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Applications of new soft intersection set on groups

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Abstract. Molodtsov's first initiate soft set theory which provides a general mathematical framework for handling with uncertainty. The aim of this paper is to lay a foundation for providing a new soft algebraic tool in considering many problems that contain uncertainties. In this paper, first we introduced the notion of (α, β)-soft intersection set and given several examples. Based on this notion a new kind of soft group structure, called (α, β)-SI group is developed which was made on some results of soft sets and intersection operations on sets. The main purpose of this paper is to introduce the concepts of (α, β)-SI group, from which we define (α, β)-SI subgroup and (α, β)-SI normal subgroup and illustrated by examples. Some properties of group theory are presented based on (α, β)-soft intersection sense and its application in group structures, and also established the relationship between SI-group and (α, β)-SI group. Finally, soft intersection product and soft characteristic function are introduced and given some characterization of their properties in details by means of (α, β)-SI group theory.

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1. Introduction

Theory of algebraic structure has a vital role in applications of many disciplines such as automata theory, graph theory, computer science, signal processing, quantum physics, control engineering, discrete mathematics and so on. From algebraic point of view, soft algebraic theory is a very useful mathematical tool in many applied fields. Bhowmik et al. [5], Jana et al. [16, 17, 18, 19], Bej and Pal [6] and Senapati et al. [35, 36, 37, 38, 39, 40, 41] have done lot of works on BCK/BCI-algebras and $B/BG/G$-algebras which are related to these algebras.

The classical methods are unsuitable to dealt with modeling of uncertain data in economics, engineering, environmental science, sociology and information sciences successfully, because some classical methods have inherent difficulties and that make usual theoretical approaches troublesome. In 1999, to overcome these difficulties, Molodtsov [31] introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties which is free from difficulties and also, pointed out for the applications of soft sets in several directions. Maji et al. [28, 29] described the application of soft set theory in a decision making problem and studied several operations on the theory of soft sets. Now a days, based on these theory and operations of soft sets which have a huge applications in wide variety of real life situation. We draw the attention of some selected works as, theory of soft sets [2, 7, 8, 14, 25, 27, 30, 43, 44], theory of soft sets applied in decision making problem [12, 13, 26, 32, 49], soft sets theory applied in fuzzy sets [4, 9, 20, 24, 45] and soft rough set theory [11, 15]. On the definition of soft set theory, that is parameterized family of subsets of the universal set many authors developed different soft algebraic structure. In 2007, Aktaş and Çağman [3] gave the notions of soft group which is a parameterized family of subgroups of a group. Based on this concept some researchers [1, 21, 22, 23, 33, 34, 42, 46, 47, 48] studied different soft algebraic properties.

To extend the soft set in group theory, Çağman et al. [10] introduced the notions of soft int-group and investigated its applications in group theory. Using their definition, in this paper, first defined (α, β)-soft intersection set and then introduced a new type of soft group on soft set, which we called (α, β)-soft intersectional group (abbreviated as (α, β)-SI group). This concept is based on soft set theory, intersections of sets and group theory together bring a development of new soft group structure. Moreover, (α, β)-SI group shows how soft set effects on a group in the sense of intersection. On basis of the definition of (α, β)-SI group, we introduced (α, β)-SI subgroup, (α, β)-SI normal subgroup. Finally, we then study abalian soft set, left soft coset, soft image, soft pre-image, soft intersection product and soft characteristic function on the basis of (α, β)-SI group.

2. Preliminaries

In this section, we define elementary definition and results of soft sets which are necessary for this paper taken from [7, 28, 29, 31].

Through this paper, we take U as an initial universal sets and E be the set of parameters, $P(U)$ is the power sets of U. Take $A, B, C \subseteq E$.

Definition 2.1 ([31])**.** A pair $(\mathcal{F}, E)$ is called a soft set over U if $\mathcal{F}$ is a mapping given by

$$\mathcal{F} : E \to \mathcal{P}(U).$$

Definition 2.2. For a non-empty subset A of E, a soft set $(\mathcal{F}, E)$ over U satisfying the following condition:

$$\mathcal{F}(x) = \varnothing \ for\ all \ \ x \notin A,$$

($\varnothing$ is a null set) is called A-soft set over U and is denoted by $\mathcal{F}_A$ so, an A-soft set $\mathcal{F}_A$ over U is a function $\mathcal{F}_A : E \to \mathcal{P}(U)$ such that $\mathcal{F}_A(x) = \varnothing$ for all $x \notin A$.

A soft set over U can be represented by the set of ordered pair:

$$\mathcal{F}_A = \{(x, \mathcal{F}_A(x)) : x \in E, \mathcal{F}_A(x) \in P(U)\}.$$

It is noted to see that a soft set is a parameterized family of subsets of the set U. A soft set $\mathcal{F}_A(x)$ may be arbitrary. Some of them may be empty, and some may have nonempty intersection. We denote the set of all soft sets over U by $S(U)$.

Definition 2.3. Let $\mathcal{F}_A \in S(U)$. If $\mathcal{F}_A(x) = \varnothing$ for all $x \in E$, then $\mathcal{F}_A$ is called an empty soft set and denoted by Φ_A. If $\mathcal{F}_A(x) = U$ for all $x \in A$, then $\mathcal{F}_A$ is called an A-universal soft set and denoted by $\mathcal{F}_{\widetilde{A}}$. If $\mathcal{F}_A(x) = U$ and $A = E$ for all $x \in E$, then $\mathcal{F}_{\widetilde{A}}$ is called a universal soft set and denoted by $\mathcal{F}_{\widetilde{E}}$.

Definition 2.4 ([8])**.** Let $\mathcal{F}_A, \mathcal{F}_B \in S(U)$ be two soft sets over U. Then $\mathcal{F}_A$ is soft subset of $\mathcal{F}_{\mathcal{B}}$ and is denoted by $\mathcal{F}_A \widetilde{\subseteq} \mathcal{F}_B$, if $\mathcal{F}_{\mathcal{A}}(x) \subseteq \mathcal{F}_{\mathcal{B}}(x)$ for all $x \in X$. $\mathcal{F}_A$ is called a proper subset of $\mathcal{F}_B$, denoted by $\mathcal{F}_A \widetilde{\subseteq} \mathcal{F}_B$ if $\mathcal{F}_A(x) \subseteq \mathcal{F}_B(x)$ for all $x \in E$. $\mathcal{F}_A(x) \neq \mathcal{F}_B(x)$ for at least one $x \in E$. $\mathcal{F}_A$ and $\mathcal{F}_B$ are soft equal, denoted by $\mathcal{F}_A = \mathcal{F}_B$ if and only if $\mathcal{F}_A(x) = \mathcal{F}_B(x)$ for all $x \in E$.

Definition 2.5 ([8])**.** Let $\mathcal{F}_A, \mathcal{F}_B \in S(U)$. Then intersection of two soft sets $\mathcal{F}_A$ and $\mathcal{F}_{\mathcal{B}}$ is defined by $\mathcal{F}_{\mathcal{A}} \tilde{\cap} \mathcal{F}_{\mathcal{B}} = \mathcal{F}_{\mathcal{A} \cap \mathcal{B}}$, where $\mathcal{F}_{\mathcal{A} \cap \mathcal{B}}(x) = \mathcal{F}_{\mathcal{A}}(x) \cap \mathcal{F}_{\mathcal{B}}(x)$ for all $x \in X$. The union of two soft sets $\mathcal{F}_A$ and $\mathcal{F}_{\mathcal{B}}$ is defined by $\mathcal{F}_{\mathcal{A}} \tilde{\cup} \mathcal{F}_{\mathcal{B}} = \mathcal{F}_{\mathcal{A} \cup \mathcal{B}}$, where $\mathcal{F}_{\mathcal{A} \cup \mathcal{B}}(x) = \mathcal{F}_{\mathcal{A}}(x) \cup \mathcal{F}_{\mathcal{B}}(x)$ for all $x \in X$.

Proposition 2.6 ([7])**.** *Let $\mathcal{F}_A \in S(U)$. Then*

(1) $\mathcal{F}_A \widetilde{\bigcup} \mathcal{F}_A = \mathcal{F}_A$, $\mathcal{F}_A \widetilde{\bigcap} \mathcal{F}_A = \mathcal{F}_A$.
(2) $\mathcal{F}_A \widetilde{\bigcup} \Phi_A = \mathcal{F}_A$, $\mathcal{F}_A \widetilde{\bigcap} \Phi_A = \Phi_A$.
(3) $\mathcal{F}_A \widetilde{\bigcup} \mathcal{F}_E = \mathcal{F}_E$, $\mathcal{F}_A \widetilde{\bigcap} \mathcal{F}_E = \mathcal{F}_A$.
(4) $\mathcal{F}_A \widetilde{\bigcup} \mathcal{F}^c_A = \mathcal{F}_E$, $\mathcal{F}^c_{\widetilde{A}} \widetilde{\bigcup} \mathcal{F}^c_A = \Phi_A$.

Proposition 2.7 ([7])**.** *Let $\mathcal{F}_A, \mathcal{F}_B, \mathcal{F}_C \in S(U)$. Then*

(5) $\mathcal{F}_A \widetilde{\bigcup} \mathcal{F}_B = \mathcal{F}_B \widetilde{\bigcup} \mathcal{F}_A$, $\mathcal{F}_A \widetilde{\bigcap} \mathcal{F}_B = \mathcal{F}_B \widetilde{\bigcup} \mathcal{F}_A$.
(6) $(\mathcal{F}_A \widetilde{\bigcup} \mathcal{F}_B)^c = \mathcal{F}^c_A \widetilde{\bigcap} \mathcal{F}^c_B$, $(\mathcal{F}_A \widetilde{\bigcap} \mathcal{F}_B)^c = \mathcal{F}^c_{\widetilde{A}} \widetilde{\bigcup} \mathcal{F}^c_{\widetilde{B}}$.
(7) $(\mathcal{F}_A \widetilde{\bigcup} \mathcal{F}_B) \widetilde{\bigcup} \mathcal{F}_C = \mathcal{F}_A \widetilde{\bigcup} (\mathcal{F}_B \widetilde{\bigcup} \mathcal{F}_C)$, $(\mathcal{F}_A \widetilde{\bigcap} \mathcal{F}_B) \widetilde{\bigcap} \mathcal{F}_C = \mathcal{F}_A \widetilde{\bigcap} (\mathcal{F}_B \widetilde{\bigcap} \mathcal{F}_C)$.
(8) $\mathcal{F}_A \widetilde{\bigcup} (\mathcal{F}_B \widetilde{\bigcap} \mathcal{F}_C) = (\mathcal{F}_A \widetilde{\bigcup} \mathcal{F}_B) \widetilde{\bigcap} (\mathcal{F}_A \widetilde{\bigcup} \mathcal{F}_C)$,
$\mathcal{F}_A \widetilde{\bigcap} (\mathcal{F}_B \widetilde{\bigcup} \mathcal{F}_C) = (\mathcal{F}_A \widetilde{\bigcap} \mathcal{F}_B) \widetilde{\bigcup} (\mathcal{F}_A \widetilde{\bigcap} \mathcal{F}_C)$.

Definition 2.8 ([10])**.** Let $\mathcal{F}_A, \mathcal{F}_B \in S(U)$. Then $\wedge$-product of $\mathcal{F}_A$ and $\mathcal{F}_B$ is defined by $\mathcal{F}_{A \wedge B}(x, y) = \mathcal{F}_A(x) \cap \mathcal{F}_B(y)$, denoted by $\mathcal{F}_A \wedge \mathcal{F}_B$. The $\vee$-product of $\mathcal{F}_A$ and $\mathcal{F}_B$ is defined by $\mathcal{F}_{A \vee B}(x, y) = \mathcal{F}_A(x) \vee \mathcal{F}_B(y)$, denoted by $\mathcal{F}_A \vee \mathcal{F}_B$ for all $x, y \in E$.

Proposition 2.9. *Let $\mathcal{F}_A, \mathcal{F}_B, \mathcal{F}_C \in S(U)$. Then*

(9) $(\mathcal{F}_A \wedge \mathcal{F}_B) \vee \mathcal{F}_C = \mathcal{F}_A \vee (\mathcal{F}_B \vee \mathcal{F}_C)$.
(10) $(\mathcal{F}_A \vee \mathcal{F}_B) \vee \mathcal{F}_C = \mathcal{F}_A \vee (\mathcal{F}_B \vee \mathcal{F}_C)$.

Definition 2.10 ([10])**.** Let G be a group and $\mathcal{F}_G \in S(U)$. Then $\mathcal{F}_G$ is called soft intersection groupoid over U if

$$\mathcal{F}_G(xy) \supseteq \mathcal{F}_G(x) \cap \mathcal{F}_G(y),$$

for all $x, y \in G$. If $\mathcal{F}_G$ is soft intersection group over U if the soft intersection groupoid satisfied $\mathcal{F}_G(x^{-1}) = \mathcal{F}_G(x)$, for all $x \in G$. For simplicity, soft intersection group as soft int-group.

Definition 2.11 ([10])**.** Let H be a subgroup of the group G, $\mathcal{F}_G$ be a soft int-group over U and $\mathcal{F}_H$ be nonempty soft subset of $\mathcal{F}_G$ over U. If $\mathcal{F}_H$ is an soft int-group over U, then $\mathcal{F}_H$ is called soft int-subgroup of $\mathcal{F}_G$ over U, which is denoted by $\mathcal{F}_H \leq \mathcal{F}_G$.

Definition 2.12 ([10])**.** Let $\mathcal{F}_G$ be a soft int-group over U and $\mathcal{F}_N$ be a soft int-subgroup of $\mathcal{F}_G$ over U. Then, $\mathcal{F}_N$ is called normal soft int-subgroup of $\mathcal{F}_G$ over U, denoted by $\mathcal{F}_N \diamondsuit \mathcal{F}_G$, it is an abelian soft subset of $\mathcal{F}_G$ over U.

Definition 2.13 ([10])**.** Let G be a group and soft set $\mathcal{F}_G$ be not necessarily a soft int-group over U. Let N be a subgroup of G and $\mathcal{F}_N$ be a nonempty soft subset of $\mathcal{F}_G$ over U. Then, $\mathcal{F}_N$ is called an abelian soft subset of $\mathcal{F}_G$ over U if $\mathcal{F}_N(xy) = \mathcal{F}_N(yx)$ for all $x, y \in G$.

Definition 2.14 ([10])**.** Let $\mathcal{F}_G$ be a soft set over U. The e-soft set of $\mathcal{F}_G$ is defined by

$$\mathcal{F}_{G(e)} = \{x \in G | \mathcal{F}_G(x) = \mathcal{F}_G(e)\}.$$

3. (α, β)-soft intersectional set

In this section, let U be the initial universe set and E be the set of parameters, and " $\circledast$ " be the binary operation. We take $S(U)$ be the set of all soft sets. From now let, $\varnothing \subseteq \alpha \subset \beta \subseteq U$.

Definition 3.1. For any non-empty subset A of E, the soft set $\mathcal{F}_A \in S(U)$. Then, for all $x, y \in A$, the soft set $\mathcal{F}_A$ is called an (α, β)-soft intersectional set over U if it satisfies the condition:

$$\mathcal{F}_A(x \circledast y) \cup \alpha \supseteq \mathcal{F}_A(x) \cap \mathcal{F}_A(y) \cap \beta.$$

Example 3.2. We consider five houses in the initial universal set U which is given by

$$U = \{h_1, h_2, h_3, h_4, h_5\}.$$

Let the set of parameters set $E = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ be the status of the set of houses which follows for the parameters "cheap", "expensive", "in the flooded area" and "in urban area" respectively, with following binary operation

$\circledast$	η_1	η_2	η_3	η_4
η_1	η_1	η_2	η_3	η_4
η_2	η_2	η_1	η_4	η_3
η_3	η_3	η_4	η_3	η_4
η_4	η_4	η_3	η_4	η_3

(1) For a subset $A = \{\eta_1, \eta_3, \eta_4\}$ of E, consider a soft set $(\mathcal{F}_A, A)$ over U defined as follows:
$\mathcal{F}_A(\eta_1) = \{h_1, h_3, h_4, h_5\}$, $\mathcal{F}_A(\eta_3) = \{h_1, h_3, h_5\}$ and $\mathcal{F}_A(\eta_4) = \{h_1, h_2, h_4, h_5\}$. Then it is examine that $(\mathcal{F}_A, A)$ is an (α, β)-soft intersectional set over U for $\beta = \{h_1, h_2, h_3, h_5\}$ and $\alpha = \{h_1, h_2, h_3\}$.

(2) Let $B = \{\eta_1, \eta_2, \eta_3\}$, then soft set $(\mathcal{F}_B, B)$ over U is defined as $\mathcal{F}_B(\eta_1) = \{h_1, h_2, h_3, h_4, h_5\}$, $\mathcal{F}_B(\eta_2) = \{h_2, h_5\}$, $\mathcal{F}_B(\eta_3) = \{h_1, h_4, h_5\}$ and $\mathcal{F}_B(\eta_4) = \varnothing$ is an (α, β)-soft intersectional set over U for $\beta = \{h_1, h_2, h_3, h_5\}$ and $\alpha = \{h_1, h_2, h_5\}$.

(3) The soft set $(\mathcal{F}_H, H)$, where $H = \{\eta_1, \eta_2, \eta_3\}$ is a subset of E defined by $\mathcal{F}_H(\eta_1) = \{h_1, h_2, h_3, h_4, h_5\}$, $\mathcal{F}_H(\eta_2) = \{h_2, h_3, h_5\}$, $\mathcal{F}_H(\eta_3) = \{h_1, h_3, h_4, h_5\}$ and $\mathcal{F}_A(\eta_4) = \varnothing$ is not an (α, β)-soft intersectional set over U, where $\beta = \{h_2, h_3, h_4, h_5\}$ and $\alpha = \{h_2, h_4, h_5\}$, because $\mathcal{F}_H(\eta_2) \cap \mathcal{F}_H(\eta_3) \cap \beta = \{h_3, h_5\} \nsubseteq \{h_2, h_4, h_5\} = \mathcal{F}_H(\eta_2 \divideontimes \eta_3) \cup \alpha$.

4. Applications of (α, β)-soft intersection set on group

In this section, we first define (α, β)-SI group and then (α, β)-SI subgroup, (α, β)-SI normal subgroup and characterized their basic properties. Through out this paper, let G be a arbitrary group and e be the identity element.

Definition 4.1. Let G be a group and $\mathcal{F}_G \in S(U)$. Then $\mathcal{F}_G$ is called a (α, β)-soft intersection groupoid over U if

$$\mathcal{F}_G(xy) \cup \alpha \widetilde{\supseteq} \mathcal{F}_G(x) \cap \mathcal{F}_G(y) \cap \beta,$$

for all $x, y \in G$. If $\mathcal{F}_G$ is an (α, β)-soft intersection group over U if the (α, β)-soft intersection groupoid satisfied $\mathcal{F}_G(x^{-1}) = \mathcal{F}_G(x)$, for all $x \in G$. For simplicity, we take (α, β)-soft intersection group as (α, β)-SI group.

We consider the order relation "$\widetilde{\subseteq}_{(\alpha,\beta)}$" on $S(U)$ as a manner: for any $\mathcal{F}_E, \mathcal{G}_E \in S(U)$ and $\varnothing \subseteq \alpha \subset \beta \subseteq U$. we define $\mathcal{F}_E \widetilde{\subseteq}_{(\alpha,\beta)} \mathcal{G}_E \Leftrightarrow \mathcal{F}_E \cap \beta \widetilde{\subseteq} \mathcal{G}_E \cup \alpha$. We define a relation "$=_{(\alpha,\beta)}$" such as $\mathcal{F}_E =_{(\alpha,\beta)} \mathcal{G}_E \Leftrightarrow \mathcal{F}_E \widetilde{\subseteq}_{(\alpha,\beta)} \mathcal{G}_E$ and $\mathcal{G}_E \widetilde{\subseteq}_{(\alpha,\beta)} \mathcal{F}_E$. Using the above notion, (α, β)- SI group defined as follows.

Definition 4.2. Let G be a group and $\mathcal{F}_G \in S(U)$. Then $\mathcal{F}_G$ is called a (α, β)-SI group over U if

$$\mathcal{F}_G(xy) \widetilde{\supseteq}_{(\alpha,\beta)} \mathcal{F}_G(x) \cap \mathcal{F}_G(y),$$

for all $x, y \in G$.

Example 4.3. Assume that $U = Z$ (set of integers) is the universal set and $G = Z_6$ is the subset of set of parameters. Let $\alpha = \{0, 1, 3, 6, 7\}$ and $\beta = \{0, 1, 3, 6, 7, 10, 11\}$. We define the soft set $\mathcal{F}_G$ over U as follows:
$\mathcal{F}_G(0) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,
$\mathcal{F}_G(1) = \{0, 2, 4, 6, 8\}$,
$\mathcal{F}_G(2) = \{1, 3, 5, 6, 7\}$,
$\mathcal{F}_G(3) = \{0, 2, 4, 7, 9, 11\}$,
$\mathcal{F}_G(4) = \{1, 3, 4, 5, 6\}$,
$\mathcal{F}_G(5) = \{0, 2, 4, 6, 8\}$.
Then, we can easily check that $\mathcal{F}_G$ is an (α, β)-SI group of G over U.

Theorem 4.4. *Let $\mathcal{F}_G$ be (α, β)-SI group over U. Then $\mathcal{F}_G(e) \widetilde{\supseteq}_{(\alpha,\beta)} \mathcal{F}_G(x)$, for all $x \in G$.*

Proof. Since $\mathcal{F}_G$ is an (α,β)-SI group over U,

$$\begin{aligned}\mathcal{F}_G(e)\cap\alpha &= \mathcal{F}_G(xx^{-1})\cup\alpha\\ &\supseteq \mathcal{F}_G(x)\cap\mathcal{F}_G(x^{-1})\cap\beta\\ &= \mathcal{F}_G(x)\cap\mathcal{F}_G(x)\cap\beta\\ &= \mathcal{F}_G(x)\cap\beta.\end{aligned}$$

Thus $\mathcal{F}_G(x)\widetilde{\subseteq}_{(\alpha,\beta)}\mathcal{F}_G(e)$ for all $x\in G$. □

Theorem 4.5. *If a soft $\mathcal{F}_A$ over U is an (α,β)-SI group, then*

$$(\mathcal{F}_A(e)\cap\beta)\cup\alpha\supseteq(\mathcal{F}_A(x)\cap\beta)\cup\alpha,$$

for all $x\in G$.

Proof. The proof of the theorem is obvious. □

Theorem 4.6. *A soft set $\mathcal{F}_G$ is an (α,β)-SI group over U if and only if*

$$\mathcal{F}_G(x)\cap\mathcal{F}_G(y)\widetilde{\subseteq}_{(\alpha,\beta)}\mathcal{F}_G(xy^{-1}),$$

for all $x,y\in G$.

Proof. Let us assume that $\mathcal{F}_G$ is an (α,β)-SI group over U. Then

$$\mathcal{F}_G(xy^{-1})\cup\alpha\supseteq\mathcal{F}_G(x)\cap\mathcal{F}_G(y^{-1})\cap\beta=\mathcal{F}_G(x)\cap\mathcal{F}_G(y)\cap\beta,$$

for all $x,y\in G$.

Conversely, suppose that $\mathcal{F}_G(xy^{-1})\cup\alpha\supseteq\mathcal{F}_G(x)\cap\mathcal{F}_G(y)\cap\beta$ for all $x,y\in G$. Let us choose $x=e$ which produced $\mathcal{F}_G(y^{-1})\cup\alpha\supseteq\mathcal{F}_G(y)\cap\beta$. Again, $\mathcal{F}_G(y)\cup\alpha=\mathcal{F}_G((y^{-1})^{-1})\cup\alpha\supseteq\mathcal{F}_G(y^{-1})\cap\beta$. Thus $\mathcal{F}_G(y)=\mathcal{F}_G(y^{-1})$ is hold only when $\alpha=\beta$. So $\mathcal{F}_G(xy)\cup\alpha=\mathcal{F}_G(x(y^{-1})^{-1})\cup\alpha\supseteq\mathcal{F}_G(x)\cap\mathcal{F}_G(y^{-1})\cap\beta=\mathcal{F}_G(x)\cap\mathcal{F}_G(y)\cap\beta$. Hence $\mathcal{F}_G(x)\cap\mathcal{F}_G(y)\widetilde{\subseteq}_{(\alpha,\beta)}\mathcal{F}_G(xy^{-1})$ for all $x,y\in G$. □

Theorem 4.7. *Let $\mathcal{F}_G$ be an (α,β)-SI group over U and $x\in G$. Then*

$$\mathcal{F}_G(y)\widetilde{\subseteq}_{(\alpha,\beta)}\mathcal{F}_G(xy) \text{ for all } y\in G \text{ if and only if } \mathcal{F}_G(x)=\mathcal{F}(e).$$

Proof. Let $\mathcal{F}_G(xy)\cup\alpha\supseteq\mathcal{F}_G(y)$ for all $y\in G$. We choose $y=e$. Then produced $\mathcal{F}_G(x)\cup\alpha\supseteq\mathcal{F}_G(e)$. thus, from Theorem 4.4, $\mathcal{F}_G(x)=\mathcal{F}_G(e)$ hold only when $\alpha=\beta$.

Conversely, let $\mathcal{F}_G(x)=\mathcal{F}_G(e)$. Then

$$\begin{aligned}\mathcal{F}_G(xy)\cup\alpha &\supseteq \mathcal{F}_G(x)\cap\mathcal{F}_G(y)\cap\beta\\ &= \mathcal{F}_G(e)\cap\mathcal{F}_G(y)\cap\beta\\ &= \mathcal{F}_G(y)\cap\beta.\end{aligned}$$

□

Theorem 4.8. *Let $\mathcal{F}_G$ be an (α,β)-SI group over U. Then, for $n\in N$, $\mathcal{F}_G(x^n)\cup\alpha\widetilde{\supseteq}\mathcal{F}_G(x)\cap\beta$ for all $x\in X$.*

Proof. It is obvious by the method of induction and Definition 4.1. □

Theorem 4.9. *Let $\mathcal{F}_G$ be an (α,β)-SI group over U. If $\mathcal{F}_G(xy^{-1})=\mathcal{F}_G(e)$, then $\mathcal{F}_G(x)=_{(\alpha,\beta)}\mathcal{F}_G(y)$ for all $x,y\in G$ over U.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned}\mathcal{F}_G(x) \cup \alpha &= \mathcal{F}_G((xy^{-1})y) \cup \alpha\\ &\supseteq \mathcal{F}_G(xy^{-1}) \cap \mathcal{F}_G(y) \cap \beta\\ &= (\mathcal{F}_G(e) \cap \mathcal{F}_G(y)) \cap \beta\\ &= \mathcal{F}_G(y) \cap \beta\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_G(y) \cup \alpha &= \mathcal{F}_G(y^{-1}) \cup \alpha\\ &= \mathcal{F}_G(x^{-1}(xy^{-1})) \cup \alpha\\ &\supseteq (\mathcal{F}_G(x^{-1}) \cap \mathcal{F}_G(xy^{-1})) \cap \beta\\ &= \mathcal{F}_G(x^{-1}) \cap \mathcal{F}_G(e) \cap \beta\\ &= \mathcal{F}_G(x) \cap \beta.\end{aligned}$$

Thus $\mathcal{F}_G(x) =_{(\alpha,\beta)} \mathcal{F}_G(y)$ holds over U. □

Theorem 4.10. *If $\mathcal{F}_G$ is an (α, β)-SI group over U and $H \leq G$, then restriction $\mathcal{F}_{G|H}$ is an (α, β)-SI group over the parameter set H.*

Proof. As $H \leq G$ and $\mathcal{F}_G(xy^{-1}) \cup \alpha \supseteq \mathcal{F}_G(x) \cap \mathcal{F}_G(y) \cap \beta$ for $x, y \in H$. Let us define $\mathcal{F}_H(x) = \mathcal{F}_G(x)$ for all $x \in H$. Since H is a group, then $xy^{-1} \in H$ for all $x, y \in H$. If $x, y \in H$, then $\mathcal{F}_H(xy^{-1}) \cup \alpha = \mathcal{F}_G(xy^{-1}) \cup \alpha \supseteq \mathcal{F}_G(x) \cap \mathcal{F}_G(y) \cap \beta = \mathcal{F}_H(x) \cap \mathcal{F}_H(y) \cap \beta$. Thus $\mathcal{F}_H$ is an (α, β)-SI group over the parameter set H. □

Theorem 4.11. *Let $\mathcal{F}_G$ and $\mathcal{F}_H$ be two (α, β)-SI group over U. Then $\mathcal{F}_G \wedge \mathcal{F}_H$ is an (α, β)-SI group over U.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in G \times H$. Then

$$\begin{aligned}\mathcal{F}_{G\wedge H}((x_1, y_1)(x_2, y_2)^{-1}) \cup \alpha &= \mathcal{F}_{G\wedge H}(x_1x_2^{-1}, y_1y_2^{-1}) \cup \alpha\\ &= \{\mathcal{F}_G(x_1x_2^{-1}) \cap \mathcal{F}_H(y_1y_2^{-1})\} \cup \alpha\\ &= \{\mathcal{F}_G(x_1x_2^{-1}) \cup \alpha\} \cap \{\mathcal{F}_H(y_1y_2^{-1}) \cup \alpha\}\\ &\supseteq \{\mathcal{F}_G(x_1) \cap \mathcal{F}_G(x_2^{-1}) \cap \beta\} \cap \{\mathcal{F}_H(y_1) \cap \mathcal{F}_H(y_2^{-1}) \cap \beta\}\\ &= \{\mathcal{F}_G(x_1) \cap \mathcal{F}_G(x_2)\} \cap \{\mathcal{F}_H(y_1) \cap \mathcal{F}_H(y_2)\} \cap \beta\\ &= \{\mathcal{F}_G(x_1) \cap \mathcal{F}_H(x_2)\} \cap \{\mathcal{F}_G(x_2) \cap \mathcal{F}_H(y_2)\} \cap \beta\\ &= \{\mathcal{F}_{G\wedge H}(x_1, x_2) \cap \mathcal{F}_{G\wedge H}(x_2, y_2)\} \cap \beta.\end{aligned}$$

Thus $\mathcal{F}_G \wedge \mathcal{F}_H$ is an (α, β)-SI group over U. □

Example 4.12. Let $U = S_3$ (symmetric 3-group) be the universal set. Let $G = Z_6$ and $H = \{1, -1, i, -i\}$ be the subsets of set of parameters. For $\beta = \{(1), (12), (13), (123)\}$ and $\alpha = \{(13), (123)\}$ are permutation.
We define (α, β)-SI group $\mathcal{F}_G$ as follows:

$\mathcal{F}_G(0) = S_3$,
$\mathcal{F}_G(1) = \{(12), (13), (132)\}$,
$\mathcal{F}_G(2) = \{(12), (13), (123), (132)\}$,
$\mathcal{F}_G(3) = \{(1), (12), (13), (132)\}$,
$\mathcal{F}_G(4) = \{(12), (13), (23), (132), (123)\}$,

$\mathcal{F}_G(5) = \{(1), ((12), (13), (132)\}$.
And we define (α, β)-SI group $\mathcal{F}_H$ as follows:
$\mathcal{F}_H(1) = S_3$,
$\mathcal{F}_H(-1) = \{(12), (23), (123), (132)\}$,
$\mathcal{F}_H(i) = \{(1), (12), (23), (132)\}$,
$\mathcal{F}_H(-i) = \{(12), (23), (132)\}$.
It is easily verify that $\mathcal{F}_G \wedge \mathcal{F}_H$ is an (α, β)-SI group over U.
But, it is justified that $\mathcal{F}_G \vee \mathcal{F}_H$ is not an (α, β)-SI group over U, because
$\mathcal{F}_{G\vee H}((2, i)(5, -1)) \cup \alpha \widetilde{\nsupseteq} (\mathcal{F}_{G\vee H}(2, i) \cap \mathcal{F}_{G\vee H}(5, -1)) \cap \beta$.

Definition 4.13. Let $\mathcal{F}_{G_1}$, $\mathcal{F}_{G_2}$ be two (α, β)-SI groups over U. Then, the product of (α, β)-SI groups $\mathcal{F}_{G_1}$, $\mathcal{F}_{G_2}$ is defined by $\mathcal{F}_{G_1} \times \mathcal{F}_{G_2} = \mathcal{F}_{G_1\times G_2}$. Then $\mathcal{F}_{G_1\times G_2}(x, y) = \mathcal{F}_{G_1}(x) \times \mathcal{F}_{G_2}(y)$ for all $(x, y) \in G_1 \times G_2$.

Proposition 4.14. *If $\mathcal{F}_{G_1}$, $\mathcal{F}_{G_2}$ are (α, β)-SI groups over U, then $\mathcal{F}_{G_1} \times \mathcal{F}_{G_2}$ is so over $U \times U$.*

Proof. For all $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2$ and by using the Definition 4.13

$$\begin{aligned}
\mathcal{F}_{G_1\times G_2}((x_1, y_1)(x_2, y_2)^{-1}) \cup \alpha &= \mathcal{F}_{G_1\times G_2}(x_1x_2^{-1}, y_1y_2^{-1}) \\
&= (\mathcal{F}_{G_1}(x_1y_2^{-1}) \times \mathcal{F}_{G_2}(x_2y_2^{-1})) \cup \alpha \\
&\supseteq ((\mathcal{F}_{G_1}(x_1) \cap \mathcal{F}_{G_1}(x_2^{-1})) \times \mathcal{F}_{G_1}(y_1) \cap \mathcal{F}_{G_1}(y_2^{-1})) \cap \beta \\
&= ((\mathcal{F}_{G_1}(x_1) \cap \mathcal{F}_{G_1}(x_2)) \times \mathcal{F}_{G_1}(y_1) \cap \mathcal{F}_{G_1}(y_2)) \cap \beta \\
&= ((\mathcal{F}_{G_1}(x_1) \times \mathcal{F}_{G_2}(x_2)) \cap (\mathcal{F}_{G_1}(y_1) \times \mathcal{F}_{G_2}(y_2))) \cap \beta \\
&= (\mathcal{F}_{G_1\times G_2}(x_1, x_2) \cap \mathcal{F}_{G_1\times G_2}(y_1, y_2)) \cap \beta.
\end{aligned}$$

Thus $\mathcal{F}_{G_1} \times \mathcal{F}_{G_2} = \mathcal{F}_{G_1\times G_2}$ is an (α, β)-SI group over $U \times U$. □

Theorem 4.15. *Let $\mathcal{F}_G$ and $\mathcal{H}_G$ be two (α, β)-SI groups over U. Then, $\mathcal{F}_G \widetilde{\cap} \mathcal{H}_G$ is an (α, β)-SI group over U.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned}
(\mathcal{F}_G \widetilde{\cap} \mathcal{H}_G)(xy^{-1}) \cup \alpha &= (\mathcal{F}_G(xy^{-1}) \cup \alpha) \cap (\mathcal{H}_G(xy^{-1} \cup \alpha) \\
&\supseteq (\mathcal{F}_G(x) \cap \mathcal{F}_G(y^{-1}) \cap \beta) \cap (\mathcal{H}_G(x) \cap \mathcal{H}_G(y^{-1}) \cap \beta) \\
&= ((\mathcal{F}_G(x) \cap \mathcal{F}_G(y)) \cap (\mathcal{H}_G(x) \cap \mathcal{H}_G(y))) \cap \beta \\
&= ((\mathcal{F}_G(x) \cap \mathcal{H}_G(x)) \cap (\mathcal{F}_G(y) \cap \mathcal{H}_G(y))) \cap \beta \\
&= (\mathcal{F}_G \widetilde{\cap} \mathcal{H}_G)(x) \cap (\mathcal{F}_G \widetilde{\cap} \mathcal{H}_G)(y) \cap \beta.
\end{aligned}$$

Thus $\mathcal{F}_G \widetilde{\cap} \mathcal{H}_G$ is an (α, β)-SI group. But $\mathcal{F}_G \widetilde{\cup} \mathcal{H}_G$ is not an (α, β)-SI group. □

The above theorem is verified in the following example.

Example 4.16. Let $U = Z$ (set of integers) be the universal set and $G = Z_6$ is the subset of set of parameters.
Let $\beta = \{0, 1, 4, 6, 7, 10, 13, 14\}$, $\alpha = \{0, 4, 6, 10\}$ where, $\varnothing \subseteq \alpha \subset \beta \subseteq U$.
Define (α, β)-SI group $\mathcal{F}_G$ as follows:
$\mathcal{F}_G(0) = Z$,
$\mathcal{F}_G(1) = \{0, 1, 4, 5\}$,
$\mathcal{F}_G(2) = \{0, 1, 4, 5, 11\}$,

$\mathcal{F}_G(3) = \{0, 4, 5, 12, 13\}$,
$\mathcal{F}_G(4) = \{0, 1, 4, 11\}$,
$\mathcal{F}_G(5) = \{0, 1, 4, 5\}$.
Now, define (α, β)-SI group $\mathcal{H}_G$ as follows:
$\mathcal{H}_G(0) = Z$,
$\mathcal{F}_G(1) = \{6, 7\}$,
$\mathcal{F}_G(2) = \{6, 7, 10, 13\}$,
$\mathcal{F}_G(3) = \{6, 7, 8, 9\}$,
$\mathcal{F}_G(4) = \{6, 7, 10, 13\}$,
$\mathcal{F}_G(5) = \{6, 7\}$.
It is clearly proved that $\mathcal{F}_G \widetilde{\cap} \mathcal{H}_G$ is an (α, β)-SI group of G over U. But, $\mathcal{F}_G \widetilde{\cup} \mathcal{H}_G$ is not an (α, β)-SI group of G over U, because $\mathcal{F}_G \widetilde{\cup} \mathcal{H}_G)(2+3) \cup \alpha \not\supseteq (\mathcal{F}_G \widetilde{\cup} \mathcal{H}_G)(2) \cap \mathcal{F}_G \widetilde{\cup} \mathcal{H}_G)(3)) \cap \beta$.

Theorem 4.17. *Let $G_i \leq G$ for all $i \in I_g$ (I_g is an index set) and $\{\mathcal{F}_{G_i} | i \in I_g\}$ be a family of (α, β)-SI groups over U. Then, $\bigcap\limits_{i \in I_g} \mathcal{F}_{G_i}$ is an (α, β)-SI group over U.*

Proof. We prove in Theorem 4.15 that the intersection of two (α, β)-SI groups over U is an (α, β)-SI group over U. Now, we prove general case, let $x, y \in G$. Then, by using Definition 4.1,

$$\begin{aligned} \bigcap_{i \in I_g} \mathcal{F}_{G_i}(xy^{-1}) \cup \alpha &= \bigcap \{\mathcal{F}_{G_i}(xy^{-1}) | i \in I_g\} \cup \alpha \\ &\supseteq \bigcap \{\mathcal{F}_{G_i}(x) \cap \mathcal{F}_{G_i}(y) | i \in I_g\} \cap \beta \\ &= (\bigcap_{i \in I_g} \mathcal{F}_{G_i}(x)) \cap (\bigcap_{i \in I_g} \mathcal{F}_{G_i}(y)) \cap \beta. \end{aligned}$$

Thus the proof of the theorem is completed. □

Lemma 4.18. *Let $\mathcal{F}_G$ be an (α, β)-SI group over U such that either $\mathcal{F}_G(x) \subseteq \mathcal{F}_G(y)$ or $\mathcal{F}_G(x) \supseteq \mathcal{F}_G(y)$ for any $x, y \in G$. If $\mathcal{F}_G(x) \neq \mathcal{F}_G(y)$, then $\mathcal{F}_G(xy) =_{(\alpha,\beta)} \mathcal{F}_G(x) \cap \mathcal{F}_G(y)$ for any $x, y \in G$.*

Proof. If $\mathcal{F}_G(x) \neq \mathcal{F}_G(y)$, then either $\mathcal{F}_G(x) \subset \mathcal{F}_G(y)$ or $\mathcal{F}_G(x) \supset \mathcal{F}_G(y)$.
Suppose that

$$(A) \qquad \mathcal{F}_G(x) \subset \mathcal{F}_G(y).$$

Then

$$\mathcal{F}_G(x) \cup \alpha = \mathcal{F}_G((xy)y^{-1}) \cup \alpha \supseteq (\mathcal{F}_G(xy) \cap \mathcal{F}_G(y^{-1})) \cap \beta = \mathcal{F}_G(xy) \cap \mathcal{F}_G(y) \cap \beta.$$

Thus

$$(B) \qquad \mathcal{F}_G(x) \cup \alpha \supseteq \mathcal{F}_G(xy) \cap \mathcal{F}_G(y) \cap \beta.$$

From (A) and (B),

$$\mathcal{F}_G(x) \cup \alpha \supseteq \mathcal{F}_G(xy) \cap \mathcal{F}_G(y) \cap \beta \supseteq \mathcal{F}_G(xy) \cap \beta = \mathcal{F}_G(x) \cap \mathcal{F}_G(y) \cap \beta = \mathcal{F}_G(x) \cap \beta.$$

That is

$$(C) \qquad \mathcal{F}_G(x) \cup \alpha \supseteq \mathcal{F}_G(x) \cap \beta.$$

So, all expression are equal by (C). Hence $\mathcal{F}_G(xy) =_{(\alpha,\beta)} \mathcal{F}_G(x) \cap \mathcal{F}_G(y)$. The proof is similar to other case. □

Theorem 4.19. *Every SI-group of G over U is an (α, β)-SI group of G over U for arbitrary $\varnothing \subseteq \alpha \subset \beta \subseteq U$, but converse is not true.*

The proof of the Theorem 4.19 is established by the following example.

Example 4.20. Let $K_4 = \{e, a, b, c\}$ (Klein four-group) be the universal set and $G = Z_6$ is the subset of set of parameters. Let $\beta = \{e, a, c\}$ and $\alpha = \{e, c\}$ Now, define soft set as follows:
$\mathcal{F}_G(0) = \{e, a\}$,
$\mathcal{F}_G(1) = \{a, c\}$,
$\mathcal{F}_G(2) = \{a, b, c\}$,
$\mathcal{F}_G(3) = \{a, b\}$,
$\mathcal{F}_G(4) = \{a, b, c\}$
and
$\mathcal{F}_G(5) = \{a, c\}$. It is justified that $\mathcal{F}_G$ is an (α, β)-SI group of G over U, but it is not an SI-group of G over U because $\mathcal{F}_G(2+4) = \{e, a\} \not\supseteq \{a, b, c\} = \mathcal{F}_G(2) \cap \mathcal{F}_G(4)$.

5. (α, β)-SI SUBGROUPS

In this section, we define (α, β)-SI subgroups, (α, β)-SI normal subgroups and characterized some of their properties, and also study e-soft set, soft left coset by means of (α, β)-SI group.

Definition 5.1. Let H be a subgroup of the group G, $\mathcal{F}_G$ be a (α, β)-SI group over U and $\mathcal{F}_H$ be nonempty soft subset of $\mathcal{F}_G$ over U. If $\mathcal{F}_H$ is an (α, β)-SI group over U, $\mathcal{F}_H$ is called (α, β)-SI subgroup of $\mathcal{F}_G$ over U, which is denoted by $\mathcal{F}_H \leq \mathcal{F}_G$.

Example 5.2. Let us consider the (α, β)-SI group $\mathcal{F}_G$ over U given in Example 4.12, and let $H = \{0, 2, 4\}$ be the subset of set of parameters. Now define soft set $\mathcal{F}_H$ as
$\mathcal{F}_H(0) = S_3$,
$\mathcal{F}_H(2) = \{((12), 123), (132)\}$,
$\mathcal{F}_H(4) = \{(13), (123), (132)\}$.
Now, for $\beta = \{(1), (12), (13), (123)\}$ and $\alpha = \{(13), (123)\}$.
It is easily check that $\mathcal{F}_H$ is an (α, β)-SI subgroup of $\mathcal{F}_G$ over U.

Theorem 5.3. *Let $\mathcal{F}_G$ be an (α, β)-SI group over U, and $\mathcal{F}_H$, $\mathcal{F}_M$ be two (α, β)-SI subgroups of $\mathcal{F}_G$ over U. Then $\mathcal{F}_H \widetilde{\cap} \mathcal{F}_M$ is an (α, β)-SI subgroup of $\mathcal{F}_G$ over U.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned}(\mathcal{F}_{H\tilde{\cap}M}(xy^{-1})) \cup \alpha &= (\mathcal{F}_{H\tilde{\cap}M}(xy^{-1}) \cup \alpha) \cap (\mathcal{F}_{H\tilde{\cap}M}(xy^{-1} \cup \alpha)\\ &\supseteq (\mathcal{F}_H(x) \cap \mathcal{F}_H(y^{-1}) \cap \beta) \cap (\mathcal{F}_M(x) \cap \mathcal{F}_M(y^{-1}) \cap \beta)\\ &= ((\mathcal{F}_H(x) \cap \mathcal{F}_H(y)) \cap (\mathcal{F}_M(x) \cap \mathcal{F}_M(y))) \cap \beta\\ &= ((\mathcal{F}_H(x) \cap \mathcal{F}_M(x)) \cap (\mathcal{F}_H(y) \cap \mathcal{F}_M(y))) \cap \beta\\ &= (\mathcal{F}_{H\tilde{\cap}M}(x) \cap \mathcal{F}_{H\tilde{\cap}M}(y)) \cap \beta.\end{aligned}$$

Thus $\mathcal{F}_{H\tilde{\cap}M}$ is an (α, β)-SI subgroup of $\mathcal{F}_G$ over U. □

The following example shows that $\mathcal{F}_{H\tilde{\cup}M}$ is not an (α, β)-SI subgroup of $\mathcal{F}_G$ over U

Example 5.4. Let us consider the (α, β)-SI group $\mathcal{F}_G$ over $U = S_3$ given in Example 4.12, and (α, β)-SI subgroup $\mathcal{F}_H$ of $\mathcal{F}_G$ over U given in Example 5.2. Let $M = \{0, 3\}$ be the subset of G. Now define (α, β)-SI subgroup $\mathcal{F}_M$ as follows:
$\mathcal{F}_M(0) = S_3$
$\mathcal{F}_M(3) = \{(1), (12), (13), (132)\}$.
Assume that $\beta = \{(1), (12), (13), (123)\}$ and $\alpha = \{(13), (123)\}$ such that $\varnothing \subseteq \alpha \subset \beta \subseteq U$. It is easy to clear that $\mathcal{F}_{H\tilde{\cup}M}$ is not an (α, β)-SI subgroup of $\mathcal{F}_G$ over U, because $\mathcal{F}_{H\tilde{\cup}M}(2+3) \cup \alpha \not\supseteq (\mathcal{F}_{H\tilde{\cup}M}(2) \cap \mathcal{F}_{H\tilde{\cup}M}(3)) \cap \beta$.

Definition 5.5. Let $\mathcal{F}_G$ be an (α, β)-SI group over U and $\mathcal{F}_N$ be a (α, β)-SI subgroup of $\mathcal{F}_G$ over U. Then, $\mathcal{F}_N$ is called an (α, β)-SI normal subgroup of $\mathcal{F}_G$ over U, denoted by $\mathcal{F}_N \diamondsuit \mathcal{F}_G$, it is an abelian soft subset of $\mathcal{F}_G$ over U.

Example 5.6. We assume that $U = Z$ (set of integers) is the universal set and $G = S_3$ is the symmetric group, and $M = A_3$ is the alternating group, are the subsets of set of parameters. Define the soft set $\mathcal{F}_G$ as follows:
$\mathcal{F}_G(1) = Z$,
$\mathcal{F}_G(12) = \{3, 5, 6, 7, 11\}$,
$\mathcal{F}_G(13) = \{2, 3, 5, 6, 12\}$,
$\mathcal{F}_G(23) = \{3, 5, 6, 7, 9\}$,
$\mathcal{F}_G(123) = \{1, 3, 5, 6, 8, 10\}$,
$\mathcal{F}_G(132) = \{1, 3, 5, 6, 8, 10\}$.
Define soft set $\mathcal{F}_M$ by
$\mathcal{F}_M(1) = Z$,
$\mathcal{F}_M(123) = \{1, 3, 5, 6\}$,
$\mathcal{F}_M(132) = \{1, 3, 5, 6\}$.
Let $\beta = \{1, 3, 5, 6, 8, 9\}$ and $\alpha = \{1, 3, 5, 6\}$, where $\varnothing \subseteq \alpha \subset \beta \subseteq U$. Then, it is verify that $\mathcal{F}_M$ is an (α, β)-SI normal subgroup of $\mathcal{F}_G$ over U.

Theorem 5.7. *Let G be a group and $\mathcal{F}_G$ be a soft set over U, and $\mathcal{F}_N$ be a nonempty soft subset of $\mathcal{F}_G$ over U. Then, the following conditions are holds:*
(1) $\mathcal{F}_N(xy) =_{(\alpha,\beta)} \mathcal{F}_N(yx)$ *for all* $x, y \in G$.
(2) $\mathcal{F}_N(xyx^{-1}) =_{(\alpha,\beta)} \mathcal{F}_N(y)$ *for all* $x, y \in G$.
(3) $\mathcal{F}_N(y) \subseteq_{(\alpha,\beta)} \mathcal{F}_N(xyx^{-1})$ *for all* $x, y \in G$.
(4) $\mathcal{F}_N(xyx^{-1}) \subseteq_{(\alpha,\beta)} \mathcal{F}_N(y)$ *for all* $x, y \in G$.

Definition 5.8 ([10]). Let $\mathcal{F}_A$ be a soft set U and $m \subset U$. Then, γ-inclusion of the soft set $\mathcal{F}_A$, denoted by $\mathcal{F}_A^{\gamma}$, is defined as

$$\mathcal{F}_A^{\gamma} = \{x \in A | \mathcal{F}_A(x) \supseteq \gamma\}.$$

Theorem 5.9. *Let $\mathcal{F}_A$ and $\mathcal{F}_B$ be two soft sets over U. Then*
(1) $\mathcal{F}_A \widetilde{\subseteq} \mathcal{F}_B$, $\alpha \in P(U) \Rightarrow \mathcal{F}_A^{\gamma} \widetilde{\subseteq}_{(\alpha,\beta)} \mathcal{F}_B^{\gamma}$.
(2) $\gamma_1 \subseteq \gamma_2$, $\gamma_1, \gamma_2 \in P(U) \Rightarrow \mathcal{F}_A^{\gamma_2} \widetilde{\subseteq}_{(\alpha,\beta)} \mathcal{F}_A^{\gamma_1}$.
(3) $\mathcal{F}_A = \mathcal{F}_B \Leftrightarrow \mathcal{F}_A^{\gamma} =_{(\alpha,\beta)} \mathcal{F}_B^{\gamma}$ *for all* $\gamma \in P(U)$.

Theorem 5.10. *Let the family of soft set $\{\mathcal{F}_{A_i} | i \in I_g\}$ over U, where I_g is the index set. Then, for any $\gamma \subseteq U$,*
(1) $\bigcup_{i \in I_g} (\mathcal{F}_{A_i}^{\gamma}) \widetilde{\subseteq}_{(\alpha,\beta)} (\bigcup_{i \in I_g} \mathcal{F}_{A_i})^{\gamma}$.
(2) $\bigcap_{i \in I_g} (\mathcal{F}_{A_i}^{\gamma}) \widetilde{\subseteq}_{(\alpha,\beta)} (\bigcap_{i \in I_g} \mathcal{F}_{A_i})^{\gamma}$.

Theorem 5.11. *Let $\mathcal{F}_G$ be a soft set over $\alpha \subseteq U$. For $\gamma \in U$, we define a non-empty set $U(\mathcal{F}_G; \gamma)$ and is defined by*

$$U(\mathcal{F}_G; \gamma) = \{x \in G | \mathcal{F}_G(x) \supseteq \gamma \cap \beta\}$$

where $\alpha \supseteq \gamma$. Then, $\mathcal{F}_G$ is an (α, β)-SI group of G over U if and only if the non-empty set $U(\mathcal{F}_G; \gamma)$ is a subgroups of G.

Proof. Let $\mathcal{F}_G$ be an (α, β)-SI group of G over U such that $\mathcal{F}_G(x) \supseteq \alpha$ for every $x \in G$ and $x, y \in U(\mathcal{F}_G; \gamma)$. Then $\mathcal{F}_G(xy^{-1}) = \mathcal{F}_G(xy^{-1}) \cup \alpha \supseteq \mathcal{F}_G(x) \cap \mathcal{F}_G(y^{-1}) \cap \beta = \mathcal{F}_G(x) \cap \mathcal{F}_G(y) \cap \beta \supseteq \gamma \cap \beta$ which indicate that $xy^{-1} \in U(\mathcal{F}_G; \gamma)$. Thus $U(\mathcal{F}_G; \gamma)$ is a subgroups of G.

Conversely, assume that each non-empty subset $U(\mathcal{F}_G; \gamma)$ is a subgroups of G. Then, by our assumption on $\mathcal{F}_G$, for $x, y \in G$ there exists $\alpha \supseteq \gamma_1, \gamma_2$ such that $\mathcal{F}_G(x) = \gamma_1$ and $\mathcal{F}_G(y) = \gamma_2$. Thus $\gamma = \gamma_1 \cap \gamma_2 \subseteq \alpha$, $\mathcal{F}_G(x) \supseteq \gamma$ and $\mathcal{F}_G(y) \supseteq \gamma$. Since $U(\mathcal{F}_G; \gamma)$ is a subgroups, $x, y \in U(\mathcal{F}_G; \gamma)$. So $xy^{-1} \in U(\mathcal{F}_G; \gamma)$. Hence $\mathcal{F}_G(xy^{-1}) \cup \alpha \supseteq \gamma$ and $\mathcal{F}_G(x) \cap \mathcal{F}_G(y) \cap \beta = \gamma_1 \cap \gamma_2 \cap \beta = \gamma \cap \beta$, which provides, $\mathcal{F}_G(xy^{-1}) \cup \alpha \supseteq \mathcal{F}_G(x) \cap \mathcal{F}_G(y^{-1}) \cap \beta$. Therefore $\mathcal{F}_G$ is an (α, β)-SI group of G over U. □

Theorem 5.12. *Let $T \subseteq G$ be a non-empty subgroups of G if and only if soft subset $\mathcal{F}_T$ defined by*

$$\mathcal{F}_T(x) = \begin{cases} \mu, & \text{if } x \in T \\ \nu, & \text{if } x \in G \setminus T. \end{cases}$$

where $\alpha \subseteq \mu \subseteq \nu \subseteq \beta \subseteq U$, is an (α, β)-SI group of G over U.

Proof. Suppose that T is a subgroups of G and $x, y \in G$. Then $xy^{-1} \in T$ and $\mathcal{F}_T(xy^{-1}) = \mathcal{F}_T(x) = \mathcal{F}_T(y) = \mu$. Thus $\mathcal{F}_T(xy^{-1}) \cup \alpha = \mu \cup \alpha = \mu$ and $\mathcal{F}_T(x) \cap \mathcal{F}_T(y) \cap \beta = \mu \cap \beta = \mu$, which produces, $\mathcal{F}_T(xy^{-1}) \cup \alpha \supseteq \mathcal{F}_T(x) \cap \mathcal{F}_T(y) \cap \beta$. Again, if $x \notin T$ or $y \notin T$, then either $xy^{-1} \in T$ or $xy^{-1} \notin T$. So $\mathcal{F}_T(xy^{-1}) \cup \alpha = \mu \cup \alpha = \mu$ or $\mathcal{F}_T(xy^{-1}) \cup \alpha = \nu \cup \alpha = \nu$. But, $\mathcal{F}_T(x) \cap \mathcal{F}_T(y) \cap \beta = \nu \cap \beta = \nu$, which imply $\mathcal{F}_T(xy^{-1}) \cup \alpha \supseteq \mathcal{F}_T(x) \cap \mathcal{F}_T(y) \cap \beta$.HenceTherefore, $\mathcal{F}_T$ is an (α, β)-SI subgroup of G over U.

Conversely, assumed that $\mathcal{F}_T$ is an (α,β)-SI subgroups of G over U. If $x,y\in T$, then $\mathcal{F}_T(xy^{-1})\cup\alpha\supseteq\mathcal{F}_T(x)\cap\mathcal{F}_T(y)\cap\beta=\mu\cap\beta=\mu$. But $\alpha\subseteq\mu\subseteq\nu\subseteq\beta$. Thus $\mathcal{F}_T(xy^{-1})=\mu$. So $xy^{-1}\in T$. Hence T is a subgroups of G. □

Theorem 5.13. *Let $\mathcal{F}_G$ be an (α,β)-SI group of G over U. Then $\mathcal{F}_{G(e)}$ is a subgroup if and only if $\mathcal{F}_G(xy^{-1})=_{(\alpha,\beta)}\mathcal{F}_(e)$.*

Proof. It is seen that $\mathcal{F}_{G(e)}\neq\varnothing$. Let $x,y\in\mathcal{F}_{G(e)}$. Then $\mathcal{F}_G(x)=\mathcal{F}_G(e)$ and $\mathcal{F}_G(y)=\mathcal{F}_G(x)=\mathcal{F}_G(e)$, and there exists $\varnothing\subseteq\alpha\subset\beta\subseteq U$ such that

$$\begin{aligned}\mathcal{F}_G(xy^{-1})\cup\alpha &\supseteq \mathcal{F}_G(x)\cap\mathcal{F}_G(y^{-1})\cap\beta\\ &= \mathcal{F}_G(x)\cap\mathcal{F}_G(y)\cap\beta\\ &= \mathcal{F}_G(e)\cap\mathcal{F}_G(e)\cap\beta\\ &= \mathcal{F}_G(e)\cap\beta.\end{aligned}$$

It is also seen that $\mathcal{F}_G(e)\cup\alpha\supseteq\mathcal{F}_G(xy^{-1})$. Thus $\mathcal{F}_G(xy^{-1})=_{(\alpha,\beta)}\mathcal{F}_G(e)$. □

Definition 5.14. [10] Let $\mathcal{F}_G$ be a soft int-group of G over U and $a\in G$. Then, soft left coset of $\mathcal{F}_G$, denoted by $a\mathcal{F}_G$, is defined by the function

$$(a\mathcal{F}_G)(x)=\mathcal{F}_G(a^{-1}x)$$

for all $x\in G$.

Theorem 5.15. *Let $\mathcal{F}_G$ be a (α,β)-SI group over U. Then, for all $a,b\in G$*

$$a\mathcal{F}_G=_{(\alpha,\beta)}b\mathcal{F}_G.$$

Proof. We assume that $a\mathcal{F}_G=b\mathcal{F}_G$. Then $a\mathcal{F}_G(x)=b\mathcal{F}_G(x)$ which indicate that $\mathcal{F}_G(a^{-1}x)=\mathcal{F}_G(b^{-1}x)$ for all $x\in G$. Let $x=b$ which yields $\mathcal{F}_G(a^{-1}b)=\mathcal{F}_G(bb^{-1})=\mathcal{F}_G(e)$ which implies that $a^{-1}b\in\mathcal{F}_{G(e)}$.

Conversely, suppose that $a\mathcal{F}_{G(e)}=b\mathcal{F}_{G(e)}$. Then $a^{-1}x,b^{-1}x\in\mathcal{F}_{G(e)}$. Thus, for $\varnothing\subseteq\alpha\subset\beta\subseteq U$, we get

$$\begin{aligned}\mathcal{F}_G(a^{-1}x)\cup\alpha &= \mathcal{F}_G(a^{-1}bb^{-1}x)\cup\alpha\\ &\supseteq \mathcal{F}_G(a^{-1}x)\cap\mathcal{F}_G(b^{-1}x)\cap\beta\\ &= \mathcal{F}_G(e)\cap\mathcal{F}_G(b^{-1}x)\cap\beta\\ &= \mathcal{F}_G\cap\beta forall x\in G.\end{aligned}$$

Similarly, we show that $\mathcal{F}_G(b^{-1}x)\cup\alpha\supseteq\mathcal{F}_G(a^{-1}x)$ for all $x\in G$.
So $\mathcal{F}_G(b^{-1}x)=_{(\alpha,\beta)}\mathcal{F}_G(a^{-1}x)$ for all $x\in G$. Hence $a\mathcal{F}_G=_{(\alpha,\beta)}b\mathcal{F}_G$ hold over U. □

Theorem 5.16. *Let $\mathcal{F}_G$ be an (α,β)-SI group over U and $\mathcal{H}_N$ be an (α,β)-SI normal subgroup of $\mathcal{F}_G$ over U. If $a\mathcal{F}_G=_{(\alpha,\beta)}\mathcal{F}_G$, then $\mathcal{H}_N(a)=_{(\alpha,\beta)}\mathcal{H}_N(b)$ for all $a,b\in N$.*

Proof. Assume that $a\mathcal{F}_G=b\mathcal{F}_G$. Then, by Theorem 5.15, $a^{-1}b,b^{-1}a\in\mathcal{F}_{G(e)}$. Since $\mathcal{H}_N$ be an (α,β)-SI normal subgroup of $\mathcal{F}_G$ over U. It is following from Theorem

5.7 that

$$\begin{aligned}\mathcal{H}_N(a)\cup\alpha &= \mathcal{H}_N(b^{-1}ab)\cup\alpha\\ &\supseteq \mathcal{H}_N(b^{-1}a)\cap\mathcal{H}_N(b)\cap\beta\\ &= \mathcal{H}_N(e)\cap\mathcal{H}_N(b)\cap\beta\\ &= \mathcal{H}_N(b)\cap\beta.\end{aligned}$$

Similarly, it can be shown that $\mathcal{H}_N(b)\cup\alpha \supseteq \mathcal{H}_N(a)\cap\beta$. Thus $\mathcal{H}_N(a) =_{(\alpha,\beta)} \mathcal{H}_N(b)$ hold over U, for all $a, b \in N$. □

Theorem 5.17. *Let G be a cyclic group of prime order and $H \subseteq G$. Then, soft set $\mathcal{F}_H$ is defined by*

$$\mathcal{F}_H = \begin{cases} \vartheta, & \text{if } \ x = e\\ \upsilon, & \text{if } \ x \neq e.\end{cases}$$

where, $\vartheta \supset \upsilon$, $\vartheta, \upsilon \in S(U)$ is an (α,β)-SI group over U.

Proof. We have to prove the four conditions for any $x, y \in G$,

(1) If $xy \neq e$, then $x \neq e$ and $y \neq e$. Thus

$$\begin{aligned}\mathcal{F}_H(xy)\cup\alpha &\supseteq \mathcal{F}_H(x)\cap\mathcal{F}_H(y)\cap\beta\\ &= (\upsilon\cap\upsilon)\cap\beta\\ &= \upsilon\cap\beta = \gamma\\ &= \mathcal{F}_H(x)\cap\mathcal{F}_H(y)\cap\beta.\end{aligned}$$

Since $x \neq e$, $x^{-1} \neq e$. So $\mathcal{F}_H(x) = \mathcal{F}_H(x^{-1}) = \upsilon$.

(2) If $xy \neq e$, then either $x = e$ or $y = e$. Let $x = e$. Then

$$\begin{aligned}\mathcal{F}_H(xy)\cup\alpha &= \mathcal{F}_H(ey)\cup\alpha = \upsilon\cup\alpha\\ &\supseteq (\vartheta\cap\upsilon)\cap\beta = \delta = \mathcal{F}_H(x)\cap\mathcal{F}_H(y)\cap\beta.\end{aligned}$$

For the second condition will be (α,β)-SI group, if $x = e$, then

$$\mathcal{F}_H(x) = \mathcal{F}_H(e) = \vartheta = \mathcal{F}_H(e^{-1}) = \mathcal{F}_H(x^{-1}).$$

Since $y \neq e$, $y^{-1} \neq e$. Thus $\mathcal{F}_H(y) = \upsilon = \mathcal{F}_H(y^{-1})$.

(3) If $xy = e$, then neither $x = e$ nor $y = e$. Thus

$$\mathcal{F}_H(xy)\cup\alpha = \vartheta\cup\alpha \supseteq (\upsilon\cap\upsilon)\cap\beta = \upsilon\cap\beta = \gamma = \mathcal{F}_H(x)\cap\mathcal{F}_H(y)\cap\beta.$$

and $\mathcal{F}_H(x) = \upsilon = \mathcal{F}_H(x^{-1})$, since $x \neq e \Rightarrow x^{-1} \neq e$.

(4) If $x = y = e$, which satisfied all the conditions as well. □

6. Homomorphism of (α,β)-SI group

In this section, we study homomorphism of a soft intersection group on the basis of (α,β) intersection.

Definition 6.1 ([10]). Let $\mathcal{F}_X, \mathcal{F}_Y \in S(U)$ be the soft sets over the common universe U and Ψ is a function from X to Y.

(i) Then soft image of $\mathcal{F}_X$ under Ψ, denoted by $\Psi(\mathcal{F}_X)$ is a soft set over U defined by

$$\Psi(\mathcal{F}_X) = \{(y, \Psi(\mathcal{F}_X)(y)) : y \in Y, \Psi(\mathcal{F}_X)(y) \in \mathcal{P}(U)\},$$

where

$$\Psi(\mathcal{F}_X)(y) = \begin{cases} \bigcup\{\mathcal{F}_X(x) |\ x \in X, \Psi(x) = y\} & \text{if} \quad \Psi^{-1}(y) \neq \varnothing \\ \varnothing & \textit{otherwise.} \end{cases}$$

(ii) The soft set

$$\Psi^{-1}(\mathcal{F}_Y) = \{(x, \Psi^{-1}(\mathcal{F}_Y)(x)) : x \in X, \Psi^{-1}(\mathcal{F}_Y)(x) \in \mathcal{P}(U)\},$$

where $\Psi^{-1}(\mathcal{F}_Y)(x) = \mathcal{F}_Y(\Psi(x))$, is called anti soft image of $\mathcal{F}_Y$ under Ψ.

Theorem 6.2. *Let Ψ be function from X to Y and I_g be an index set. For all $i \in I_g$, $X_i \subseteq X$ and $\mathcal{F}_{X_i} \in S(U)$, then*

$$\Psi(\widetilde{\bigcup}_{i\in I_g} \mathcal{F}_{X_i}) =_{(\alpha,\beta)} \widetilde{\bigcup}_{i\in I_g} \Psi(\mathcal{F}_{X_i}).$$

Theorem 6.3. *Let Ψ be a function from X to Y and $X_1, X_2 \subseteq X$, $\mathcal{F}_{X_1}, \mathcal{F}_{X_2} \in S(U)$. Then*

$$\mathcal{F}_{X_1} \widetilde{\subseteq} \mathcal{F}_{X_2} \Rightarrow \Psi(\mathcal{F}_{X_1}) \widetilde{\subseteq}_{(\alpha,\beta)} \mathcal{F}_{X_2}.$$

Proof. For all $y \in Y$,

$$\begin{aligned} \Psi(\mathcal{F}_{X_1})(y) \cap \beta &= \cup\{\mathcal{F}_{X_1}(x) | x \in X_1, \Psi(x) = y\} \cap \beta \\ &= \cup\{\mathcal{F}_{X_1}(x) \cap \beta | x \in X_1, \Psi(x) = y\} \\ &\subseteq \cup\{\mathcal{F}_{X_2}(x) \cup \alpha | x \in X_2, \Psi(x) = y\} \\ &= \Psi(\mathcal{F}_{X_2})(y) \cup \alpha. \end{aligned}$$

Then $\Psi(\mathcal{F}_{X_1}) \widetilde{\subseteq}_{(\alpha,\beta)} \mathcal{F}_{X_2}$ holds for $\mathcal{F}_{X_1} \widetilde{\subseteq} \mathcal{F}_{X_2}$ over U. □

Theorem 6.4. *Let Ψ be a function from X to Y and J_g be a nonempty index set. For all $j \in J_g$, $Y_j \subseteq Y$ and $\mathcal{F}_{Y_j} \in S(U)$,*

(1)(*i*) $\Psi^{-1}(\widetilde{\cup}_{j\in J_g} \mathcal{F}_{Y_j}) =_{(\alpha,\beta)} \widetilde{\cup}_{j\in J_g} \Psi^{-1}(\mathcal{F}_{Y_j})$.

(2) $\Psi^{-1}(\widetilde{\cap}_{j\in J_g} \mathcal{F}_{Y_j}) =_{(\alpha,\beta)} \widetilde{\cap}_{j\in J_g} \Psi^{-1}(\mathcal{F}_{Y_j})$.

Theorem 6.5. *Let Ψ be a function from X to Y. For all $\mathcal{F}_X \in S(U)$,*

$$\Psi^{-1}(\Psi(\mathcal{F}_X)) \widetilde{\supseteq} \mathcal{F}_X$$

for a particular case, if Ψ is an injective function, then $\Psi^{-1}(\Psi(\mathcal{F}_X)) = \mathcal{F}_X$.

Proof. For all $x \in X$,

$$\begin{aligned} \Psi^{-1}(\Psi(\mathcal{F}_X))(x) \cup \alpha &= \Psi(\mathcal{F}_X)(\Psi(x)) \cup \alpha \\ &= \cup\{\mathcal{F}_X(x') \cup \alpha | x' \in X, \Psi(x') = \Psi(x)\} \\ &\supseteq \mathcal{F}_X(x) \cap \beta. \end{aligned}$$

Thus $\Psi^{-1}(\Psi(\mathcal{F}_X)) \widetilde{\supseteq}_{(\alpha,\beta)} \mathcal{F}_X$.
From which it is clear that if Ψ is an injective function, then $\Psi^{-1}(\Psi(\mathcal{F}_X)) = \mathcal{F}_X$. □

Theorem 6.6. *Let $\mathcal{F}_G$ be an (α, β)-SI group over U and Ψ be an isomorphism from the groups G to H, then $\Psi(\mathcal{F}_G)$ is an (α, β)-SI group over U.*

Proof. Since Ψ is surjective function, for $x, y \in G$, there exist $u, v \in H$ such that $u = \Psi(x)$ and $v = \Psi(y)$. Then, it follows that

$$\begin{aligned}
\Psi(\mathcal{F}_G)(uv) \cup \alpha &= \cup\{\mathcal{F}_G(z)|z \in G, \Psi(z) = uv\} \cup \alpha \\
&= \cup\{\mathcal{F}_G(xy) \cup \alpha | x, y \in G, \Psi(x) = u, \Psi(y) = v\} \\
&\supseteq \cup\{\mathcal{F}_G(x) \cap \mathcal{F}_G(y) \cap \beta | x, y \in G, \Psi(x) = u, \Psi(y) = v\} \\
&= \cup(\{\mathcal{F}_G(x)|x \in G, \Psi(x) = u\} \cap \{\mathcal{F}_G(y)|y \in G, \Psi(y) = v\}) \cap \beta \\
&= (\Psi(\mathcal{F}_G)(u) \cap \Psi(\mathcal{F}_G)(v)) \cap \beta
\end{aligned}$$

and

$$\begin{aligned}
\Psi(\mathcal{F}_G)(u^{-1}) &= {}_{(\alpha,\beta)} \cup \{\mathcal{F}_G(z)|z \in H, \Psi(z) = u^{-1}\} \\
&= {}_{(\alpha,\beta)} \cup \{\mathcal{F}_G(z^{-1})|z \in H, \Psi(z^{-1}) = u\} \\
&= {}_{(\alpha,\beta)}(\Psi(\mathcal{F}_G))(u).
\end{aligned}$$

Thus soft image $\Psi(\mathcal{F}_G)$ is an (α, β)-SI group over U. □

Theorem 6.7. *Let $\mathcal{F}_H$ be an (α, β)-SI group over U and Ψ be a homomorphism from G to H. Then, $\Psi^{-1}(\mathcal{F}_H)$ is an (α, β)-SI group over U.*

Proof. For $x, y \in G$, we have

$$\begin{aligned}
\Psi^{-1}(\mathcal{F}_H)(xy) \cup \alpha &= \mathcal{F}_H(\Psi(xy)) \cup \alpha \\
&= \mathcal{F}_H(\Psi(x)\Psi(y)) \cup \alpha \\
&\supseteq (\mathcal{F}_H(\Psi(x)) \cap \mathcal{F}_H(\Psi(y))) \cap \beta \\
&= (\Psi^{-1}(\mathcal{F}_H(x)) \cap \Psi^{-1}(\mathcal{F}_H(y))) \cap \beta
\end{aligned}$$

and

$$\begin{aligned}
\Psi^{-1}(\mathcal{F}_H)(x^{-1}) &= {}_{(\alpha,\beta)}\mathcal{F}_H(\Psi(x^{-1})) \\
&= {}_{(\alpha,\beta)}\mathcal{F}_H(\Psi(x^{-1})) \\
&= {}_{(\alpha,\beta)}(\mathcal{F}_H(\Psi(x)) \\
&= {}_{(\alpha,\beta)}\Psi^{-1}(\mathcal{F}_H(x)).
\end{aligned}$$

Thus anti soft image $\psi^{-1}(\mathcal{F}_H)$ is an (α, β)-SI group over U. □

7. Soft intersection product and soft characteristic function

In this section, soft intersection product and soft characteristic function are defined and prove some properties.

Definition 7.1. Let G be a group and $\mathcal{F}_G$, $\mathcal{H}_G$ be two soft groups over U. Then, (α, β)-SI product $\mathcal{F}_G * \mathcal{H}_G$ over U is defined by

$$(\mathcal{F}_G * \mathcal{H}_G)(x) \cup \alpha \supseteq \bigcup_{x=yz} \{\mathcal{F}_G(y) \cap \mathcal{H}_G(z)\} \cap \beta$$

for all $x \in G$ such that $yz = x \in G$.

By using the order relation in Section 4, then above Definition 7.1 is equivalent to the following.

Definition 7.2. Let G be a group and $\mathcal{F}_G$, $\mathcal{H}_G$ be two soft groups over U. Then, (α,β)-SI product $\mathcal{F}_G * \mathcal{H}_G$ over U is defined by

$$(\mathcal{F}_G * \mathcal{H}_G)(x) \supseteq_{(\alpha,\beta)} \bigcup_{x=yz} \{\mathcal{F}_G(y) \cap \mathcal{H}_G(z)\}$$

for all $x \in G$ such that $yz = x \in G$.

Theorem 7.3. *Let $\mathcal{F}_G, \mathcal{G}_G, \mathcal{H}_G \in S(U)$. Then*

(1) $(\mathcal{F}_G * \mathcal{G}_G) * \mathcal{H}_G =_{(\alpha,\beta)} \mathcal{F}_G * (\mathcal{G}_G * \mathcal{H}_G)$.

(2) $\mathcal{F}_G * \mathcal{G}_G =_{(\alpha,\beta)} \mathcal{G}_G * \mathcal{F}_G$ *if G is commutative.*

(3) $\mathcal{F}_G * (\mathcal{G}_G \widetilde{\cup} \mathcal{H}_G) \widetilde{\supseteq}_{(\alpha,\beta)} (\mathcal{F}_G * \mathcal{G}_G) \widetilde{\cup} (\mathcal{F}_G * \mathcal{H}_G)$ *and* $(\mathcal{F}_G \widetilde{\cup} \mathcal{G}_G) * \mathcal{H}_G \widetilde{\supseteq}_{(\alpha,\beta)} (\mathcal{F}_G * \mathcal{H}_G) \widetilde{\bigcup} (\mathcal{G}_G * \mathcal{H}_G)$.

(4) $\mathcal{F}_G * (\mathcal{G}_G \widetilde{\cap} \mathcal{H}_G) \widetilde{\supseteq}_{(\alpha,\beta)} (\mathcal{F}_G * \mathcal{G}_G) \widetilde{\cap} (\mathcal{F}_G * \mathcal{H}_G)$ *and* $(\mathcal{F}_G \widetilde{\cap} \mathcal{G}_G) * \mathcal{H}_G \widetilde{\supseteq}_{(\alpha,\beta)} (\mathcal{F}_G * \mathcal{H}_G) \widetilde{\cap} (\mathcal{G}_G * \mathcal{H}_G)$.

(5) *If* $\mathcal{F}_G \widetilde{\subseteq} \mathcal{G}_G$, *then* $\mathcal{F}_G * \mathcal{H}_G \widetilde{\subseteq}_{(\alpha,\beta)} \mathcal{G}_G * \mathcal{H}_G$ *and* $\mathcal{H}_G * \mathcal{F}_G \widetilde{\subseteq}_{(\alpha,\beta)} \mathcal{H}_G * \mathcal{G}_G$.

(6) *If* $\mathcal{T}_G, \mathcal{L}_G \in S(U)$ *such that* $\mathcal{T}_G \widetilde{\subseteq} \mathcal{F}_G$ *and* $\mathcal{L}_G \widetilde{\subseteq} \mathcal{G}_G$, *then* $\mathcal{T}_G * \mathcal{L}_G) \widetilde{\subseteq}_{(\alpha,\beta)} \mathcal{F}_G * \mathcal{G}_G$.

Proof. (1) and (2) can be proved from the Definition 7.1.
Since proofs of (3) and (4)[resp. (5) and (6)] are similar, we will show that (3) and (5) hold.

(3) Let $x \in G$. If there is no $y, z \in G$ such that $x = yz$, then $\mathcal{F}_G * (\mathcal{H}_G \widetilde{\cup} \mathcal{H}_G)(x) = \varnothing$.

Similarly, $(\mathcal{F}_G * \mathcal{G}_G) \widetilde{\cup} (\mathcal{F}_G * \mathcal{H}_G)(x) = (\mathcal{F}_G * \mathcal{G}_G)(x) \widetilde{\cup} (\mathcal{F}_G * \mathcal{H}_G)(x) = \varnothing \cup \varnothing = \varnothing$.
Now, let there exists $y, z \in G$ such that $x = yz$. Then

$$\begin{aligned}
(\mathcal{F}_G * (\mathcal{G}_G \widetilde{\cup} \mathcal{H}_G))(x) \cup \alpha &\supseteq \bigcup_{x=yz} (\mathcal{F}_G(y) \cap (\mathcal{G}_G \widetilde{\cup} \mathcal{H}_G)(z)) \cap \beta \\
&= \bigcup_{x=yz} \mathcal{F}_G(y) \cap (\mathcal{G}_G(z) \cup \mathcal{H}_G(z)) \cap \beta \\
&= \bigcup_{x=yz} \{(\mathcal{F}_G(y) \cap \mathcal{G}_G(z) \cap \beta\} \cup \{(\mathcal{F}_G(y) \cap \mathcal{H}_G(z) \cap \beta)\} \\
&= [\bigcup_{x=yz} (\mathcal{F}_G(y) \cap \mathcal{G}_G(z)) \cap \beta] \cup [\bigcup_{x=yz} (\mathcal{F}_G(y) \cap \mathcal{H}_G(z)) \cap \beta] \\
&= [(\mathcal{F}_G * \mathcal{G}_G)(x) \cup (\mathcal{F}_G * \mathcal{H}_G)(x)] \cap \beta \\
&= [(\mathcal{F}_G * \mathcal{G}_G) \widetilde{\cup} (\mathcal{F}_G * \mathcal{H}_G)](x) \cap \beta.
\end{aligned}$$

Thus $(\mathcal{F}_G * (\mathcal{G}_G \widetilde{\cup} \mathcal{H}_G)) \widetilde{\supseteq}_{(\alpha,\beta)} (\mathcal{F}_G * \mathcal{G}_G) \widetilde{\cup} (\mathcal{F}_G * \mathcal{H}_G)$.
So $(\mathcal{F}_G \widetilde{\cup} \mathcal{G}_G) * \mathcal{H}_G \widetilde{\supseteq}_{(\alpha,\beta)} (\mathcal{F}_G * \mathcal{H}_G) \widetilde{\bigcup} (\mathcal{G}_G * \mathcal{H}_G)$.

(5) Let $x \in G$. If x is not expressed as $x = yz$, then

$$(\mathcal{F}_G * \mathcal{H}_G)(x) = (\mathcal{G}_G * \mathcal{H}_G)(x) = \varnothing.$$

Otherwise,

$$\begin{aligned}(\mathcal{F}_G * \mathcal{H}_G)(x) \cap \beta &= \bigcup_{x=yz} (\mathcal{F}_G(y) \cap \mathcal{H}_G(z)) \cap \beta \\ &\subseteq \bigcup_{x=yz} (\mathcal{G}_G(y) \cap \mathcal{H}_G(z)) \cap \beta \\ &\subseteq \bigcup_{x=yz} (\mathcal{G}_G(y) \cap \mathcal{H}_G(z)) \cup \alpha \\ &= ((\mathcal{G}_G(y) * \mathcal{H}_G(z)) \cup \alpha.\end{aligned}$$

Thus $(\mathcal{F}_G * \mathcal{H}_G)\widetilde{\subseteq}_{(\alpha,\beta)}(\mathcal{G}_G(y) * \mathcal{H}_G(z)$.

Similarly, one can proved that $(\mathcal{H}_G * \mathcal{F}_G)\widetilde{\subseteq}_{(\alpha,\beta)}(\mathcal{H}_G(y) * \mathcal{G}_G(z)$. $\square$

Definition 7.4. Let G be group and $X \subseteq G$. Then, soft characteristic function of X on G is denoted by G_X, and is defined by

$$G_X(x) = \begin{cases} U, & \text{if } \ x \in X \\ \varnothing, & \text{if } \ x \notin X. \end{cases}$$

The soft characteristic function is a soft set over U for

$$G_X : G \to \mathcal{P}(U).$$

Theorem 7.5. *Let G be a group and $X, Y \subseteq G$. Then, the following properties hold:* (1) *If $X \subseteq Y$, then $G_{G\setminus Y}\widetilde{\subseteq}_{(\alpha,\beta)}G_{G\setminus X}$.*

(2) *$G_{G\setminus X}\widetilde{\cap}G_{G\setminus Y} =_{(\alpha,\beta)} G_{G\setminus X\cap Y}$ and $G_{G\setminus X}\widetilde{\cup}G_{G\setminus Y} =_{(\alpha,\beta)} G_{G\setminus X\cup Y}$.*

(3) *$G_{G\setminus X} * G_{G\setminus Y} =_{(\alpha,\beta)} G_{G\setminus XY}$.*

Proof. (1) We consider three different cases: (a) $x \in X \subseteq Y \subseteq G$, (b) $x \in Y \setminus X$ and (c) $x \in G \setminus Y$.

In the first and third cases, $(G_{G\setminus X}(x)\cap\beta)\cup\alpha = (G_{G\setminus Y}(x)\cap\beta)\cup\alpha$ and in the second case, $(G_{G\setminus Y}(x) \cap \beta) \cup \alpha = (\varnothing \cap \beta) \cup \alpha = \alpha \subseteq \beta = (U \cap \beta) \cup \alpha = (G_{G\setminus X}(x) \cap \beta) \cup \alpha$. Then, for any case, $G_{G\setminus Y}\widetilde{\subseteq}_{(\alpha,\beta)}G_{G\setminus X}$.

(2) If $x \in (G \setminus X) \cap (G \setminus Y)$, then $x \in (G \setminus X)$ and $x \in (G \setminus Y)$, from which

$$\begin{aligned}((G_{G\setminus X} \cap G_{G\setminus Y})(x) \cap \beta) \cup \alpha &= ((G_{G\setminus X}(x) \cap G_{G\setminus Y}(x)) \cap \beta) \cup \alpha \\ &= (U \cap \beta) \cup \alpha \\ &= ((G_{(G\setminus Y)} \cap G_{(G\setminus Y)})(x) \cap \beta) \cup \alpha.\end{aligned}$$

If $x \notin (G \setminus X) \cap (G \setminus Y)$, then $x \in X$ and $x \in Y$. Thus

$$\begin{aligned}((G_{G\setminus X}\widetilde{\cap}G_{G\setminus Y})(x) \cap \beta) \cup \alpha &= ((G_{G\setminus X}(x) \cap G_{G\setminus Y}(x)) \cap \beta) \cup \alpha \\ &= (\varnothing \cap \beta) \cup \alpha \\ &= \alpha \subseteq \beta \\ &= (U \cap \beta) \cup \alpha \\ &= ((G_{(G\setminus Y)} \cap G_{(G\setminus Y)})(x) \cap \beta) \cup \alpha.\end{aligned}$$

So $G_{G\setminus X}\widetilde{\cap}G_{G\setminus Y} =_{(\alpha,\beta)} G_{G\setminus X\cap Y}$.
Again, If $x \in (G \setminus X) \cup (G \setminus Y)$, then $x \in (G \setminus X)$ and $x \in (G \setminus Y)$, from which

$$\begin{aligned}((G_{G\setminus X} \cup G_{G\setminus Y})(x) \cap \beta) \cup \alpha &= ((G_{G\setminus X}(x) \cup G_{G\setminus Y}(x)) \cap \beta) \cup \alpha \\ &= (U \cap \beta) \cup \alpha \\ &= ((G_{(G\setminus Y)} \cup G_{(G\setminus Y)})(x) \cap \beta) \cup \alpha.\end{aligned}$$

If $x \notin (G \setminus X) \cap (G \setminus Y)$, then $x \in X$ and $x \in Y$. Thus

$$\begin{aligned}((G_{G\setminus X} \cup G_{G\setminus Y})(x) \cap \beta) \cup \alpha &= ((G_{G\setminus X}(x) \cup G_{G\setminus Y}(x)) \cap \beta) \cup \alpha \\ &= (\varnothing \cap \beta) \cup \alpha \\ &= \alpha \subseteq \beta \\ &= (U \cap \beta) \cup \alpha \\ &= ((G_{(G\setminus Y)} \cap G_{(G\setminus Y)})(x) \cap \beta) \cup \alpha.\end{aligned}$$

So $G_{G\setminus X}\widetilde{\cup}G_{G\setminus Y} =_{(\alpha,\beta)} G_{G\setminus X\cap Y}$.
(3) If $x \in G \setminus XY$, then $G_{G\setminus XY}(x) = U$. Thus

$$\begin{aligned}G_{G\setminus X} * G_{G\setminus Y}(x)) \cap \beta) \cup \alpha &= (G_{G\setminus XY}(x) \cap \beta) \cup \alpha \\ &= (U \cap \beta) \cup \alpha \\ &= (G_{G\setminus XY}(x) \cap \beta) \cup \alpha.\end{aligned}$$

If $x \notin G \setminus XY$, then $G_{G\setminus XY}(x) = \varnothing$. Thus

$$\begin{aligned}G_{G\setminus X} * G_{G\setminus Y}(x)) \cap \beta) \cup \alpha &= (G_{G\setminus XY}(x) \cap \beta) \cup \alpha \\ &= (\varnothing \cap \beta) \cup \alpha \\ &= \alpha \subseteq \beta \\ &= (U \cap \beta) \cup \alpha \\ &= (G_{G\setminus XY}(x) \cap \beta) \cup \alpha.\end{aligned}$$

So $G_{G\setminus X} * G_{G\setminus Y} = (\alpha, \beta)G_{G\setminus XY}$. □

Corollary 7.6. *Let $H \subseteq G$, then H is a subgroup of G if and only if the soft characteristic function $G_{G\setminus H}$ is an (α, β)-SI subgroups of G over U.*

8. Conclusions

Soft set theory proposed by Molodstov [31] is an important mathematical tool for dealing with uncertainties and vagueness. After the introduction of soft set, it has rapidly applied in mathematics and real life situations. Soft algebraic mathematics is an important one. In this paper, first we defined (α, β)-soft intersection set, and then based on this notion define (α, β)-SI group which is a new type of soft group theory and developed a new theoretical studies of (α, β)-SI group theory. Then we defined (α, β)-SI subgroups, (α, β)-SI normal subgroups and study e-soft set, soft left coset, soft image, and anti soft image by means of (α, β)-SI group. Lastly, we study soft intersection product and characteristic function with respect to (α, β)-SI group.

We hope that this work will give a deep impact on the upcoming research in this field and other soft algebraic study to open up a new horizons of interest and innovations. It is our hope that this work will serve as a foundation for further study of the theory of group structures. One may be applied this concept to study some application fields like decision making, knowledge base system, data analysis, etc.

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