Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 6, (June 2016), pp. 885–
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Reducibility and complete reducibility of intuitionistic fuzzy G-modules

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Received 19 September 2015; Revised 12 December 2015; Accepted 23 December 2015

Abstract. In this paper, we will study reducibility and complete reducibility of intuitionistic fuzzy G-modules. A method of constructing an intuitionistic fuzzy G-module of a given G-module M in terms of double pinned flags is given. It is proved that for any finite dimensional G-module M of dimension at least two, there exist infinite many completely reducible intuitionistic fuzzy G-modules. Moreover, it is shown that union of intuitionistic fuzzy completely reducible G-modules is an intuitionistic fuzzy completely reducible G-module but intersection of intuitionistic fuzzy completely reducible G-modules is not an intuitionistic fuzzy completely reducible G-module.

2010 AMS Classification: 08A72, 03F55

Keywords: Intuitionistic fuzzy G-module, Direct sum of intuitionistic fuzzy G-modules, Complete reducibility, Key chain, Double pinned flags.

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1. Introduction

Algebraic structures play a vital role in mathematics, and numerous applications of these structures are seen in many disciplines such as computer sciences, information sciences, theoretical physics, control engineering, etc. After the introduction of fuzzy sets by Zadeh [20], the researchers have been carrying out research in various concepts of abstract algebra in fuzzy setting. Rosenfeld [12] was the first one to define the concept of fuzzy subgroups of a group. The literature of various fuzzy algebraic concepts have been growing rapidly. In particular, Nagoita and Ralescu [9] introduced and examined the notion of fuzzy submodule of a module. Fernadez introduce and studied the notion of fuzzy G-modules in [18] and [19]. Abraham and Sebastian studied the representation of fuzzy G-modules in [1].

One of the interesting generalisations of the theory of fuzzy sets is the theory of intuitionistic fuzzy sets introduced by Atanassov in [2, 3, 4]. Using the Atanassov$'$s idea, Biswas [5] established the intuitionistic fuzzification of the concept of subgroup of a group. Later on many mathematician work on it and introduced the notion of intuitionistic fuzzy subring, intuitionistic fuzzy submodule etc. see [7, 8, 8, 11, 13, 14, 15]. The notion of intuitionistic fuzzy G- modules was introduced and studied by the author et al. in [16, 17]. In this paper, we will define reducibility and complete reducibility of intuitionistic fuzzy G-modules and study their properties. A method of constructing an intuitionistic fuzzy G-module of a given G-module M in terms of double pinned flags is also given.

2. Preliminaries

In this section, we list some basic concepts and well known results on reducibility and complete reducibility of G-modules and intuitionistic fuzzy G-modules for the sake of completeness of the topic under study. Throughout the paper, R and C denote the field of real numbers and field of complex numbers respectively. Unless otherwise stated all G-modules are assumed to be taken over the field K, where K is a subfield of the field of complex numbers.

Definition 2.1 ([6])**.** Let G be a group and M be a vector space over a field K(a subfield of C). Then M is called a G-module if for every g∈G and m∈M, ∃ a product(called the action of G on M), gm ∈M satisfies the following axioms :
(i) $1_G m = m, \forall m$ ∈M (1_G being the identity of G),
(ii)$(gh)m = g(hm), \forall m \in M, g, h \in$ G,
(iii)$g(k_1m_1 + k_2m_2) = k_1(gm_1) + k_2(gm_2), \forall k_1, k_2 \in K; m_1, m_2$ ∈M and g ∈G.

Definition 2.2 ([6])**.** Let G be a group and let M be a G-module over the field K. Let N be a subspace of the vector space over K. Then N is called a G-submodule of M if $an_1 + bn_2$ ∈N, for all a, b ∈K and n_1, n_2 ∈N.

Definition 2.3 ([6])**.** Let M and M* be G-modules. A mapping f : M → M* is a G-module homomorphism if
(i) $f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$,
(ii)$f(gm) = gf(m), \forall k_1, k_2 \in K; m, m_1, m_2$ ∈M and g ∈G.

Definition 2.4 ([6])**.** A non-zero G-module M is said to be irreducible if the only G-submodules of M are M and {0}. Otherwise M is said to be reducible.

Example 2.5 ([6])**.** Let G ={1, −1}, M = C is a vector space over Q. Then M is a G-module having G-submodules Q and R and therefore, it is reducible.

Example 2.6 ([6])**.** For any prime p, we have M = (Z_p , $+_p$, $\times_p$) is a field . Let G = M - {0}. Then under the field operation of M, M is a G-module. Since the only G-submodules of M are M and {0}. So, M is an irreducible G-module.

Definition 2.7 ([6])**.** A non-zero G-module M is said to be completely reducible if for every G-submodule N of M there exists a G-submodule N* such that M = N ⊕ N*.

Remark 2.8. (i) Any finite dimensional G-module is completely reducible.
(ii) All completely reducible G-submodules of dimension atleast two is reducible, but all reducible G-modules are not completely reducible. (See the following example)

Example 2.9 ([6])**.** Let G = $\{1, -1\}$, M = C is a vector space over the field Q. Then M is reducible module (as it has proper G-submodules Q and R). But M is not completely reducible, for the G-submodule N = Q($\sqrt{2}$) of M, there does not exist G-submodule N* such that M = N$\bigoplus$N*.(the set N* = C -[Q($\sqrt{}$ 2)-{0}] is not a G-submodule of M because N* does not contain G)

Definition 2.10 ([16])**.** Let G be a group and M be a G-module over K, which is a subfield of C. Then an intuitionistic fuzzy G-module on M is an intuitionistic fuzzy set A = (μ_A, ν_A) of M such that following conditions are satisfied :

(i) $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y), \forall a, b \in$K and $x, y \in$M,

(ii) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m), \forall g \in$ G; $m \in$M.

Example 2.11 ([16])**.** Let G = $\{1, -1\}$, M = R^n over R. Then M is a G-module. Define the intuitionistic fuzzy set A = (μ_A, ν_A) on M by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.5 & \text{if } x \neq 0 \end{cases} \quad ; \nu_A(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0.25 & \text{if } x \neq 0, \end{cases}$$

where $x = (x_1, x_2, ..., x_n) \in R^n$. Then A is an intuitionistic fuzzy G-module on M.

Proposition 2.12 ([16])**.** *Let M be a G-module over K and A be an intuitionistic fuzzy set of M, then A is an intuitionistic fuzzy G-module on M if and only if either* $C_{(\alpha,\beta)}(A) = \varnothing$ *or* $C_{(\alpha,\beta)}(A)$*, for all* $\alpha, \beta \in [0, 1]$ *such that* $\alpha+\beta \leq 1$*, is a G-submodule of M, where* $C_{(\alpha,\beta)}(A) = \{x \in M : \mu_A(x) \geq \alpha$ *and* $\nu_A(x) \leq \beta\}$.

3. Flags, double key chain and double pinned flags for the intuitionistic fuzzy set

In this section, we first define the notion of double pinned flag for the intuitionistic fuzzy set and by using this we will construct intuitionistic fuzzy G-modules of a given G-module M. We shall also define the notion of direct sum of intuitionistic fuzzy G-modules.

Definition 3.1. A flag is a maximal chain of submodules of G-module M of the form

$$M_0 \subset M_1 \subset M_2 \subset .... \subset M_n = M \ ,$$

in which $M_0 = \{0\}$ and all $M_i^{'}$s are called the components of the flag.

Definition 3.2. Let A be an intuitionistic fuzzy subset of a G-module M. Put $\wedge(A) = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), ..., (\alpha_n, \beta_n)\}$, where $\alpha_i, \beta_i \in [0, 1]$ such that $\alpha_i + \beta_i \leq 1$ for all $i = 1, 2, ..., n$, then we call the chain $(\alpha_0, \beta_0) \geq (\alpha_1, \beta_1) \geq .... \geq (\alpha_n, \beta_n)$ a double keychain if and only if $1 = \alpha_0 \geq \alpha_1 \geq ...... \geq \alpha_n$ and $0 = \beta_0 \leq \beta_1 \leq ......, \leq \beta_n$ and the pair (α_i, β_i) are called double pins and the set $\wedge(A)$ is called the set of double pinned flags for the intuitionistic fuzzy set A of M.

Definition 3.3. With the combination of flag and double keychain, we denote the chain

$$M_0^{(\alpha_0,\beta_0)} \subset M_1^{(\alpha_1,\beta_1)} \subset M_2^{(\alpha_2,\beta_2)} \subset .... \subset M_n^{(\alpha_n,\beta_n)} \ ,$$

as double pinned flags.

The purpose of defining the double pinned flag is to define intuitionistic fuzzy G-module A $= (\mu_A, \nu_A)$ of M in term of double pinned flags.
Moreover, for any $x \in M$ and $i, j \in \{0, 1, 2, ..., n\}$, we have
$\mu_A(x) = \alpha_i$ and $\nu_A(x) = \beta_i$ if and only if $i = \max\{j : x \in M_j\}$ if and only if $x \in M_i \backslash M_{i-1}$.

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in M_0 \\ \alpha_1 & \text{if } x \in M_1 \backslash M_0 \\ \alpha_2 & \text{if } x \in M_2 \backslash M_1 \\ .... & \\ \alpha_n & \text{if } x \in M_n \backslash M_{n-1} \end{cases} ; \nu_A(x) = \begin{cases} 0 & \text{if } x \in M_0 \\ \beta_1 & \text{if } x \in M_1 \backslash M_0 \\ \beta_2 & \text{if } x \in M_2 \backslash M_1 \\ .... & \\ \beta_n & \text{if } x \in M_n \backslash M_{n-1} \end{cases}$$

The converse of this result is also true. That is, given an intuitionistic fuzzy G-module A of M, then A can be represented in the form of a double pinned flags. For this see the following Theorem 3.4.

Theorem 3.4. *Consider a maximal chain of submodules of G-module M over the field K*

$$M_0 \subset M_1 \subset M_2 \subset .... \subset M_n = M \ ,$$

where $\subset$ denotes proper inclusion. Then there exists an intuitionistic fuzzy G-module A of M whose (α, β) - cut sets are exactly the G-submodules of M in the above chain.

Proof. Let $\{(\alpha_0, \beta_0), (\alpha_1, \beta_1), ..., (\alpha_n, \beta_n)\}$, where $\alpha_i, \beta_i \in [0, 1]$ such that $\alpha_i + \beta_i \leq 1$ for all $i = 1, 2, ..., n$, be a double keychain, where the pair (α_i, β_i) are double pins. Let the intuitionistic fuzzy set A $= (\mu_A, \nu_A)$ of M defined in term of double pinned flags is given by $\mu_A(x) = \alpha_i$ and $\nu_A(x) = \beta_i$ if and only if $i = \max\{j : x \in M_j\}$ if and only if $x \in M_i \backslash M_{i-1}$ for any $x \in M$ and for all $i, j \in \{1, 2, ..., n\}$, we have

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in M_0 \\ \alpha_1 & \text{if } x \in M_1 \backslash M_0 \\ \alpha_2 & \text{if } x \in M_2 \backslash M_1 \\ .... & \\ \alpha_n & \text{if } x \in M_n \backslash M_{n-1} \end{cases} ; \nu_A(x) = \begin{cases} 0 & \text{if } x \in M_0 \\ \beta_1 & \text{if } x \in M_1 \backslash M_0 \\ \beta_2 & \text{if } x \in M_2 \backslash M_1 \\ .... & \\ \beta_n & \text{if } x \in M_n \backslash M_{n-1}. \end{cases}$$

We claim that A is an intuitionistic fuzzy G-module of M.
Let $x, y \in$ M. If $x, y \in M_i \backslash M_{i-1}$, then $x, y \in M_i, gx, ax + by \in M_i$, for all $a, b \in$ K and $g \in$ G. Thus

$$\mu_A(x) = \alpha_i = \mu_A(y)$$

and

$$\nu_A(x) = \beta_i = \mu_A(y).$$

So

$$\mu_A(ax+by) \geq \alpha_i = \mu_A(x) \wedge \mu_A(y)$$

and

$$\nu_A(ax+by) \leq \beta_i = \nu_A(x) \vee \nu_A(y).$$

Also $\mu_A(gx) \geq \alpha_i = \mu_A(x)$ and $\nu_A(gx) \leq \beta_i = \nu_A(x)$.

For $i > j$, if $x \in M_i \setminus M_{i-1}$ and $y \in M_j \setminus M_{j-1}$, then $\mu_A(x) = \alpha_i < \alpha_j = \mu_A(y)$ and $\nu_A(x) = \beta_i > \beta_j = \mu_A(y)$. Thus $x, y \in M_i$. Since each M_i is G-submodule of M, $ax+by, gx \in M_i, a, b \in K, g \in$G. So

$$\mu_A(ax+by) \geq \alpha_i = \mu_A(x) \wedge \mu_A(y)$$

and

$$\nu_A(ax+by) \leq \beta_i = \nu_A(x) \vee \nu_A(y)$$

Also $\mu_A(gx) \geq \alpha_i = \mu_A(x)$ and $\nu_A(gx) \leq \beta_i = \nu_A(x)$.
Since $(\alpha_0, \beta_0) \geq (\alpha_1, \beta_1) \geq .... \geq (\alpha_n, \beta_n)$ is a double keychain, it follows that the (α, β)-cuts submodules of A are given by the following chain of G-submodules of M as

$$C_{(\alpha_0,\beta_0)}(A) \subset C_{(\alpha_1,\beta_1)}(A) \subset C_{(\alpha_2,\beta_2)}(A) \subset .... \subset C_{(\alpha_n,\beta_n)}(A) = M.$$

Obviously, we have $C_{(\alpha_0,\beta_0)}(A) = \{x \in M : \mu_A(x) \geq \alpha_0 \text{ and } \nu_A(x) \leq \beta_0\} = M_0$.

Now, we prove that $C_{(\alpha_i,\beta_i)}(A) = M_i$ for $0 < i < n$.
Clearly, $M_i \subseteq C_{(\alpha_i,\beta_i)}(A)$. For other inclusion, let $x \in C_{(\alpha_i,\beta_i)}(A)$. Then $\mu_A(x) \geq \alpha_i$ and $\nu_A(x) \leq \beta_i$ and $x \notin M_k$ for $k > i$. Thus $\mu_A(x) \in \{\alpha_1, \alpha_2, ..., \alpha_i\}$ and $\nu_A(x) \in \{\beta_1, \beta_2, ..., \beta_i\}$. So $x \in M_j$ for all $j \leq i$. Since $M_j \subseteq M_i$, $x \in M_i$. Hence $C_{(\alpha_i,\beta_i)}(A) = M_i$ for all $0 \leq i \leq n$. Therefore the result follows by proposition 2.11. □

Proposition 3.5. *Any n-dimensional G-module M over K has an intuitionistic fuzzy G-module A with $\mid \wedge(A) \mid = n+1$, where $\mid \wedge(A) \mid$ = the number of double pinned flags for the intuitionistic fuzzy set A.*

Proof. Let $\{m_1, m_2, ..., m_n\}$ be the basis of G-module M. Let M_i be the G-submodule of M span by $\{m_1, m_2, ..., m_i\}$. Take $M_0 = \{0\}$. Then we get a maximal chain of G-submodules of M as $M_0 \subset M_1 \subset M_2 \subset .... \subset M_n = M$.
Let $\wedge(A) = \{(1, 0), (\frac{1}{2}, \frac{1}{n+1}), (\frac{1}{3}, \frac{1}{n}), ..., (\frac{1}{n}, \frac{1}{3}), (\frac{1}{n+1}, \frac{1}{2})\}$ be the set of double pinned flags for the intuitionistic fuzzy set A $= (\mu_A, \nu_A)$ defined by

$$\mu_A(m) = \begin{cases} 1 & \text{if } m \in M_0 \\ \frac{1}{2} & \text{if } m \in M_1 \backslash M_0 \\ \frac{1}{3} & \text{if } m \in M_2 \backslash M_1 \\ .... & \\ \frac{1}{n} & \text{if } m \in M_{n-1} \backslash M_{n-2} \\ \frac{1}{n+1} & \text{if } m \in M_n \backslash M_{n-1} \end{cases} ; \nu_A(m) = \begin{cases} 0 & \text{if } m \in M_0 \\ \frac{1}{n+1} & \text{if } m \in M_1 \backslash M_0 \\ \frac{1}{n} & \text{if } m \in M_2 \backslash M_1 \\ .... & \\ \frac{1}{3} & \text{if } m \in M_{n-1} \backslash M_{n-2} \\ \frac{1}{2} & \text{if } m \in M_n \backslash M_{n-1} \end{cases}$$

i.e., if $m = c_1m_1 + c_2m_2 + ... + c_nm_n$, then

$$\mu_A(c_1m_1 + c_2m_2 + ... + c_nm_n) = \begin{cases} 1 & \text{if } c_i = 0 \forall i \\ \frac{1}{2} & \text{if } c_1 \neq 0, c_2 = c_3 = ... = c_n = 0 \\ \frac{1}{3} & \text{if } c_2 \neq 0, c_3 = c_4 = ... = c_n = 0 \\ .... & \\ \frac{1}{n} & \text{if } c_{n-1} \neq 0, c_n = 0 \\ \frac{1}{n+1} & \text{if } c_n \neq 0 \end{cases}$$

$$\nu_A(c_1m_1 + c_2m_2 + ... + c_nm_n) = \begin{cases} 0 & \text{if } c_i = 0 \forall i \\ \frac{1}{n+1} & \text{if } c_1 \neq 0, c_2 = c_3 = ... = c_n = 0 \\ \frac{1}{n} & \text{if } c_2 \neq 0, c_3 = c_4 = ... = c_n = 0 \\ .... & \\ \frac{1}{3} & \text{if } c_{n-1} \neq 0, c_n = 0 \\ \frac{1}{2} & \text{if } c_n \neq 0. \end{cases}$$

Then by Theorem 3.4, A is an intuitionistic fuzzy G-module with $| \wedge(A) | = n+1$. □

Example 3.6. Let G = $\{1, -1\}$, M is the G-module R^4 over R. Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ be the standard ordered basis for M. Let M_0, M_1, M_2, M_3, M_4 be G-submodule of R^4 spanned by $\{0\}, \{\varepsilon_1\}, \{\varepsilon_1, \varepsilon_2\}, \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ respectively. Define an intuitionistic fuzzy set A = (μ_A, ν_A) of M by

$$\mu_A(m) = \begin{cases} 1 & \text{if } m \in M_0 \\ \frac{1}{2} & \text{if } m \in M_1 \backslash M_0 \\ \frac{1}{3} & \text{if } m \in M_2 \backslash M_1 \\ \frac{1}{4} & \text{if } m \in M_3 \backslash M_2 \\ \frac{1}{5} & \text{if } m \in M_4 \backslash M_3 \end{cases} ; \nu_A(m) = \begin{cases} 0 & \text{if } m \in M_0 \\ \frac{1}{5} & \text{if } m \in M_1 \backslash M_0 \\ \frac{1}{4} & \text{if } m \in M_2 \backslash M_1 \\ \frac{1}{3} & \text{if } m \in M_3 \backslash M_2 \\ \frac{1}{2} & \text{if } m \in M_4 \backslash M_3 \end{cases}$$

i.e., if $m = c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3 + c_4\varepsilon_4$, then

$$\mu_A(c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3 + c_4\varepsilon_4) = \begin{cases} 1 & \text{if } c_i = 0 \forall i \\ \frac{1}{2} & \text{if } c_1 \neq 0, c_2 = c_3 = c_4 = 0 \\ \frac{1}{3} & \text{if } c_2 \neq 0, c_3 = c_4 = 0 \\ \frac{1}{4} & \text{if } c_3 \neq 0, c_4 = 0 \\ \frac{1}{5} & \text{if } c_4 \neq 0 \end{cases}$$

$$\nu_A(c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3 + c_4\varepsilon_4) = \begin{cases} 0 & \text{if } c_i = 0 \forall i \\ \frac{1}{5} & \text{if } c_1 \neq 0, c_2 = c_3 = c_4 = 0 \\ \frac{1}{4} & \text{if } c_2 \neq 0, c_3 = c_4 = 0 \\ \frac{1}{3} & \text{if } c_3 \neq 0, c_4 = 0 \\ \frac{1}{2} & \text{if } c_4 \neq 0. \end{cases}$$

Then by Theorem 3.4, A is an intuitionistic fuzzy G-module, where $| \wedge(A) | = 4 + 1 = 5$.

Remark 3.7. The above construction can be extended to an infinite dimensional G-modules also.

Proposition 3.8. *Let M be a G-module over K and $M = \oplus_{i=1}^{n} M_i$, where $M_i^{'}s$ are G-submodules of M. If $A_i^{'}s$ $(1 \leq i \leq n)$ are intuitionistic fuzzy G-modules on $M_i^{'}s$, then an intuitionistic fuzzy set A of M defined by*

$$\mu_A(m) = \wedge\{\mu_{A_i}(m_i) : i = 1, 2, ..., n\} \text{ and } \nu_A(m) = \vee\{\nu_{A_i}(m_i) : i = 1, 2, ..., n\},$$

where $m = \Sigma_{i=1}^{n} m_i \in M$, is an intuitionistic fuzzy G-module on M.

Proof. Since each A_i is an intuitionistic fuzzy G-modules on M_i, for every $m_i, m_i^{'} \in M_i, g \in$ G and $a, b \in$ K, we have

(i) $\mu_{A_i}(am_i + bm_i^{'}) \geq \mu_{A_i}(m_i) \wedge \mu_{A_i}(m_i^{'})$ and $\nu_{A_i}(am_i + bm_i^{'}) \leq \nu_{A_i}(m_i) \vee \nu_{A_i}(m_i^{'}), \forall a, b \in$K and $m_i, m_i^{'} \in M_i$,

(ii) $\mu_{A_i}(gm_i) \geq \mu_{A_i}(m_i)$ and $\nu_{A_i}(gm_i) \leq \nu_{A_i}(m_i), \forall g \in$ G; $m_i \in M_i$.

Let $m = \Sigma_{i=1}^{n} m_i, m^{'} = \Sigma_{i=1}^{n} m_i^{'} \in$ M, where $m_i, m_i^{'} \in M_i$ and $a, b \in$ K. Then

$$\begin{aligned} \mu_A(am + bm^{'}) &= \mu_A(\Sigma(am_i + bm_i^{'})) \\ &= \wedge\{\mu_{A_i}(am_i + bm_i^{'}) : i = 1, 2, ..., n\} \\ &= \mu_{A_j}(am_j + bm_j^{'}), \text{for some j}(1 \leq j \leq n) \\ &\geq \mu_{A_j}(m_j) \wedge \mu_{A_j}(m_j^{'}) \\ &\geq \mu_A(m) \wedge \mu_A(m^{'}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \nu_A(am + bm^{'}) &= \nu_A(\Sigma(am_i + bm_i^{'})) \\ &= \vee\{\nu_{A_i}(am_i + bm_i^{'}) : i = 1, 2, ..., n\} \\ &= \nu_{A_k}(am_k + bm_k^{'}), \text{for some k}(1 \leq k \leq n) \\ &\leq \nu_{A_k}(m_k) \vee \nu_{A_k}(m_k^{'}) \\ &\leq \nu_A(m) \vee \mu_A(m^{'}). \end{aligned}$$

Also, for $g \in$ G and $m = \Sigma_{i=1}^{n} m_i \in$ M, we have

$$\begin{aligned} \mu_A(gm) &= \mu_A(\Sigma(gm_i)) \\ &= \wedge\{\mu_{A_i}(gm_i) : i = 1, 2, ..., n\} \\ &= \mu_{A_j}(gm_j), \text{for some j}(1 \leq j \leq n) \\ &\geq \mu_{A_j}(m_j) \\ &\geq \mu_A(m). \end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu_A(gm) &= \nu_A(\Sigma(gm_i))\\ &= \vee\{\nu_{A_i}(gm_i) : i = 1, 2, ..., n\}\\ &= \nu_{A_k}(gm_k), \text{for some k}(1 \leq k \leq n)\\ &\leq \nu_{A_k}(m_k)\\ &\leq \mu_A(m).\end{aligned}$$

Therefore, A is an intuitionistic fuzzy G-module. □

Remark 3.9. In the above proposition, if $\mu_{A_i}(0)$ are all equal and $\nu_{A_i}(0)$ are all equal, for each $(i = 1, 2, ..., n)$, then we have $\mu_A(0) = \wedge\{\mu_{A_i}(0) : i = 1, 2, ..., n\} = \mu_{A_i}(0)$ and $\nu_A(0) = \vee\{\nu_{A_i}(0) : i = 1, 2, ..., n\} = \nu_{A_i}(0)$.

Definition 3.10. An intuitionistic fuzzy G-module A of M = $\oplus_{i=1}^n M_i$, where $M_i^{'}$s are G-submodules of M. If $A_i^{'}$s $(1 \leq i \leq n)$ are intuitionistic fuzzy G-modules on $M_i^{'}$s, as in the above proposition such that $\mu_A(0) = \mu_{A_i}(0)$ and $\nu_A(0) = \nu_{A_i}(0)$, for all i, then an intuitionistic fuzzy G-module A of M is called the direct sum of A_i and it is written as A = $\oplus_{i=1}^n A_i$.

Example 3.11. Let G ={1, -1} and M = C over R. Then M is G-module. We have $M = M_1 \bigoplus M_2$, where $M_1 = R, M_2 = iR$. Define an intuitionistic fuzzy set A = (μ_A, ν_A) on M by

$$\mu_A(x+iy) = \begin{cases} 1 & \text{if } x = y = 0\\ \frac{1}{2} & \text{if } x \neq 0, y = 0\\ \frac{1}{3} & \text{if } y \neq 0 \end{cases} ; \nu_A(x+iy) = \begin{cases} 0 & \text{if } x = y = 0\\ \frac{1}{3} & \text{if } x \neq 0, y = 0.\\ \frac{1}{2} & \text{if } y \neq 0. \end{cases}$$

Then A is an intuitionistic fuzzy G-module on M.
Also, the intuitionistic fuzzy sets A_1 and A_2 on M_1 and M_2 respectively are defined by

$$\mu_{A_1}(x) = \begin{cases} 1 & \text{if } x = 0\\ \frac{1}{2} & \text{if } x \neq 0 \end{cases} ; \nu_{A_1}(x) = \begin{cases} 0 & \text{if } x = 0\\ \frac{1}{3} & \text{if } x \neq 0, \end{cases}$$

$$\mu_{A_2}(iy) = \begin{cases} 1 & \text{if } y = 0\\ \frac{1}{3} & \text{if } y \neq 0 \end{cases} ; \nu_{A_2}(iy) = \begin{cases} 0 & \text{if } y = 0\\ \frac{1}{2} & \text{if } y \neq 0 \end{cases}$$

are intuitionistic fuzzy G-modules on M_1 and M_2 respectively and A = $A_1 \oplus A_2$.

4. Reducibility and complete reducibility of intuitionistic fuzzy G-module

In this section, we define the notion of reducibility and complete reducibility of intuitionistic fuzzy G-modules. We show that there exists an infinite number of completely reducible intuitionistic fuzzy G-modules on any finite dimensional G-module M of atleast dimension 2. The intuitionistic fuzzy G-modules considered in this section are assumed to be non-trivial (i.e., non-constant).

Definition 4.1. An intuitionistic fuzzy G-module A of a G-module M is said to be reducible if M is reducible as a G-module otherwise it is said to be irreducible.

Example 4.2. Let G = $\{1, -1\}$ and M = C, regarded as a vector space over Q. Then by Example 2.5, M is a G-module having proper G-submodule Q and R and therefore M is reducible. Define an intuitionistic fuzzy set A = (μ_A, ν_A) of M by

$$\mu_A(m) = \begin{cases} 1 & \text{if } m = 0 \\ \frac{1}{2} & \text{if } (m \neq 0) \text{is real} \\ \frac{1}{4} & \text{otherwise} \end{cases} \quad ; \nu_A(m) = \begin{cases} 0 & \text{if } m = 0 \\ \frac{1}{4} & \text{if } (m \neq 0) \text{is real.} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Then A is an intuitionistic fuzzy G-module on M and hence A is reducible.

Example 4.3. For any prime p, we have M = $(Z_p, +_p, \times_p)$ is a field. Let G = M - $\{0\}$. Then under the field operation of M, M is a G-module. Since the only G-submodule of M are M and $\{0\}$, so any intuitionistic fuzzy G-module A on M is irreducible.

Definition 4.4. An intuitionistic fuzzy G-module A on M is completely reducible if

(i) M is completely reducible,

(ii) M has atleast one proper G-submodule,

(iii) Corresponding to any proper decomposition $M_1 \oplus M_2$, there exists intuitionistic fuzzy G-submodules $A_i{}'$s of $M_i{}'$s such that A = $A_1 \oplus A_2$ with $\wedge(A_1) \neq \wedge(A_2)$ [i.e., set of double pinned flags for the intuitionistic fuzzy G-module $A_1 \neq$ set of double pinned flags for the intuitionistic fuzzy G-module A_2].

Example 4.5. Let G = $\{1, -1\}$ and M = Q($\sqrt{2}$) over Q. Then M is a G-module and also the only G-submodules of M are $\{0\}, M_1 = Q, M_2 = \sqrt{2}Q = \{b\sqrt{2} : b \in Q\}$ and M = Q($\sqrt{2}$). Therefore the only decompositions of M are M = $M \oplus \{0\}$ and M = $M_1 \oplus M_2$ and hence M is completely reducible. Here the only proper decomposition is M = $M_1 \oplus M_2$. Define an intuitionistic fuzzy set A = (μ_A, ν_A) on M as

$$\mu_A(a + \sqrt{2}b) = \begin{cases} 1 & \text{if } a = b = 0 \\ \frac{4}{5} & \text{if } a \neq 0, b = 0 \\ \frac{1}{2} & \text{if } b \neq 0 \end{cases} \quad ; \nu_A(a + \sqrt{2}b) = \begin{cases} 0 & \text{if } a = b = 0 \\ \frac{1}{2} & \text{if } a \neq 0, b = 0 \\ \frac{4}{5} & \text{if } b \neq 0. \end{cases}$$

Then A is an intuitionistic fuzzy G-module of M. Also, the intuitionistic fuzzy sets A_1 and A_2 on M_1 and M_2 respectively are defined by

$$\mu_{A_1}(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{4}{5} & \text{if } x \neq 0 \end{cases} \quad ; \nu_{A_1}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{10} & \text{if } x \neq 0 \end{cases}$$

$$\mu_{A_2}(\sqrt{2}y) = \begin{cases} 1 & \text{if } y = 0 \\ \frac{1}{10} & \text{if } y \neq 0 \end{cases} \quad ; \nu_{A_2}(\sqrt{2}y) = \begin{cases} 0 & \text{if } y = 0 \\ \frac{4}{5} & \text{if } y \neq 0. \end{cases}$$

Clearly, A_1, A_2 are intuitionistic fuzzy G-modules on M_1 and M_2 respectively such that A = $A_1 \oplus A_2$. Also $\wedge(A_1) \neq \wedge(A_2)$ and therefore the intuitionistic fuzzy G-module A is completely reducible.

Theorem 4.6. *Any finite dimensional G-module with dimension atleast 2, has an intuitionistic fuzzy completely reducible G-module.*

Proof. Let M be a G-module and let dim M = n. Since M is finite dimensional, it is completely reducible. Let $\mathbb{B} = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis for M. Then any proper G-submodule of M is the span of some proper subset of $\mathbb{B}$. Let A = (μ_A, ν_A) be an intuitionistic fuzzy set of M defined by

$$\mu_A(c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n) = \begin{cases} 1 & \text{if } c_i = 0 \forall i \\ \frac{1}{2} & \text{if } c_1 \neq 0, c_2 = c_3 = ... = c_n = 0 \\ \frac{1}{3} & \text{if } c_2 \neq 0, c_3 = c_4 = ... = c_n = 0 \\ .... & \\ \frac{1}{n} & \text{if } c_{n-1} \neq 0, c_n = 0 \\ \frac{1}{n+1} & \text{if } c_n \neq 0 \end{cases}$$

$$\nu_A(c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n) = \begin{cases} 0 & \text{if } c_i = 0 \forall i \\ \frac{1}{n+1} & \text{if } c_1 \neq 0, c_2 = c_3 = ... = c_n = 0 \\ \frac{1}{n} & \text{if } c_2 \neq 0, c_3 = c_4 = ... = c_n = 0 \\ .... & \\ \frac{1}{3} & \text{if } c_{n-1} \neq 0, c_n = 0 \\ \frac{1}{2} & \text{if } c_n \neq 0. \end{cases}$$

Then A is an intuitionistic fuzzy G-module on M. We will prove that A is the required intuitionistic fuzzy completely reducible G-module.
Let M_i be any proper G-submodule of M. Let M_1 be G-submodule of M spanned by $\{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_r}\}$, where $1 \leq r \leq n$ and $1 \leq i_1 \leq i_2 \leq .... \leq i_r \leq n$. Then $M = M_1 \oplus M_2$, where M_2 is a G-submodule of M spanned by the remaining base elements i.e., spanned by $\{\alpha_{i_{r+1}}, \alpha_{i_{r+2}}, ..., \alpha_{i_n}\}$, where $1 \leq i_{r+1} \leq i_{r+2} \leq .... \leq i_n \leq n$. Define the intuitionistic fuzzy sets A_1, A_2 of M_1, M_2 respectively by

$$\mu_{A_1}(c_{i_1}\alpha_{i_1} + c_{i_2}\alpha_{i_2} + ... + c_{i_r}\alpha_{i_r}) = \begin{cases} 1 & \text{if } c_{i_1} = c_{i_1} = c_{i_2} = ... = c_{i_r} = 0 \\ \frac{1}{i_1+1} & \text{if } c_{i_1} \neq 0, c_{i_2} = c_{i_3} = ... = c_{i_r} = 0 \\ \frac{1}{i_1+2} & \text{if } c_{i_2} \neq 0, c_{i_3} = c_{i_4} = ... = c_{i_r} = 0 \\ .... & \\ \frac{1}{i_{r-1}+1} & \text{if } c_{i_{r-1}} \neq 0, c_{i_r} = 0 \\ \frac{1}{i_r+1} & \text{if } c_{i_r} \neq 0, \end{cases}$$

$$\nu_{A_1}(c_{i_1}\alpha_{i_1} + c_{i_2}\alpha_{i_2} + ... + c_{i_r}\alpha_{i_r}) = \begin{cases} 0 & \text{if } c_{i_1} = c_{i_1} = c_{i_2} = ... = c_{i_r} = 0 \\ \frac{1}{i_r+1} & \text{if } c_{i_1} \neq 0, c_{i_2} = c_{i_3} = ... = c_{i_r} = 0 \\ \frac{1}{i_{r-1}+1} & \text{if } c_{i_2} \neq 0, c_{i_3} = c_{i_4} = ... = c_{i_r} = 0 \\ .... & \\ \frac{1}{i_2+1} & \text{if } c_{i_{r-1}} \neq 0, c_{i_r} = 0 \\ \frac{1}{i_1+1} & \text{if } c_{i_r} \neq 0 \end{cases}$$

and

$$\mu_{A_2}(c_{i_{r+1}}\alpha_{i_{r+1}} + c_{i_{r+2}}\alpha_{i_{r+2}} + ... + c_{i_n}\alpha_{i_n})$$
$$= \begin{cases} 1 & \text{if } c_{i_{r+1}} = c_{i_{r+2}} = c_{i_{r+3}} = ... = c_{i_n} = 0 \\ \frac{1}{i_r+2} & \text{if } c_{i_{r+1}} \neq 0, c_{i_{r+2}} = c_{i_{r+3}} = ... = c_{i_n} = 0 \\ \frac{1}{i_r+3} & \text{if } c_{i_{r+2}} \neq 0, c_{i_{r+3}} = c_{i_{r+4}} = ... = c_{i_n} = 0 \\ .... & \\ \frac{1}{i_{n-1}+1} & \text{if } c_{i_{n-1}} \neq 0, c_{i_n} = 0 \\ \frac{1}{i_n+1} & \text{if } c_{i_n} \neq 0, \end{cases}$$

$$\nu_{A_2}(c_{i_{r+1}}\alpha_{i_{r+1}} + c_{i_{r+2}}\alpha_{i_{r+2}} + ... + c_{i_n}\alpha_{i_n})$$
$$= \begin{cases} 0 & \text{if } c_{i_{r+1}} = c_{i_{r+2}} = c_{i_{r+3}} = ... = c_{i_n} = 0 \\ \frac{1}{i_n+1} & \text{if } c_{i_{r+1}} \neq 0, c_{i_{r+2}} = c_{i_{r+3}} = ... = c_{i_n} = 0 \\ \frac{1}{i_{n-1}+1} & \text{if } c_{i_{r+2}} \neq 0, c_{i_{r+3}} = c_{i_{r+4}} = ... = c_{i_n} = 0 \\ .... & \\ \frac{1}{i_r+3} & \text{if } c_{i_{n-1}} \neq 0, c_{i_n} = 0 \\ \frac{1}{i_r+2} & \text{if } c_{i_n} \neq 0. \end{cases}$$

Then A_1 and A_2 are intuitionistic fuzzy G-modules on M_1 and M_2 respectively such that $A = A_1 \oplus A_2$ and $\wedge(A_1) \neq \wedge(A_2)$. Thus A is completely reducible and hence the theorem proved. □

Corollary 4.7. *Any intuitionistic fuzzy G-module A on an n-dimensional* $(n \geq 2)$ *G-module M is completely reducible only if* $|\wedge(A)| \geq 3$.

Proof. Since M is a completely reducible G-module of dimension $n \geq 2$, it has atleast one proper decomposition $M = M_1 \oplus M_2$. If A is an intuitionistic fuzzy completely reducible G-module on M, $\exists$ an intuitionistic fuzzy G-module A_i on $M_i (i = 1, 2)$ such that A = $A_1 \oplus A_2$ with $\wedge(A_1) \neq \wedge(A_2)$ and $|\wedge(A_i)| \geq 2$ for $i = 1, 2$. This is possible only if $|\wedge(A_1) \cup \wedge(A_2)| \geq 3$ i.e., only if $|\wedge(A)| \geq 3$. □

Example 4.8. Let G = {1, -1} and Q be the field of rational numbers. Let M = Q $(\sqrt{2}, \sqrt{3}) = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} : a, b, c, d \in Q\}$. Then M is a G-module such that dim M = 4 and $\mathbb{B} = \{\alpha_1 = 1, \alpha_2 = \sqrt{2}, \alpha_3 = \sqrt{3}, \alpha_4 = \sqrt{6}\}$ is a basis for M over Q. Define intuitionistic fuzzy set A on M by

$$\mu_A(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4) = \begin{cases} 1 & \text{if } c_i = 0 \forall i \\ \frac{1}{2} & \text{if } c_1 \neq 0, c_2 = c_3 = c_4 = 0 \\ \frac{1}{3} & \text{if } c_2 \neq 0, c_3 = c_4 = 0 \\ \frac{1}{4} & \text{if } c_3 \neq 0, c_4 = 0 \\ \frac{1}{5} & \text{if } c_4 \neq 0 \end{cases}$$

$$\nu_A(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4) = \begin{cases} 0 & \text{if } c_i = 0 \forall i \\ \frac{1}{5} & \text{if } c_1 \neq 0, c_2 = c_3 = c_4 = 0. \\ \frac{1}{4} & \text{if } c_2 \neq 0, c_3 = c_4 = 0 \\ \frac{1}{3} & \text{if } c_3 \neq 0, c_4 = 0 \\ \frac{1}{2} & \text{if } c_4 \neq 0. \end{cases}$$

Then A is an intuitionistic fuzzy G-module on M. Consider two proper submodules of M (say) M_1 spanned by $\{\alpha_1, \alpha_3\}$ and M_2 spanned by $\{\alpha_2, \alpha_4\}$. Thus $M = M_1 \oplus M_2$. Define intuitionistic fuzzy sets A_1 and A_2 on M_1 and M_2 respectively by

$$\mu_{A_1}(c_1\alpha_1 + c_3\alpha_3) = \begin{cases} 1 & \text{if } c_1 = c_3 = 0 \\ \frac{1}{2} & \text{if } c_1 \neq 0, c_3 = 0 \\ \frac{1}{4} & \text{if } c_1 \neq 0 \end{cases} ; \nu_{A_1}(c_1\alpha_1 + c_3\alpha_3) = \begin{cases} 0 & \text{if } c_1 = c_3 = 0 \\ \frac{1}{4} & \text{if } c_1 \neq 0, c_3 = 0 \\ \frac{1}{2} & \text{if } c_1 \neq 0 \end{cases}$$

$$\mu_{A_2}(c_2\alpha_2 + c_4\alpha_4) = \begin{cases} 1 & \text{if } c_2 = c_4 = 0 \\ \frac{1}{3} & \text{if } c_2 \neq 0, c_4 = 0 \\ \frac{1}{5} & \text{if } c_2 \neq 0 \end{cases} ; \nu_{A_2}(c_2\alpha_2 + c_4\alpha_4) = \begin{cases} 0 & \text{if } c_2 = c_4 = 0 \\ \frac{1}{5} & \text{if } c_2 \neq 0, c_4 = 0. \\ \frac{1}{3} & \text{if } c_2 \neq 0. \end{cases}$$

So it is easy to check that A_1 and A_2 are intuitionistic fuzzy G-modules on M_1 and M_2 respectively such that $A = A_1 \oplus A_2$ and $\wedge(A_1) \neq \wedge(A_2)$.

Proposition 4.9. *Every intuitionistic fuzzy completely reducible G-module is intuitionistic fuzzy reducible.*

Proof. Let A be an intuitionistic fuzzy completely reducible G-module on M. Then M is a completely reducible G-module of dimension atleast two. Therefore by Remark 2.8, M is reducible and from Corollary 4.7, we have $| \wedge(A) | \geq 3$. Thus A is an intuitionistic fuzzy reducible G-module on M. □

Proposition 4.10. *There exists an infinite number of completely reducible intuitionistic fuzzy G-modules on any finite dimensional G-module M of atleast dimension 2.*

Proof. Let M be a finite dimensional G-module of atleast dimension 2. Then, by Theorem 4.6, ∃ an intuitionistic fuzzy completely reducible G-module A $=(\mu_A, \nu_A)$ on M.

Let $r \in (0, 1]$. Then it is easy to check that the intuitionistic fuzzy set A_r on M defined by $\mu_{A_r}(x) = r\mu_A(x)$ and $\nu_{A_r}(x) = (1 - r)\nu_A(x), \forall x \in$ M is an intuitionistic fuzzy G-module on M.

In the definition of intuitionistic fuzzy G-module A and intuitionistic fuzzy G-submodules $A_i{}'$s in the Theorem 4.6, replace 1 in the numerator by r. Then the intuitionistic fuzzy G-module A_r on M and the intuitionistic fuzzy G-submodules $A_{i_r}{}'$s on $M_i{}'$s ($\mu_{A_{i_r}}(x) = r.\mu_{A_r}(x)$ and $\nu_{A_{i_r}}(x) = (1 - r).\nu_{A_r}(x), \forall x \in M_i$) satisfies the conditions of the Theorem 4.6. Thus for every $r \in (0, 1]$, A_r is an intuitionistic fuzzy completely reducible G-module on M. □

Remark 4.11. (1) If $r = 0$, then A_r is the constant intuitionistic fuzzy G-module $\mu_{A_r}(x) = 0$ and $\nu_{A_r}(x) = 1, \forall x \in$ M. So A_r with $r = 0$ is not an intuitionistic fuzzy reducible G-module.

(2) The intuitionistic fuzzy completely reducible G-module A in Theorem 4.6 is an intuitionistic fuzzy completely reducible G-module A_1 in the above Proposition 4.10.

Corollary 4.12. *Let M be a finite dimensional G-module of dimension atleast 2. Then the union of intuitionistic fuzzy completely reducible G-modules $A_r, r \in (0,1]$ in the above Proposition 4.10 is an intuitionistic fuzzy completely reducible G-module. But the intersection of ${A_r}'s$ and intuitionistic fuzzy complement of each A_r are not intuitionistic fuzzy completely reducible G-modules.*

Proof. Let B be the intuitionistic fuzzy union and C be the intuitionistic fuzzy intersection of the intuitionistic fuzzy completely reducible G-module A_r. Then For each $x \in$ M, we define

$$\mu_B(x) = \text{Sup.}\{\mu_{A_r}(x) : r \in (0,1]\},\ \nu_B(x) = \text{Inf.}\{\nu_{A_r}(x) : r \in (0,1]\}$$

and

$$\mu_C(x) = \text{Inf.}\{\mu_{A_r}(x) : r \in (0,1]\},\ \nu_C(x) = \text{Sup.}\{\nu_{A_r}(x) : r \in (0,1]\}.$$

Then by the above Remark 4.11, $B = A_1$ is an intuitionistic fuzzy completely reducible G-module and $C = A_0$ is not an intuitionistic fuzzy completely reducible G-module. Let $r \in (0,1]$. Then in the intuitionistic fuzzy completely reducible G-module A_r, we have $\mu_{A_r}(0) = r$ and $\nu_{A_r}(0) = 1$ is maximal. Thus, in the intuitionistic fuzzy complement of A_r (say A_r^c), we have $\mu^c_{A_r}(0) = 1 - \mu_{A_r}(0) = 1 - r$ and $\nu^c_{A_r}(0) = 1 - \nu_{A_r}(0) = 1 - 1 = 0$, i.e., A_r^c is minimal among the grades of all $m \in$M. So A_r^c is not an intuitionistic fuzzy completely reducible G-module. [Because in an intuitionistic fuzzy G-module, the membership grade of the zero element is maximal 1 and non-membership grade of zero element is minimal 0.] □

Acknowledgements. Author is very thankful to the university grant commission, New Delhi for providing necessary financial assistance to carry out the present work under major research project file no. F. 42-2 / 2013 (SR).

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