k}(u^\alpha_{i_k})(x) > 1 - r,\ k\,(1 \leq k \leq n)$$

or

$$\sup(u^\alpha_{i_k})(x_{i_k}) > 1 - r,$$

where $f_{i_k}(x_{i_k}) = x$, as f_{i_k} is bijective. In particular

$$\sup(u^\alpha_{i_1})(x_{i_1}) > 1 - r.$$

Now let $u_1 = \sup u^\alpha_{i_1}$. Then $u_1 \in s_{i_1}$ and $u_1(x_{i_1}) > 1 - r$. This implies that $(x_{i_1})_r q u_1$. Also as $u(y) = 0$, from (1), we get

$$f_{i_1}(u^\alpha_{i_1})(y) = 0,\ \forall \alpha \in (0, 1-r).$$

Thus

$$\sup f_{i_1}(u^\alpha_{i_1})(y) = 0,\ \forall \alpha \in (0, 1-r).$$

or

$$\sup(u^\alpha_{i_1})(y_{i_1}) = 0$$

where $f_{i_1}(y_{i_1}) = y$, since f_{i_1} is bijective. Hence $u_1(y_{i_1}) = 0$.

Since $(X_{i_1}, s_{i_1}, t_{i_1})$ is $FPR_0(iii)$, then for every two fuzzy points $(x_{i_1})_r$, $(y_{i_1})_s$ of X_{i_1}, there exists fuzzy set $v_1 \in t_{i_1}$ such that $(y_{i_1})_s q v_1$ and $(x_{i_1})_r \cap v_1 = 0$. That is, $v_1(y_{i_1}) > 1 - s$ and $v_1(x_{i_1}) = 0$.
Now let $v = f_{i_1}(v_1)$. Then

$$v(y) = f_{i_1}(v_1)(y) = v_1(y_{i_1}) > 1 - s,$$

where $f_{i_1}(y_{i_1}) = y$, since f_{i_1} is bijective. so, $v(y) + s > 1$. Hence $y_s q v$. Also $x_r \cap v = 0$, since

$$v(x) = f_{i_1}(v_1)(x) = v_1(x_{i_1}) = 0.$$

Therefore (X, s, t) is $FPR_0(iii)$.

Similarly, we can prove for $j = i, ii, iv, v$.

□

4. Conclusions

The concepts of fuzzy pairwise R_0-type axioms of fuzzy bitopological spaces are introduced and studied. We observed that all of our concepts are good extension of their bitopological counter parts. Further, it is clear that all notions are preserved under one-one, onto and continuous mappings. Finally initial and final properties are introduced and studied in fuzzy bitopological spaces.

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