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Review of some anti fuzzy properties of some fuzzy subgroups

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Abstract. This paper is to study some anti fuzzy group properties. The aim is to review some fuzzy group properties and provide the anti fuzzy versions of some existing theorems in fuzzy group theory. The research therefore focuses on the properties of fuzzy subgroup, fuzzy cosets, fuzzy conjugacy and fuzzy normal subgroups of a group which are mimicked in anti fuzzy group theory.

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1. Introduction

L. A. Zadeh[18] started the research in fuzzy set and he worked on the properties of fuzzy sets in which every element $x \in X$ is assigned a membership value between 0 and 1 depending on to what degree does x belong to X. This was given a group structure by A. Rosenfeld[15]. Furthermore, P. S. Das[3] developed the concept of level subgroups.

In 1984, N. P. Mukherjee and P. Bhattacharya[9] also introduced the concept of fuzzy left and right cosets. The concept of fuzzy normal subgroup was improved on by the work of W.B. Vasantha Kandasamy and D. Meiyappan[17] in 1997, which studied the concepts such as fuzzy middle cosets, pseudo fuzzy cosets and pseudo fuzzy double coset. The link between fuzzy subgroup and anti fuzzy subgroup was made by R. Biswas[1] in 1990.

It is interesting to say that, in the recent time, researchers such as K. H. Manikandan et al and P. Corsini et al as in [7, 8] and [2] respectively, among many others, have started studying the fuzzy properties of some hyperstructures. In particular, K. H Manikan-dan and R. Muthuraj[8] have extended the concept to anti fuzzy properties of some hyperstructures. Moreover, there is a generalization of the concept of fuzzy subgroup, anti fuzzy subgroup and anti fuzzy subfields to (λ, μ)-fuzzy type as in the work of Y. Feng and B. Yao [5, 4] to mention just some.

2. Preliminaries

Definition 2.1 ([16])**.** Let X be a non-empty set. A fuzzy subset μ of the set X is a function $\mu : X \to [0, 1]$.

Definition 2.2 ([15, 16])**.** Let G be a group and μ a fuzzy subset of G. Then μ is called a fuzzy subgroup of G if
(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x^{-1}) = \mu(x)$,
(iii) μ is called a fuzzy normal subgroup if $\mu(xy) = \mu(yx)$ for all x and y in G.

Definition 2.3 ([16])**.** Let μ and λ be any two fuzzy subsets of a set X. Then
(i) λ and μ are equal if $\mu(x) = \lambda(x)$ for every x in X.
(ii) λ and μ are disjoint if $\mu(x) \neq \lambda(x)$ for every x in X.
(iii) $\lambda \subseteq \mu$ if $\mu(x) \geq \lambda(x)$

Definition 2.4 ([16, 17])**.** Let μ be a fuzzy subset (subgroup) of X. Then, for some $t \in [0, 1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a t-level subset (subgroup) of the fuzzy subset (subgroup) μ.

Remark 2.5. If μ is a fuzzy subgroup of G and μ_t is a subgroup of G, the set μ_t can be represented as G^t_μ.

Definition 2.6 ([17])**.** Let μ be a fuzzy subgroup of a group G. The set

$$H = \{x \in G : \mu(x) = \mu(e)\}$$

is such that $o(\mu) = o(H)$ and the order of μ is $\mid H \mid$.

Theorem 2.7 ([15])**.** *Let G be a group and μ a fuzzy subset of G. Then μ is a fuzzy subgroup of G if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$.*

Proposition 2.8 ([1])**.** *Let G be a group. A fuzzy subset μ is an anti fuzzy subgroup of G if and only if $\forall x, y \in G, \mu(xy^{-1}) \leq max\{\mu(x), \mu(y)\}$.*

Definition 2.9 ([9, 16])**.** Let μ be a fuzzy (or an anti fuzzy) subgroup of a group G. For a in G, the fuzzy (or anti fuzzy) coset $a\mu$ of G determined by a and μ is defined by $(a\mu)(x) = \mu(a^{-1}x)$ for all x in G.

Definition 2.10 ([12])**.** Let μ be an anti fuzzy subgroup of G. For $a, b \in G$, anti fuzzy middle coset $(a\mu b)$ is defined by $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$.

Definition 2.11 ([12])**.** Let μ be an anti fuzzy subgroup of G. Then it is anti fuzzy normal if $\mu(xy) = \mu(yx)\ \forall x, y \in G$ or $\mu(x) = \mu(yxy^{-1})\ \forall x, y \in G$.

Definition 2.12 ([9])**.** Let μ be a fuzzy subgroup of G. Then it is fuzzy normal if $\mu(xy) = \mu(yx)\ \forall x, y \in G$ or $\mu(x) = \mu(yxy^{-1})\ \forall x, y \in G$.

Theorem 2.13 ([12])**.** *Let $a^{-1}\mu a$ be a fuzzy middle coset of G for some $a \in G$. Then all such a form the normaliser $N(\mu)$ of fuzzy subgroup μ of G if and only if μ is fuzzy normal.*

Definition 2.14 ([12])**.** Let μ and λ be any two anti fuzzy subgroup of G for any x and some $g \in G$. Then, they are said to be anti fuzzy conjugate if $\lambda(x) = \mu(g^{-1}xg)$

Definition 2.15 ([17])**.** Let μ be an anti fuzzy subgroup of a group G. Then the set

$$H = \{x \in G \mid \mu(x) = \mu(e)\}$$

is such that $o(\mu) = o(H)$.

Theorem 2.16 ([11])**.** *Any subgroup H of G as in Definition 2.15 can be realised as a lower level subgroup of an anti fuzzy subgroup μ of G.*

Definition 2.17 ([1])**.** Let μ be an fuzzy subgroup of X. Then, for some $t \in [0,1]$, the set $\mu_t = \{x \in X : \mu(x) \leq t\}$ is called a lower level subgroup of the anti fuzzy subgroup μ.

3. Anti fuzzy cosets and conjugates

Many of the results here have been treated in fuzzy version by B. O. Onasanya and S. A. Ilori[12]

Theorem 3.1. *Let $a^{-1}\mu a$ be an anti fuzzy middle coset of G for some $a \in G$. Then all such a form the normalizer $N(\mu)$ of the anti fuzzy subgroup μ of G if and only if μ is anti fuzzy normal.*

Proof. We define the normaliser of the anti fuzzy subgroup μ by $N(\mu) = \{a \in G : \mu(axa^{-1}) = \mu(x)\}$ [11]. Then $\mu(axa^{-1}) = \mu(x) \Leftrightarrow \mu$ is anti fuzzy normal so that $\mu(axa^{-1}a) = \mu(xa) \Leftrightarrow \mu(ax) = \mu(xa)$.

Conversely, let μ be anti fuzzy normal and $a^{-1}\mu a$ an anti fuzzy middle coset in G. Then for all $x \in G$ and some $a \in G$,

$$(a^{-1}\mu a)(x) = \mu(axa^{-1}) = \mu(aa^{-1}x) = \mu(x).$$

Thus $\mu(axa^{-1}) = \mu(x)$. So $\{a\} = N(\mu)$. □

It is important to remark that by Theorems 2.13 and 3.1, a fuzzy subgroup and its complements have the same normalizer.

Proposition 3.2. *Let μ be an anti fuzzy normal subgroup of G. Then every anti fuzzy middle coset $a\mu b$ by a and b coincides with some left and right anti fuzy cosets $c\mu$ and μc respectively, where c is the product $b^{-1}a^{-1}$.*

Proof. By associativity in G and Definition 2.11, we have that

$$(a\mu b)(x) = \mu((a^{-1}x)b^{-1}) = \mu(b^{-1}(a^{-1}x))$$

$$\mu(b^{-1}a^{-1}x) = \mu(cx) = \mu(xc) \text{ still by Definition 2.11.}$$

Thus $(a\mu b)(x) = (c\mu)(x) = (\mu c)(x)$. So $(a\mu b) = c\mu = \mu c$. □

Theorem 3.3. *Let G be a group of order 2 and μ an anti fuzzy normal subgroup of G. Then, for some $a \in G$ and $\forall x \in G$, the anti fuzzy middle coset $a\mu a$ coincides with the anti fuzzy subgroup μ.*

Proof. In the anti fuzzy middle coset $a\mu b$, take $a = b$. By associativity in G,

$$(a\mu a)(x) = \mu((a^{-1}x)a^{-1}).$$

By Theorem 3.2, $\mu((a^{-1}x)a^{-1}) = \mu(a^{-2}x)$. Since $a^{-1} \in G$ and G is of order 2,

$$\mu(a^{-2}x) = \mu((a^{-1})^2 x) = \mu(ex) = \mu(x).$$

Thus $a\mu a = \mu$. □

Remark 3.4. B. O. Onasanya and S. A. Ilori[12] has define that any two elements $a, b \in G$ are fuzzy μ-commutative if μ is a fuzzy subgroup when $a\mu b = b\mu a$. Similarly, we extend that this is also the definition when μ is an anti fuzzy subgroup of G.

Theorem 3.5. *Let μ be an anti fuzzy normal subgroup of G. Then any two elements a and b in G are anti fuzzy μ-commutative.*

Proof. Clearly, $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$. Then, by Definitions 2.10, 2.11 and associativity of G,

$$\mu(a^{-1}xb^{-1}) = \mu(b^{-1}xa^{-1}) = (b\mu a)(x).$$

Thus $a\mu b = b\mu a$. □

Theorem 3.6. *Every anti fuzzy middle coset $a\mu b$ of a group G is an anti fuzzy subgroup if μ is anti fuzzy conjugate to some anti fuzzy subgroup λ of G.*

Proof. Let $b = a^{-1}$ for some $a, b \in G$ and μ and λ be anti fuzzy conjugate subgroups of G. Then

$$(a\mu b)(xy^{-1}) = (a\mu a^{-1})(xy^{-1}) = \mu(a^{-1}xy^{-1}a) = \lambda(xy^{-1}) \leq max\{\lambda(x), \lambda(y)\}.$$

Thus

$$max\{\lambda(x), \lambda(y)\} = max\{\mu(a^{-1}xa), \mu(a^{-1}ya)\} = max\{(a\mu a^{-1})(x), (a\mu a^{-1})(y)\}.$$

So $(a\mu b)(xy^{-1}) \leq max\{(a\mu b)(x), (a\mu b)(y)\}$. □

Remark 3.7. If μ is fuzzy normal, then it is self conjugate, $\mu(x) = \mu(a^{-1}xa)$ for some $a \in G$.

Theorem 3.8. *Let G be a finite group of prime order p and μ an anti fuzzy subgroup of G. Then the following are equivalent:*

(1) *G is cyclic.*

(2) $o(\mu) = 1$.

(3) *The only lower level subgroup μ_t such that $o(\mu) < p$ is the trivial subgroup $\{e\}$ of G.*

Proof. (1) $\Longrightarrow$ (2): Assume G is cyclic, then, $G = \langle a \rangle = \{a^m : m \in \mathbf{Z}\}$. Since G is of order p, $a^p = e$ and $\mu(a^p) = \mu(e)$. The element a^p in G is unique. Thus $H = \{e\}$ and the $o(H) = o(\mu) = 1$, by Definition 2.15.

(2) $\Longrightarrow$ (3): Assume that $o(\mu) = 1$. The subgroup $H = \{x \in G : \mu(x) = \mu(e)\}$ of G so that $o(H) = o(\mu) = 1$ is the singleton group $\{e\}$. By Theorem 2.16, $\{e\}$ is a lower level subgroup of μ and a trivial subgroup of G.

(3) $\Longrightarrow$ (1): Assume G has only one lower level subgroup $\{e\}$ of μ such that $o(\mu) < p$. For if not, there is at least an element $a \in G$ and $m \in \mathbf{Z}$ such that $a^m = e$ with $m \leq p$ since elements of a cyclic group are powers of an element. For $m < p$, there is a subgroup of G with order m so that $m \mid p$ by Lagrange theorem. But p is prime so that $m = 1$ or p. If $m = 1$, that subgroup such that $o(\mu) < p$ is $\{e\}$. But, if $m \neq 1$, no such m such that $m < p$ exists. So it can be claimed that $m = p$

in which case $o(a) =| \langle a\rangle |= p =| G |$ and $G = \{a^1, a^2, a^3, ..., a^p = e\} = \langle a\rangle$. So a generates G. Hence G is cyclic. □

Theorem 3.9. *Let μ and λ be any two anti fuzzy subgroups of any group G. Then, μ and λ are conjugate anti fuzzy subgroups of G if and only if $\mu = \lambda$.*

Proof. Assume that μ and λ are conjugate anti fuzzy subgroups of G. For $y, g \in G$,

$$\lambda(x) = \mu((gy)^{-1}x(gy)) = \mu(y^{-1}(g^{-1}xg)y) = \lambda(g^{-1}xg) = \mu(x).$$

Thus $\mu = \lambda$.

Conversely, assume that $\mu = \lambda$. For some $g = e \in G,\ \mu(x) = \lambda(x) = \lambda(e^{-1}xe)$. So μ and λ are conjugate fuzzy subgroups of G. □

Theorem 3.10. *An anti fuzzy middle coset $a\mu b$ is anti fuzzy normal if and only if $b = a^{-1}$ and μ is anti fuzzy normal.*

Proof. Let μ be anti fuzzy normal. By definition, $(a\mu b)(xy) = \mu(a^{-1}xyb^{-1})$.
By Theorem 3.6, Remark 3.7, Theorem 3.9 and Definition 2.14, if we take $b = a^{-1}$, then $a\mu b$ and μ are anti fuzzy conjugate so that

$$\mu(a^{-1}xya) = \mu(a^{-1}xyb^{-1}) = (a\mu b)(xy) = \mu(xy).$$

Since μ is anti fuzzy normal,

$$\mu(xy) = \mu(yx) = \mu(a^{-1}yxb^{-1}) = (a\mu b)(yx).$$

Thus $(a\mu b)(xy) = \mu(xy) = \mu(yx) = (a\mu b)(yx)$.

Conversely, assume that $a\mu b$ is anti fuzzy normal. Then $(a\mu b)(xy) = (a\mu b)(yx)$. Thus, by Definition 2.11,

$$(a\mu b)(xy) = \mu(a^{-1}xyb^{-1}) = \mu(xy) \Leftrightarrow b = a^{-1}$$

and μ is anti fuzzy normal. Also,

$$(a\mu b)(yx) = \mu(a^{-1}yxb^{-1}) = \mu(yx) \Leftrightarrow b = a^{-1} \text{ and } \mu \text{ is anti fuzzy normal.}$$

So

$$\mu(xy) = (a\mu b)(xy) = (a\mu b)(yx) = \mu(yx) \Leftrightarrow b = a^{-1} \text{ and } \mu \text{ is anti fuzzy normal.}$$

Hence μ is anti fuzzy normal and $b = a^{-1}$ □

Conclusion

Combining the results of [6, 7, 8, 10, 12, 13, 16] and this work, it may then be concluded that many algebraic properties of some fuzzy subgroups will hold similarly for their respective complements, which are anti fuzzy subgroups. Variations may only occur when level subsets are involved as can be seen in [13] and [14]. This is because, if μ is a fuzzy subgroup and $t_1, t_2 \in [0, 1]$ such that $t_1 \leq t_2$, then $\mu_{t_2} \subseteq \mu_{t_1}$. But if μ is an anti fuzzy subgroup, for such t_1 and t_2, $\mu_{t_1} \subseteq \mu_{t_2}$.

As a matter of fact, the anti fuzzy versions of some results in fuzzy algebra can be established easily by just taking the complement of the fuzzy subgroup in the following sense: If μ is a fuzzy subgroup of a group, then, $1 - \mu$, its complement, is an anti fuzzy subgroup.

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