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Fuzzy soft N-subgroups and N-ideals over right ternary N-groups

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Abstract. A right ternary N-group (RTNG) over a right ternary near-ring N is a generalization of its binary counterpart and fuzzy soft sets are generalization of soft sets which are parameterized family of subsets of a universal set. In this paper fuzzy soft N-subgroups and N-ideals over right ternary N-groups are defined and their basic algebraic properties are studied. Fuzzy soft N-subgroups and N-ideals are characterized in terms of their level sets. The homomorphic image and inverse homomorphic image of a fuzzy soft N-subgroup (N-ideal) are proved to be fuzzy soft N-subgroup (N-ideal). The basic structural properties of fuzzy soft congruence over a right ternary N-group are studied. The main result of this paper is that a normal fuzzy soft ideal can be obtained from a fuzzy soft congruence relation and vice versa. Lattice structure of the set of all fuzzy soft congruence relation over an RTNG is given.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [18] in 1965. In 1999, Molodtsov [11] introduced the soft set to deal with the uncertainties present in most of our real life situations. In 2001, Maji et al [10] expanded soft set theory to fuzzy soft set theory. In [3] Celik et al. studied about fuzzy soft rings. An encyclopaedic account of the algebraic theory of near-rings is provided by Pilz in [12]. The concept of fuzzy subnear-ring, fuzzy left and right ideals was introduced by Abou-Zaid [1]. Fuzzy R-subgroups and normal fuzzy subgroups in near-rings were introduced by Kim and Jun in [8, 9]. Dutta and Biswas introduced fuzzy congruences of a near-ring module and obtained the correspondence between fuzzy congruences and fuzzy submodules in [5]. Hur et al. introduced the notion of intuitionistic fuzzy submodules and intuitionistic fuzzy weak congruences on a near-ring module [6]. Srinivas et al. discussed anti fuzzy ideals and T-fuzzy ideals in Γ-near-rings in [13, 14]. The concept of fuzzy congruences of hemirings is discussed in [19] by Zhu et al. Fuzzy soft similarity relations over a set were studied in [15, 17]. In [16] the authors introduced the notion of right ternary N-groups and their substructures.

In this paper fuzzy soft right ternary N-groups and their ideals are defined and their basic algebraic properties are studied. Fuzzy soft N-subgroups and N-ideals are characterized in terms of their level sets. The homomorphic image and inverse homomorphic image of a fuzzy soft N-subgroup (N-ideal) are shown to be a fuzzy soft N-subgroup (N-ideal). Moreover fuzzy soft congruences over right ternary N-groups are defined and their structural properties are studied. A normal fuzzy soft ideal is obtained from a fuzzy soft congruence relation and vice versa.

2. Preliminaries

In this section the basic definitions that are necessary for the following sections of this paper are given.

Definition 2.1 ([4])**.** Let N be a non-empty set together with a binary operation $+$ and a ternary operation $[\ ] : N \times N \times N \to N$. Then $(N, +, [\ ])$ is a right ternary near-ring (RTNR) if

(i) $(N, +)$ is a group
(ii) $[[x\ y\ z]\ u\ v] = [x\ [y\ z\ u]\ v] = [x\ y\ [z\ u\ v]] = [x\ y\ z\ u\ v]$
(iii) $[(x + y)\ z\ u] = [x\ z\ u] + [y\ z\ u]$ for every $x, y, z, u, v \in N$.

An RTNR is an abelian RTNR if $(N, +)$ is abelian.

Definition 2.2 ([16])**.** Let $(N, +, [\ ])$ be an RTNR and $(\Gamma, +)$ be a group. Then Γ is said to be a right ternary N-group (RTNG) if there exists a mapping $[\ ]_\Gamma$: $N \times N \times \Gamma \to \Gamma$ satisfying the conditions :

(RTNG-1) $[n + m\ x\ \gamma]_\Gamma = [n\ x\ \gamma]_\Gamma + [m\ x\ \gamma]_\Gamma$,

(RTNG-2) $[[n\ m\ u]\ x\ \gamma]_\Gamma = [n\ [m\ u\ x]\ \gamma]_\Gamma = [n\ m\ [u\ x\ \gamma]_\Gamma]_\Gamma$ for all $\gamma \in \Gamma$ and $n, m, u, x \in N$.

An RTNG ${}_N\Gamma$ is an abelian RTNG if Γ is abelian under $+$.

A subgroup Δ of ${}_N\Gamma$ is said to be an N-subgroup of ${}_N\Gamma$ if $[N\ N\ \Delta]_\Gamma \subseteq \Delta$. A subgroup Δ of Γ is called a normal subgroup of Γ if $\forall\ \gamma \in \Gamma$, $\delta \in \Delta$, $\gamma + \delta - \gamma \in \Delta$. A normal subgroup Δ of Γ is called an N-ideal of ${}_N\Gamma$ if $[n\ x(\gamma+\delta)]_\Gamma - [n\ x\ \gamma]_\Gamma \in \Delta$, $\forall\ \gamma \in \Gamma$, $\delta \in \Delta$ and $n, x \in N$. If ${}_N\Gamma$ and ${}_N\Gamma'$ are any two right ternary N-groups then $h : {}_N\Gamma \to {}_N\Gamma'$ is an N-homomorphism if $h(\gamma + \delta) = h(\gamma) + h(\delta)\ \forall\ \gamma, \delta \in \Gamma$ and $h([x\ y\ \gamma]_\Gamma) = [x\ y\ h(\gamma)]_{\Gamma'}\ \forall\ x, y \in N$.

Definition 2.3 ([18])**.** If X is a universal set then a fuzzy subset of X is a map $\mu : X \to [0, 1]$ which is denoted by $\mu = \{(x, \mu(x))|x \in X\}$.

Definition 2.4 ([10])**.** Let U be a universal set and let A be a subset of a set of parameters E. Let I^U (where $I = [0, 1]$) be the set of fuzzy subsets of U. Then (f, A) is called a fuzzy soft set over U where $f : A \to I^U$ and $f(a) = f_a : U \to I$ is a fuzzy subset of U.

Definition 2.5 ([10])**.** Let U be a universal set and let A and B any two non-empty subsets of the set of parameters E of U. Let (f, A) and (g, B) be any two fuzzy soft sets over U. Then

(i) (f, A) is a fuzzy soft subset of (g, B), i.e., $(f, A)\tilde{\subseteq}(g, B)$ if $A \subseteq B$ and $f_a(x) \leq g_a(x)$ for every $a \in A$ and $x \in U$.

(ii) (f, A) AND (g, B) denoted by $(f, A)\tilde{\wedge}(g, B) = (h, C)$ where $C = A \times B$ is defined by $h_{(a,b)}(x) = \min\{f_a(x), g_b(x)\}$ for every $x \in U$ and $(a, b) \in A \times B$.

(iii) (f, A) OR (g, B) denoted by $(f, A)\tilde{\vee}(g, B)$ is defined by $(f, A)\tilde{\vee}(g, B) = (h, C)$, where $C = A \times B$ and $h_{(a,b)}(x) = \max\{f_a(x), g_b(x)\}$ for every $x \in U$ and $(a, b) \in A \times B$.

(iv) The union of (f, A) and (g, B) denoted by $(f, A)\tilde{\cup}(g, B) = (h, C)$,

where $C = A \cup B$ and $h(c) = h_c = \begin{cases} f_c & \text{if } c \in A - B \\ g_c & \text{if } c \in B - A \\ f_c \vee g_c & \text{if } c \in A \cap B. \end{cases}$

(v) The intersection of (f, A) and (g, B) such that $A \cap B \neq \varnothing$ is defined to be the fuzzy soft set (h, C), where $C = A \cap B$ and $h(c) = h_c = f_c \cap g_c$ for all $c \in C$ and is denoted by $(h, C) = (f, A)\tilde{\cap}(g, B)$.

Definition 2.6 ([10])**.** Let U and V be any two non-empty universal sets. Let E_1 and E_2 be parameter sets for U and V respectively. Let $A \subseteq E_1$, $B \subseteq E_2$. Let (f, A) and (g, B) be any two non-empty fuzzy soft sets over U and V respectively. Then their cartesian product over $U \times V$ is defined by $(f, A) \times (g, B) = (h, C)$, where $C = A \times B$ and $h_{(a,b)}(u, v) = \min\{f_a(u), g_b(v)\}$ for every $(u, v) \in U \times V$ and $(a, b) \in A \times B$.

Definition 2.7 ([7])**.** Let X and Y be any two non-empty sets and E_1 and E_2 be their parameter sets. Let $A \subseteq E_1$, $B \subseteq E_2$. Let (f, A) and (g, B) be any two non-empty fuzzy soft sets over X and Y respectively. Let $\phi : X \to Y$ and $\psi : A \to B$. Then $(\phi, \psi) : (f, A) \to (g, B)$ is called a fuzzy soft function and the image set $(\phi(f), B)$ of (f, A) under (ϕ, ψ) is defined as follows.

For every $y \in Y$ and $b \in B$

$$(\phi(f))_b(y) = \begin{cases} \bigvee_{x \in \phi^{-1}(y)} (\vee_{a \in \psi^{-1}(b) \cap A} f_a(x)), & \text{if } \phi^{-1}(y) \neq \varnothing, b \in \psi(A) \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.8 ([7])**.** Inverse image of fuzzy soft set (g, B) is defined by $(\phi, \psi)^{-1}((g, B)) = (\phi^{-1}(g), \psi^{-1}(B))$, where $(\phi^{-1}(g))_a(x) = g_{\psi(a)}(\phi(x))$ for every $a \in \psi^{-1}(B)$ and $x \in X$.

Definition 2.9 ([2])**.** Let (ϕ, ψ) be a fuzzy soft function from X to Y. If ϕ is a homomorphism from X to Y then (ϕ, ψ) is said to be a fuzzy soft homomorphism.

Definition 2.10 ([2])**.** Let U be a universal set. Let A be a subset of the parameter set E of U. Let (f, A) be a fuzzy soft set over U. Then for each f_a, the level subset $(f_a)_t$, $t \in (0, 1]$ is defined as $(f_a)_t = \{x \in U | f_a(x) \geq t\}$.

Definition 2.11. A relation R on a set X which is a subset of $X \times X$ is (i) reflexive if $(x, x) \in R$ (ii) symmetric if $(x, y) \in R$ implies $(y, x) \in R$ (iii) transitive if $(x, y) \in R$, $(y, z) \in R$ implies $(x, z) \in R$, for all $x, y, z \in X$.

Definition 2.12 ([18]). Let X and Y be any two non-empty sets. Then R is called a fuzzy relation from X to Y if R is a fuzzy subset of $X \times Y$,
i.e., $R = \{((x, y), \mu_R(x, y))|(x, y) \in X \times Y\}$.

Definition 2.13 ([15]). Fuzzy soft relations from X to Y are defined as fuzzy soft sets over $X \times Y$ i.e., if X and Y are any two non-empty sets and E is a set of parameters of $X \times Y$ and $C \subseteq E$. Then (ρ, C) is defined as a fuzzy soft relation where ρ is a mapping from C to $I^{X \times Y}$.

Definition 2.14 ([15]). Let X be a universal set and C be a subset of a set of parameters E of $X \times X$ and let $C \subseteq E$. Then a fuzzy soft relation (ρ, C) over $X \times X$ is a fuzzy soft similarity relation if
(i) Fuzzy soft reflexive i.e., $\rho_c(x, x) = 1$,
(ii) Fuzzy soft symmetric i.e., $\rho_c(x, y) = \rho_c(y, x)$
(iii) Fuzzy soft transitive i.e., $\rho_c \circ \rho_c \subseteq \rho_c$,
where $(\rho_c \circ \rho_c)(x, z) = \bigvee_{y \in X} (\rho_c(x, y) \wedge \rho_c(y, z))$ for each $c \in C$ and $x, y, z \in X$.

Definition 2.15 ([17]). Let X, Y and Z be any three non-empty sets. Let E and F be parameter sets of $X \times Y$ and $Y \times Z$ respectively. Let $C \subseteq E$ and $D \subseteq F$. Let (ρ, C) and (γ, D) be fuzzy soft relations over $X \times Y$ and $Y \times Z$ respectively. Then fuzzy soft max-min composition is denoted by $(\rho, C)\tilde{\circ}(\gamma, D) = (h, K)$ and is defined by $(h, K) = \{h_k|k \in K\}$ where $K = C \cap D$ with $h_k = \rho_c \circ \gamma_d = \{((x, z), \max_{y \in Y}\{\min\{\rho_c(x, y), \gamma_d(y, z)\}\})|(x, z) \in X \times Z\}$ for all $k \in K$, $c \in C$ and $d \in D$.

Definition 2.16 ([17]). Let X and Y be any two non-empty sets. Let E be a set of parameters of $X \times Y$. Let (ρ, A) and (γ, B) where $A, B \subseteq E$ be fuzzy soft relations over $X \times Y$. Then their intersection is $(\rho, A)\tilde{\cap}(\gamma, B) = (h, C)$ where $C = A \cap B$ and $(h, C) = \{\rho_c \wedge \gamma_c|c \in C\}$ and $\rho_c \wedge \gamma_c = \{((x, y), \rho_a(x, y) \wedge \gamma_b(x, y))|(x, y) \in X \times Y\}$.

Notation 2.17 ([17]). Let $X = Y = Z$ and E be a parameter set of $X \times X$. Let $C \subseteq E$ and (ρ, C) be fuzzy soft relation over $X \times X$. Then $(\rho, C)\tilde{\circ}(\rho, C) = \{\rho_c \circ \rho_c|c \in C\}$.

Proposition 2.18 ([17]). *Let X be a universal set and A and C be subsets of a set of parameters E of $X \times X$. If (ρ, C) and (γ, A) are fuzzy soft similarity relations over $X \times X$, then*
(i) $(\rho, C)\tilde{\cap}(\gamma, A)$ is a fuzzy soft similarity relation over $X \times X$.
(ii) $(\rho, C)\tilde{\circ}(\gamma, A)$ is a fuzzy soft similarity relation over $X \times X$,
if $(\rho, C)\tilde{\circ}(\gamma, A) = (\gamma, A)\tilde{\circ}(\rho, C)$.

Theorem 2.19 ([17]). *Let X be a universal set and C be a subset of set of parameters of $X \times X$. Let $(\rho_c)_\alpha$ be the α-level set of a fuzzy soft similarity relation (ρ, C). Then (ρ, C) is a fuzzy soft similarity relation iff for $\alpha \in (0, 1]$, $(\rho_c)_\alpha$ is an equivalence relation on X.*

Theorem 2.20 ([17]). *Let X be a universal set and E be a set of parameters of $X \times X$. Then a relation R is an equivalence relation on X iff (Ψ_R, E) is a fuzzy soft similarity relation over $X \times X$.*

Lemma 2.21 ([17]). *Let (ρ, C) be a fuzzy soft similarity relation over $X \times X$. Let $\rho_c^{-1}(1) = \{(x,y) \in X \times X | \rho_c(x,y) = 1\}$. Then for each $c \in C$, $\rho_c^{-1}(1)$ is an equivalence relation on X.*

Proposition 2.22 ([17]). *Let X be a universal set and let A and C be subsets of a set of parameters E of $X \times X$. If (ρ, C) and (γ, A) are fuzzy soft similarity relations over $X \times X$ and if $(\rho, C)\tilde{\circ}(\gamma, A) = (\gamma, A)\tilde{\circ}(\rho, C)$ then $(\rho, C)\tilde{\circ}(\gamma, A)$ is generated by $(\rho, C)\tilde{\cup}(\gamma, A)$.*

Lemma 2.23 ([17]). *If $FSSR(X)$ is the set of all fuzzy soft similarity relations then $(FSSR(X), +, .)$ is a lattice where $+$ and $.$ are defined by $(\rho, C) + (\gamma, A) = (\rho, C)\tilde{\circ}(\gamma, A)$ and $(\rho, C).(\gamma, A) = (\rho, C)\tilde{\cap}(\gamma, A)$ and $(\rho, C)\tilde{\circ}(\gamma, A) = (\gamma, A)\tilde{\circ}(\rho, C)$.*

Definition 2.24. A lattice $(L, +, .)$ is called modular if $(x+y).z \leq x + (y.z)$ for all $x, y, z \in L$ with $x \leq z$.

Theorem 2.25 ([17]). *$(FSSR(X), +, .)$ with $(\rho, C)\tilde{\circ}(\gamma, A) = (\gamma, A)\tilde{\circ}(\rho, C)$ for all (ρ, C), $(\gamma, A) \in FSSR(X)$ is a modular lattice.*

3. Fuzzy soft N-subgroups and N-ideals

In this section fuzzy soft N-subgroups and fuzzy soft N-ideal over a right ternary N-group are defined and their basic algebraic properties are studied.

Throughout this section ${}_N\Gamma$ denotes a right ternary N-group and E denotes a set of parameters associated with ${}_N\Gamma$, A is a subset of E and (f, A) denotes a fuzzy soft set over ${}_N\Gamma$. The zero element in Γ is denoted as 0_Γ.

Definition 3.1. (f, A) is a fuzzy soft N-subgroup if
- (i) $f_a(\gamma + \delta) \geq \min\{f_a(\gamma), f_a(\delta)\}$,
- (ii) $f_a(-\gamma) \geq f_a(\gamma)$,
- (iii) $f_a([x\ y\ \gamma]_\Gamma) \geq f_a(\gamma)$ for every $a \in A$, $\gamma \in \Gamma$ $x, y \in N$.

Remark 3.2. By combining (i) and (ii), $f_a(\gamma - \delta) \geq \min\{f_a(\gamma), f_a(\delta)\}$.

Example 3.3. Let $N = \{0, x, y, z\} = \Gamma$. Define $+$ on Γ as in Table 1 and the ternary operation $[\ ]_\Gamma$ on ${}_N\Gamma$ by $[x\ y\ z]_\Gamma = (x.y).z$ for every $x, y, z \in N$ where '.' is defined as in Table 2. Then $({}_N\Gamma, +, [\ ]_\Gamma)$ is a right ternary N-group. Let $A = \{0, x\}$

Table 1.

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table 2.

.	0	x	y	z
0	0	0	0	0
x	0	0	0	x
y	0	x	y	y
z	0	x	y	z

and $f : A \to I^{{}_N\Gamma}$ be defined by

$$(f_a)(t) = \begin{cases} 1 & \text{if } t \in \{0, x\} \\ 0.6 & \text{otherwise} \end{cases}$$

for every $a \in A$. Then (f, A) is a fuzzy soft N-subgroup over ${}_N\Gamma$.

Definition 3.4. (f, A) is a fuzzy soft N-ideal over ${}_N\Gamma$ if
(i) $f_a(\gamma - \delta) \geq \min\{f_a(\gamma), f_a(\delta)\}$
(ii) $f_a(\gamma + \delta - \gamma) \geq f_a(\delta)$
(iii) $f_a([x\ y\ (\gamma + \delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \geq f_a(\delta)$ for every $a \in A$ $x, y \in N$, $\gamma, \delta \in \Gamma$.

Example 3.5. Let $N = \{0, x, y, z\} = \Gamma$. Define $+$ as in Table 3 and $[\ ]_\Gamma$ by $[x\ y\ z] = (x.y).z$ for every $x, y, z \in N$ where . is defined as in Table 4. Then $({}_N\Gamma, +, [\ ]_\Gamma)$ is a right ternary N-group and $\{0, x\}$ and $\{0, y\}$ are N-ideals of ${}_N\Gamma$. Let $B = \{0, x\}$ and $f : B \to I^{{}_N\Gamma}$ be defined by

Table 3.

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table 4.

.	0	x	y	z
0	0	0	0	0
x	0	0	0	0
y	0	0	0	0
z	0	x	y	x

$$(g_b)(t) = \begin{cases} 1 & \text{if } t \in \{0, x\} \\ 0.7 & \text{otherwise} \end{cases}$$

for every $b \in B$ then (g, B) is a fuzzy soft N-ideal over ${}_N\Gamma$.

Lemma 3.6. *If (f, A) is a fuzzy soft N-subgroup over ${}_N\Gamma$, then*
(1) $f_a(-\gamma) = f_a(\gamma)$
(2) $f_a(0_\Gamma) \geq f_a(\gamma)$,
(3) *if* $f_a(\gamma - \delta) = f_a(0_\Gamma)$, *then* $f_a(\gamma) = f_a(\delta)$, *for every* $a \in A$ *and* $\gamma, \delta \in \Gamma$.

Proof. Let (f, A) be a fuzzy soft N-subgroup over ${}_N\Gamma$ and let $\gamma, \delta \in \Gamma$ and $a \in A$. Then (1) $f_a(\gamma) = f_a(-(-\gamma)) \geq f_a(-\gamma) \geq f_a(\gamma)$.
(2) $f_a(0_\Gamma) = f_a(\gamma - \gamma) \geq \min\{f_a(\gamma), f_a(-\gamma)\} \geq f_a(\gamma)$.
(3) $f_a(\gamma) = f_a(\gamma - \delta + \delta) \geq \min\{f_a(\gamma - \delta), f_a(\delta)\} = \min\{f_a(0_\Gamma), f_a(\delta)\} = f_a(\delta)$. Similarly it can be proved that $f_a(\delta) \geq f_a(\gamma)$ and hence (3). □

Theorem 3.7. *Let (f, A) is a fuzzy soft set over ${}_N\Gamma$ and let for $a \in A$, $\Delta = \{\gamma \in \Gamma | f_a(\gamma) = f_a(0_\Gamma)\}$. Then*
(1) *Δ is an N-subgroup of ${}_N\Gamma$ if (f, A) is a fuzzy soft N-subgroup over ${}_N\Gamma$.*
(2) *Δ is an N-ideal of ${}_N\Gamma$ if (f, A) is a fuzzy soft N-ideal over ${}_N\Gamma$.*

Proof. (1) Let $\gamma, \delta \in \Delta$ and $a \in A$. Then $f_a(\gamma - \delta) \geq \min\{f_a(\gamma), f_a(\delta)\} = f_a(0_\Gamma)$ by hypothesis. Thus, by (2) of the above lemma, $f_a(\gamma - \delta) = f_a(0_\Gamma)$ which implies that $\gamma - \delta \in \Delta$. Now let $\delta \in \Delta$ and $x, y \in N$. Then $f_a([x\ y\ \delta]_\Gamma) \geq f_a(\delta) = f_a(0_\Gamma)$. So, using (2) of the above lemma, $f_a([x\ y\ \delta]_\Gamma) = f_a(0_\Gamma)$ which implies that $[x\ y\ \delta]_\Gamma \in \Delta$. Hence $[N\ N\ \Delta]_\Gamma \subseteq \Gamma$ showing that Δ is an N-subgroup of ${}_N\Gamma$.

(2) Let $\delta \in \Delta, \gamma \in \Gamma$ and $a \in A$. Then $f_a(\gamma + \delta - \gamma) \geq f_a(\delta)$ and thus $f_a(\gamma + \delta - \gamma) \geq f_a(0_\Gamma)$. So $f_a(\gamma + \delta - \gamma) = f_a(0_\Gamma)$ which implies that $\gamma + \delta - \gamma \in \Delta$. Now let $\delta \in \Delta$, $\gamma \in \Gamma$ and $x, y \in N$. Then $f_a([x\ y\ (\gamma + \delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \geq f_a(\delta) = f_a(0_\Gamma)$. Thus, by (2) of Lemma 3.6, $f_a([x\ y\ (\gamma + \delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) = f_a(0_\Gamma)$ which implies $[x\ y\ (\gamma + \delta)]_\Gamma - [x\ y\ \gamma]_\Gamma \in \Delta$. Hence Δ is an N-ideal of ${}_N\Gamma$. □

Corollary 3.8. *Let (f, A) be a fuzzy soft set over ${}_N\Gamma$ and $\Delta' = \{\gamma \in \Gamma | f_a(\gamma) = 1\}$. Then Δ' is an N-subgroup (N-ideal) of ${}_N\Gamma$ if (f, A) is a fuzzy soft N-subgroup (N-ideal) over ${}_N\Gamma$.*

Proof. By taking $f_a(0_\Gamma) = 1$ in Δ in the above theorem the proof follows. □

Remark 3.9. In general the converse of the above corollary is not true. For example, let N and Γ be as in Example 3.5. Then $\{0, x\}$ is an N-ideal and N-subgroup of ${}_N\Gamma$.

Let $A = \{0, x\}$ and $f : A \to I^{{}_N\Gamma}$ be defined as $f_a(t) = \begin{cases} 1 & \text{if } t = 0, x \\ 0.7 & \text{if } t = y \\ 0.3 & \text{if } t = z \end{cases}$
for every $a \in A$. Then $f_a(x - y) = f_a(z) = 0.3$ and $\min\{f_a(x), f_a(y)\} = 0.7$. Thus $f_a(x - y) \leq \min\{f_a(x), f_a(y)\}$ which implies that (f, A) is not a fuzzy soft N-ideal and N-subgroup over ${}_N\Gamma$. The converse holds if $|Im\ f_a| = 2$ which is established in the following theorem.

Theorem 3.10. *If (f, A) is a fuzzy soft set over ${}_N\Gamma$, $|Im\ f_a| = 2$ where $a \in A$ and if $\Delta' = \{\gamma \in \Gamma | f_a(\gamma) = 1\}$ is an N-subgroup (N-ideal) of ${}_N\Gamma$ then (f, A) is a fuzzy soft N-subgroup (N-ideal) over ${}_N\Gamma$.*

Proof. Let (f, A) be a fuzzy soft set over ${}_N\Gamma$, $\gamma, \delta \in \Gamma$ and $a \in A$. Let $Im\ f_a = \{\alpha, 1\}$ where $\alpha \in [0, 1)$. Then we have the following three cases: (i) $\gamma, \delta \in \Delta'$, (ii) $\gamma \in \Delta'$, $\delta \notin \Delta'$, (iii) $\gamma, \delta \notin \Delta'$.

Case (i) : Let $\gamma, \delta \in \Delta'$. Suppose $f_a(\gamma-\delta) < \min\{f_a(\gamma), f_a(\delta)\}$. Then $f_a(\gamma-\delta) < 1$ which implies that $\gamma-\delta \notin \Delta'$, a contradiction. Thus $f_a(\gamma-\delta) \geq \min\{f_a(\gamma), f_a(\delta)\}$.

Case (ii) : Let $\gamma \in \Delta'$, $\delta \notin \Delta'$. Then $\gamma - \delta \notin \Delta'$. This implies that $f_a(\gamma - \delta) = \alpha = f_a(\delta) = \min\{f_a(\gamma), f_a(\delta)\}$.

Case (iii) : Let $\gamma, \delta \notin \Delta'$. Then $\gamma - \delta \notin \Delta'$. This implies that

$$f_a(\gamma - \delta) = \alpha = \min\{f_a(\gamma), f_a(\delta)\}.$$

Now let $x, y \in N$ and $\gamma \in \Gamma$. If $\gamma \in \Delta'$, then $f_a(\gamma) = 1 = f_a([x\ y\ \gamma]_\Gamma)$. If $\gamma \notin \Delta'$ then $f_a(\gamma) = \alpha = f_a([x\ y\ \gamma]_\Gamma)$. Thus in both the cases $f_a([x\ y\ \gamma]_\Gamma) \geq f_a(\gamma)$. So (f, A) is a fuzzy soft N-subgroup over ${}_N\Gamma$.

By arguing in the same manner it can be proved that if $\Delta' = \{\gamma \in \Gamma | f_a(\gamma) = 1\}$ is an N-ideal of ${}_N\Gamma$, then (f, A) is a fuzzy soft N-ideal over ${}_N\Gamma$. □

Proposition 3.11. *The intersection of two non-empty fuzzy soft N-subgroups is a fuzzy soft N-subgroup. Similarly the intersection of two non-empty fuzzy soft N-ideals over ${}_N\Gamma$ is a fuzzy soft N-ideal over ${}_N\Gamma$.*

Proof. Let ${}_N\Gamma$ be a right ternary N-group and A, B be two subsets of the parameter set E of ${}_N\Gamma$. Let (f, A) and (g, B) be any two non-empty fuzzy soft N-subgroups over ${}_N\Gamma$. Let $(f, A)\tilde{\cap}(g, B) = (h, C)$. Then for $\gamma, \delta \in \Gamma$

$$\begin{aligned} h_c(\gamma - \delta) &= \min\{f_c(\gamma - \delta), g_c(\gamma - \delta)\} \\ &\geq \min\{\min\{f_c(\gamma), f_c(\delta)\}, \min\{g_c(\gamma), g_c(\delta)\}\} \\ &= \min\{\min\{f_c(\gamma), g_c(\gamma)\}, \min\{f_c(\delta), g_c(\delta)\}\} \\ &= \min\{h_c(\gamma), h_c(\delta)\}. \end{aligned}$$

Now

$$\begin{aligned} h_c([x\ y\ \gamma]_\Gamma) &= \min\{f_c([x\ y\ \gamma]_\Gamma), g_c([x\ y\ \gamma]_\Gamma)\} \\ &\geq \min\{f_c(\gamma), g_c(\gamma)\} \\ &= h_c(\gamma) \end{aligned}$$

for $c \in C$ and $x, y \in N$, $\gamma \in \Gamma$. This implies that (h, C) is also a fuzzy soft N-subgroup over ${}_N\Gamma$.

Let (f, A) and (g, B) be any two non-empty fuzzy soft N-ideals over ${}_N\Gamma$. Consider

$$\begin{aligned} h_c(\gamma + \delta - \gamma) &= \min\{f_c(\gamma + \delta - \gamma), g_c(\gamma + \delta - \gamma)\} \\ &\geq \min\{f_c(\delta), g_c(\delta)\} \\ &= h_c(\delta). \end{aligned}$$

Now

$$\begin{aligned} h_c([x\ y\ \gamma]_\Gamma) &= \min\{f_c(([x\ y\ \gamma]_\Gamma), g_c(([x\ y\ \gamma]_\Gamma)\} \\ &\geq \min\{f_c(\gamma), g_c(\gamma)\} \\ &= h_c(\gamma). \end{aligned}$$

Also

$$\begin{aligned} &h_c([x\ y\ (\gamma + \delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \\ &\quad = \min\{f_c([x\ y\ (\gamma + \delta)]_\Gamma - [x\ y\ \gamma]_\Gamma, g_c([x\ y\ (\gamma + \delta)]_\Gamma - [x\ y\ \gamma]_\Gamma)\} \\ &\quad \geq \min\{f_c(\delta), g_c(\delta)\} \\ &\quad = h_c(\delta). \end{aligned}$$

Thus (h, C) is a fuzzy soft N-ideal over ${}_N\Gamma$. □

Proposition 3.12. (1) *The $\tilde{\wedge}$ intersection of any two non-empty fuzzy soft N-subgroups is fuzzy soft N-subgroups over ${}_N\Gamma$.*

(2) *The $\tilde{\wedge}$ intersection of any two non-empty fuzzy soft N-ideals is a fuzzy soft N-ideal over ${}_N\Gamma$.*

Proof. (1) Let ${}_N\Gamma$ be an RTNG and A, B be two subsets of the parameter set E of ${}_N\Gamma$. Let (f, A) and (g, B) be any two non-empty fuzzy soft N-subgroups over ${}_N\Gamma$. Let $(f, A)\tilde{\wedge}(g, B) = (h, C)$, where $C = A \times B$ and $h : C \to I^{{}_N\Gamma}$. Consider

$$\begin{aligned} h_c(\gamma - \delta) &= \min\{f_a(\gamma - \delta), g_b(\gamma - \delta)\} \\ &\geq \min\{\min\{f_a(\gamma), f_a(\delta)\}, \min\{g_b(\gamma), g_b(\delta)\}\} \\ &= \min\{\min\{f_a(\gamma), g_b(\gamma)\}, \min\{f_a(\delta), g_b(\delta)\}\} \\ &= \min\{h_c(\gamma), h_c(\delta)\}. \end{aligned}$$

Now $h_c([x\ y\ \gamma]_\Gamma) = \min\{f_a([x\ y\ \gamma]_\Gamma),\ g_b([x\ y\ \gamma]_\Gamma)\} \geq \min\{f_a(\gamma), g_b(\gamma)\} = h_c(\gamma)$, for every $c \in C$, $x, y \in N$ and $\gamma \in \Gamma$. Hence (h, C) is a fuzzy soft N-subgroup over ${}_N\Gamma$.

(2) Let (f, A) and (g, B) be any two non-empty fuzzy soft N-ideals over ${}_N\Gamma$. Consider

$$\begin{aligned} h_c(\gamma+\delta-\gamma) &= \min\{f_a(\gamma+\delta-\gamma), g_b(\gamma+\delta-\gamma)\} \\ &\geq \min\{f_a(\delta), g_b(\delta)\} = h_c(\delta). \end{aligned}$$

Also

$$\begin{aligned} &h_c([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \\ &= \min\{f_a([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma), g_b(([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma)\} \\ &\geq \min\{f_a(\delta), g_b(\delta)\} = h_c(\delta). \end{aligned}$$

Thus (h, C) is a fuzzy soft N-ideal over ${}_N\Gamma$. □

Proposition 3.13. (1) *The cartesian product of any two non-empty fuzzy soft N-subgroups over ${}_N\Gamma$ and ${}_N\Gamma'$ is a fuzzy soft N-subgroup over ${}_N\Gamma \times {}_N\Gamma'$.*

(2) *The cartesian product of any two non-empty fuzzy soft N-ideals over ${}_N\Gamma$ and ${}_N\Gamma'$ is a fuzzy soft N-ideal over ${}_N\Gamma \times {}_N\Gamma'$.*

Proof. (1) Let ${}_N\Gamma$ and ${}_N\Gamma'$ be any two non-empty right ternary N-groups. Let E_1, E_2 be parameter sets of ${}_N\Gamma$ and ${}_N\Gamma'$ respectively. Let $A \subseteq E_1$, $B \subseteq E_2$ and (f, A) and (g, B) be any two non-empty fuzzy soft N-subgroups over ${}_N\Gamma$ and ${}_N\Gamma'$ respectively. Then $(f, A) \times (g, B) = (h, C)$ where $C = A \times B$ is a fuzzy soft N-subgroup over ${}_N\Gamma \times {}_N\Gamma'$. Now

$$\begin{aligned} h_{(a,b)}((\gamma,\gamma') - (\delta,\delta')) &= h_{(a,b)}((\gamma-\delta), (\gamma'-\delta')) \\ &= \min\{f_a(((\gamma-\delta)), g_b((\gamma'-\delta'))\} \\ &\geq \min\{\min\{f_a(\gamma), f_a(\delta)\}, \min\{g_b(\gamma'), g_b(\delta')\}\} \\ &= \min\{\min\{f_a(\gamma), g_b(\gamma')\}, \min\{f_a(\delta), g_b(\delta')\}\} \\ &= \min\{h_c(\gamma,\gamma')), h_c(\delta,\delta')\} \end{aligned}$$

for $\gamma, \delta \in N$ and $\gamma', \delta' \in M$. Also

$$\begin{aligned} h_{(a,b)}([x\ y\ (\gamma,\gamma')]_{\Gamma\times\Gamma'}) &= h_{(a,b)}([x\ y\ \gamma]_\Gamma, [x\ y\ \gamma']_{\Gamma'}) \\ &= \min\{f_a([x\ y\ \gamma]_\Gamma), g_b([x\ y\ \gamma']_{\Gamma'})\} \\ &\geq \min\{f_a(\gamma)\}, g_b(\gamma')\} \\ &= h_{(a,b)}(\gamma,\gamma') \end{aligned}$$

for $x, y, u, v \in N$ and $\gamma \in \Gamma$, $\gamma' \in \Gamma'$. Hence (h, C) is a fuzzy soft N-subgroup over ${}_N\Gamma \times {}_N\Gamma'$.

(2) Now consider

$$\begin{aligned} h_c((\gamma,\gamma') + (\delta,\delta') - (\gamma,\gamma')) &= h_c((\gamma+\delta-\gamma, \gamma'+\delta'-\gamma')) \\ &= \min\{f_a(\gamma+\delta-\gamma), g_b(\gamma'+\delta'-\gamma')\} \\ &\geq \min\{f_a(\delta), g_b(\delta')\} = h_c((\delta,\delta')). \end{aligned}$$

Now

$$\begin{aligned} &h_c([x\ y\ (\gamma+\delta,\gamma'+\delta')]_{\Gamma\times\Gamma'}-[x\ y\ (\gamma,\gamma')]_{\Gamma\times\Gamma'}) \\ &\qquad = h_c(([x\ y\ (\gamma+\delta)]_{\Gamma},[x\ y\ (\gamma'+\delta')]_{\Gamma'})-([x\ y\ \gamma]_{\Gamma},[x\ y\ \gamma']_{\Gamma'})) \\ &\qquad = (h_c([x\ y\ (\gamma+\delta)]_{\Gamma}-[x\ y\ \gamma]_{\Gamma},[x\ y\ (\gamma'+\delta')]_{\Gamma'}-[x\ y\ \gamma']_{\Gamma'}) \\ &\qquad = \min\{f_a(([x\ y\ (\gamma+\delta)]_{\Gamma}-[x\ y\ \gamma]_{\Gamma}),g_b(([x\ y\ (\gamma'+\delta')]_{\Gamma'}-[x\ y\ \gamma']_{\Gamma'})\} \\ &\qquad \geq \min\{f_a(\delta),g_b(\delta')\}=h_c((\delta,\delta')). \end{aligned}$$

Thus (h,C) is a fuzzy soft N-ideal over ${}_N\Gamma\times{}_N\Gamma'$. □

Proposition 3.14. *Let (ϕ,ψ) be a fuzzy soft onto homomorphism from ${}_N\Gamma$ to ${}_N\Gamma'$. Then*

(1) *$(\phi,\psi)(f,A)$ is a fuzzy soft N-subgroup over ${}_N\Gamma'$ if (f,A) is a fuzzy soft N-subgroup over ${}_N\Gamma$.*

(2) *$(\phi,\psi)(f,A)$ is a fuzzy soft N-ideal over ${}_N\Gamma'$ if (f,A) is a fuzzy soft N-ideal over ${}_N\Gamma$.*

Proof. Let $\phi:{}_N\Gamma\to{}_N\Gamma'$ be an onto N-homomorphism. Let E_1 and E_2 be parameter sets for ${}_N\Gamma$ and ${}_N\Gamma'$ respectively. Let $\psi:A\to B$ where $A\subseteq E_1$, $B\subseteq E_2$ be a mapping such that $\psi(a)=b$, where $a\in A$, $b\in B$.

Let $\gamma',\delta'\in\Gamma'$. Since ϕ is onto there exists γ,δ respectively in Γ such that $\phi(\gamma)=\gamma'$, $\phi(\delta)=\delta'$. Also

$$\phi(\gamma-\delta)=\phi(\gamma)-\phi(\delta)=\gamma'-\delta'$$

and

$$\phi([x\ y\ \gamma]_{\Gamma})=[x\ y\ \phi(\gamma)]_{\Gamma'}=[x\ y\ \gamma']_{\Gamma'}.$$

(1) Consider

$$\begin{aligned} (\phi(f))_b(\gamma'-\delta') &= \bigvee_{\theta\in\phi^{-1}(\gamma'-\delta')}\Big(\bigvee_{e\in\psi^{-1}(b)\cap A} f_e(\theta)\Big) \\ &\geq \bigvee_{\theta\in\phi^{-1}(\gamma'-\delta')} f_a(\theta) \\ &\geq f_a(\gamma-\delta) \\ &\geq \min\{f_a(\gamma),f_a(\delta)\}. \end{aligned}$$

Then $(\phi(f))_b((\gamma'-\delta'))\geq\min\{(\phi(f))_b(\gamma'),(\phi(f))_b(\delta')\}$. Also

$$(\phi(f))_b([x\ y\ \gamma']_{\Gamma})=\bigvee\nolimits_{\lambda\in\phi^{-1}([x\ y\ \gamma']_{\Gamma'}])} f_a(\lambda)\geq f_a([x\ y\ \gamma]_{\Gamma})\geq f_a(\gamma),$$

for every $x,y\in N$, $\gamma\in\Gamma$.

Thus $(\phi(f))_b([x\ y\ \gamma']_{\Gamma'})\geq\phi(f))_b(\gamma')$ proving that $(\phi,\psi)(f,A)$ is a fuzzy soft N-subgroup over ${}_N\Gamma'$.

(2) Consider

$$
\begin{aligned}
(\phi(f))_b((\gamma'+\delta'-\gamma')) &= \bigvee_{\theta\in\phi^{-1}(\gamma'+\delta'-\gamma')} \Big(\bigvee_{e\in\psi^{-1}(b)\cap A} f_e(\theta)\Big) \\
&\geq \bigvee_{\theta\in\phi^{-1}(\gamma'+\delta'-\gamma')} f_a(\theta) \geq f_a(\gamma+\delta-\gamma) \\
&\geq f_a(\delta).
\end{aligned}
$$

Then $(\phi(f))_b(\gamma'+\delta'-\gamma') \geq (\phi(f))_b(\delta')$. Also for $x, y \in N$,

$$
\begin{aligned}
(\phi(f))_b([x\ y\ (\gamma'+\delta')]_{\Gamma'} - [x\ y\ \gamma']_{\Gamma'}) &= \bigvee_{\theta\in\phi^{-1}([x\ y\ (\gamma'+\delta')]_{\Gamma'})} \Big(\bigvee_{e\in\psi^{-1}(b)\cap A} f_e(\theta)\Big) \\
&\geq \bigvee_{\theta\in\phi^{-1}([x\ y\ (\gamma'+\delta')]_{\Gamma'})} (f_a(\theta)) \\
&\geq f_a([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \\
&\geq f_a(\delta).
\end{aligned}
$$

Thus $(\phi(f))_b([x\ y\ (\gamma'+\delta')]_{\Gamma'} - [x\ y\ \gamma']_{\Gamma'}) \geq (\phi(f))_b(\delta')$ which completes the proof. □

Proposition 3.15. *Let (ϕ, ψ) be an onto fuzzy soft N-homomorphism from ${}_N\Gamma$ to ${}_N\Gamma'$. Then*

(1) $(\phi, \psi)^{-1}(g, B)$ is a fuzzy soft N-subgroup over ${}_N\Gamma$ if (g, B) is a fuzzy soft N-subgroup over ${}_N\Gamma'$.

(2) $(\phi, \psi)^{-1}(g, B)$ is a fuzzy soft N-ideal over ${}_N\Gamma$ if (g, B) is a fuzzy soft N-ideal over ${}_N\Gamma'$.

Proof. Let $\phi : {}_N\Gamma \to {}_N\Gamma'$ be an onto N-homomorphism. Let E_1 and E_2 be parameter sets for ${}_N\Gamma$ and ${}_N\Gamma'$ respectively. Let $\psi : A \to B$ where $A \subseteq E_1$, $B \subseteq E_2$. Let (g, B) be a fuzzy soft set over ${}_N\Gamma'$. To prove that $(\phi, \psi)^{-1}(g, B) = (\phi^{-1}(g), \psi^{-1}(B))$ is a fuzzy soft N-subgroup over ${}_N\Gamma$ let $\phi^{-1}(g) = h$, $\psi^{-1}(B) = C$. Then $h : C \to I^{{}_N\Gamma}$ and $h_c : {}_N\Gamma \to I$.

(1) Consider

$$
\begin{aligned}
h_c(\gamma-\delta) &= g_{\psi(c)}(\phi(\gamma)-\phi(\delta)) \\
&\geq \min\{g_{\psi(c)}(\phi(\gamma)), g_{\psi(c)}(\phi(\delta))\} \\
&= \min\{h_c(\gamma), h_c(\delta)\}.
\end{aligned}
$$

Also for $x, y \in N$, $h_c([x\ y\ \gamma]_\Gamma) = g_{\psi(c)}(\phi([x\ y\ \gamma]_\Gamma)]) \geq g_{\psi(c)}(\phi(\gamma))\} = h_c(\gamma)$. Thus $(\phi, \psi)^{-1}(g, B)$ is a fuzzy soft N-subgroup fuzzy soft over ${}_N\Gamma$.

(2) Consider

$$
\begin{aligned}
h_c(\gamma+\delta-\gamma) &= g_{\psi(c)}(\phi(\gamma+\delta-\gamma)) \\
&= g_{\psi(c)}(\phi(\gamma)+\phi(\delta)-\phi(\gamma)) \\
&\geq g_{\psi(c)}(\phi(\delta)).
\end{aligned}
$$

Also for $x, y \in N$,

$$\begin{aligned} h_c([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) &= g_{\psi(c)}(\phi([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \\ &= g_{\psi(c)}([x\ y\ (\phi(\gamma)+\phi(\delta))]_{\Gamma'} - [x\ y\ \phi(\gamma)]_{\Gamma'} \\ &\geq g_{\psi(c)}(\phi(\delta) \\ &= h_c(\delta). \end{aligned}$$

Thus $(\phi, \psi)^{-1}(g, B)$ is a fuzzy soft N-ideal over ${}_N\Gamma$. □

Definition 3.16. If Δ is a non-empty subset of ${}_N\Gamma$ then the characteristic function (Ψ_Δ, E) is defined by

$$(\Psi_\Delta)_e(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Delta \\ 0 & \text{if } \gamma \notin \Delta \end{cases}, \quad \text{for every } e \in E.$$

Next we consider a subset Δ of ${}_N\Gamma$ and define a fuzzy soft set (f, A) as

$$(f_a)(\gamma) = \begin{cases} r & \text{if } \gamma \in \Delta \\ t & \text{otherwise} \end{cases} \quad \text{where } r > t, a \in A$$

and prove a necessary and sufficient condition for Δ to be an N-subgroup (N-ideal) of ${}_N\Gamma$.

Theorem 3.17. (1) *A non-empty subset Δ of ${}_N\Gamma$ is an N-subgroup (N-ideal) of ${}_N\Gamma$ iff (f, A) is a fuzzy soft N-subgroup (N-ideal) over ${}_N\Gamma$.*

(2) *In particular a non-empty subset Δ of ${}_N\Gamma$ is an N-subgroup (N-ideal) of ${}_N\Gamma$ iff the characteristic function (Ψ_Δ, E) is a fuzzy soft N-subgroup (N-ideal) over ${}_N\Gamma$.*

Proof. (1) Let Δ be an N-subgroup of ${}_N\Gamma$ and $\gamma, \delta \in \Gamma$. Then we have the following three cases: (i) $\gamma, \delta \in \Delta$, (ii) $\delta \in \Gamma$, $\gamma \notin \Delta$, (iii) $\gamma, \delta \notin \Delta$.

Case (i): Since Δ is an N-subgroup of ${}_N\Gamma$, $\gamma - \delta \in \Delta$. Thus $f_a(\gamma - \delta) = \min\{f_a(\gamma), f_a(\delta)\}$.

Case (ii): Let $\delta \in \Gamma$, $\gamma \notin \Delta$. Then $f_a(\gamma - \delta) = t$ and $\min\{f_a(\gamma), f_a(\delta)\} = t$. Thus $f_a(\gamma - \delta)) = \min\{f_a(\gamma), f_a(\delta)\}$.

Case (iii): Let $\gamma, \delta \notin \Delta$. Then $f_a(\gamma - \delta)) = t$, $\min\{f_a(\gamma), f_a(\delta)\} = t$. Thus $f_a(\gamma - \delta) = \min\{f_a(\gamma), f_a(\delta)\}$.

Similarly it can be shown that $f_a([x\ y\ \gamma]_\Gamma) \geq f_a(\gamma)\}$. Thus (f, A) is a fuzzy soft N-subgroup over ${}_N\Gamma$.

Let Δ be an N-ideal and $\gamma, \delta \in \Gamma$. If $\delta \in \Delta$, then $\gamma + \delta - \gamma \in \Delta$. Thus $f_a(\gamma + \delta - \gamma) = r = f_a(\gamma)$, as $\delta \in \Delta$.

If $\delta \notin \Delta$, then $\gamma + \delta - \gamma \notin \Delta$. Thus $f_a(\gamma + \delta - \gamma) = t = f_a(\delta)$. Again arguing in the same manner it can be established that for $x, y \in N$, $f_a([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \geq f_a(\delta)$. Thus (f, A) is a fuzzy soft N-ideal over ${}_N\Gamma$.

Conversely let $\gamma, \delta \in \Delta$. Then $f_a(\gamma) = r$, $f_a(\delta) = r$ and hence $\min\{f_a(\gamma), f_a(\delta)\} = r$ and $f_a(\gamma - \delta)\} = r$ as (f, A) is a fuzzy soft N-ideal over ${}_N\Gamma$. Thus $\gamma - \delta \in \Delta$.

Now let $\delta \in \Delta$ and $\gamma \in \Gamma$. Then $f_a(\gamma + \delta - \gamma) \geq r$ which implies $\gamma + \delta - \gamma \in \Delta$; otherwise if $\gamma + \delta - \gamma \notin \Delta$, then $f_a(\gamma + \delta - \gamma) = t$ and we get $t \geq r$, a contradiction.

A similar argument holds to establish that Δ is an N-ideal of ${}_N\Gamma$.

(2) The proof follows from (1) by taking $r = 1$, $t = 0$, $A = E$ and $f_a = (\Psi_\Delta)_e$. □

The following theorem characterizes a fuzzy soft N-subgroup (N-ideal) in terms of its level sets.

Theorem 3.18. *Let $_N\Gamma$ be a right ternary N-group. Let A be a subset of the parameter set E of $_N\Gamma$. Then (f,A) is a fuzzy soft N-subgroup (N-ideal) over $_N\Gamma$ iff for each f_a, each non-empty level subset $(f_a)_t$, $t \in (0,1]$ is an N-subgroup (N-ideal) of $_N\Gamma$.*

Proof. Let (f,A) be a fuzzy soft N-subgroup over $_N\Gamma$. Let $t \in (0,1]$ be such that $(f_a)_t \neq \varnothing$. Let $\gamma,\delta \in (f_a)_t$. Then $f_a(\gamma) \geq t$, $f_a(\delta) \geq t$. Since $f_a(\gamma - \delta) \geq \min\{f_a(\gamma), f_a(\delta)\}$, we have $f_a(\gamma-\delta) \geq t$ and for $x,y \in N$, $f_a([x\ y\ \gamma]_\Gamma) \geq f_a(\gamma)$, we have $f_a([x\ y\ \gamma]_\Gamma) \geq t$, and therefore $\gamma - \delta \in (f_a)_t$ and $[x\ y\ \gamma]_\Gamma \in (f_a)_t$. Let $\delta \in (f_a)_t$. Since $f_a(\gamma+\delta-\gamma) \geq f_a(\delta) \geq t$, $\gamma+\delta-\gamma \in (f_a)_t$. Also for $x,y \in N$ $\gamma \in \Gamma$, $\delta \in (f_a)_t$ and $f_a([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \geq f_a(\delta) \geq t$. Thus $[x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma \in (f_a)_t$. So $(f_a)_t$ is an N-ideal of $_N\Gamma$.

Conversely let $\gamma,\delta \in \Gamma$. Suppose $f_a(\gamma-\delta) < \min\{f_a(\gamma), f_a(\delta)\}$. Let $s = \min\{f_a(\gamma), f_a(\delta)\}$. Then $\gamma \in (f_a)_s$, $\delta \in (f_a)_s$. Thus $\gamma - \delta \in (f_a)_s$ which implies that $f_a(\gamma-\delta) \geq \min\{f_a(\gamma), f_a(\delta)\}$, a contradiction.

Now suppose $f_a([x\ y\ \gamma]_\Gamma) < f_a(\gamma)$. Let $r = f_a(\gamma)$. Then $\gamma \in (f_a)_r$. Thus $[x\ y\ \gamma]_\Gamma \in (f_a)_r$. So $f_a([x\ y\ \gamma]_\Gamma) \geq f_a(\gamma)$, a contradiction. Hence for each $a \in A$ and $t \in (0,1]$, $f_a(\gamma-\delta) \geq \min\{f_a(\gamma), f_a(\delta)\}$ and $f_a([x\ y\ \gamma]_\Gamma) \geq f_a(\gamma)$. Therefore (f,A) is a fuzzy soft N-subgroup over $_N\Gamma$.

Now for $\gamma,\delta \in \Gamma$ if $f_a(\gamma+\delta-\gamma) < f_a(\delta)$. Let $r = f_a(\delta)$. Then $\delta \in (f_a)_r$ which implies $\gamma+\delta-\gamma \in (f_a)_r$ and thus $f_a(\gamma+\delta-\gamma) \geq r = f_a(\delta)$, a contradiction. So $f_a(\gamma+\delta-\gamma) \geq f_a(\delta)$.

A similar argument holds to prove that $f_a([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \geq f_a(\delta)$ for $x,y \in N$ and $\gamma,\delta \in \Gamma$. Thus (f,A) is a fuzzy soft N-ideal over $_N\Gamma$. □

Proposition 3.19. *Let (f,A) be a fuzzy soft set over $_N\Gamma$. Let $\phi : {_N\Gamma} \to {_N\Gamma}$ be an N-homomorphism and define $f_a^\phi(\gamma) = f_a(\phi(\gamma))$ for every $\gamma \in {_N\Gamma}$ and $a \in A$. Then*

(1) If (f,A) is a fuzzy soft N-subgroup over $_N\Gamma$ then (f^ϕ,A) is a fuzzy soft N-subgroup over $_N\Gamma$.

(2) If (f,A) is a fuzzy soft N-ideal over $_N\Gamma$ then (f^ϕ,A) is a fuzzy soft N-ideal over $_N\Gamma$.

Proof. (1) Consider for $\gamma,\delta \in {_N\Gamma}$,

$$\begin{aligned} f_a^\phi(\gamma-\delta) &= f_a(\phi(\gamma-\delta)) \\ &= f_a(\phi(\gamma)-\phi(\delta)) \\ &\geq \min\{f_a(\phi(\gamma)), f_a(\phi(\delta))\} \\ &= \min\{f_a^\phi(\gamma), f_a^\phi(\delta)\}. \end{aligned}$$

Now for $x,y \in N$, $\gamma \in {_N\Gamma}$,

$$\begin{aligned} f_a^\phi([x\ y\ \gamma]_\Gamma) &= f_a(\phi([x\ y\ \gamma]_\Gamma)) \\ &= f_a([x\ y\ \phi(\gamma)]_\Gamma) \\ &\geq f_a(\phi(\gamma)), \end{aligned}$$

by hypothesis. Thus (f,A) is a fuzzy soft N-subgroup over $_N\Gamma$.

(2) Consider for $\gamma, \delta \in \Gamma$,

$$\begin{aligned} f_a^{\phi}(\gamma+\delta-\gamma) &= f_a(\phi(\gamma+\delta-\gamma)) \\ &= f_a(\phi(\gamma)+\phi(\delta)-\phi(\gamma)) \\ &\geq f_a(\phi(\delta)) = f_a^{\phi}(\delta). \end{aligned}$$

Moreover for $x, y \in N$ and $\gamma, \delta \in \Gamma$,

$$\begin{aligned} f_a^{\phi}([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) &= f_a(\phi([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \\ &= f_a(([x\ y\ (\phi(\gamma)+\phi(\delta))]_\Gamma - [x\ y\ \phi(\gamma)]_\Gamma) \\ &\geq f_a(\phi(\delta). \end{aligned}$$

Thus (f^{ϕ}, A) is a fuzzy soft N-ideal over ${}_N\Gamma$. □

Definition 3.20. A fuzzy soft N-subgroup (N-ideal) (f, A) over ${}_N\Gamma$ is said to be normal if $f_a(0_\Gamma) = 1$ for every $a \in A$.

Proposition 3.21. *Let (f, A) be a fuzzy soft N-subgroup over ${}_N\Gamma$ and (f^+, A) be a fuzzy soft set over ${}_N\Gamma$ where $f_a^+(\gamma) = f_a(\gamma) + 1 - f_a(0_\Gamma)$ for every $a \in A$ and $\gamma \in \Gamma$. Then*

(1) *(f^+, A) is a normal fuzzy soft N-subgroup over ${}_N\Gamma$.*
(2) *(f^+, A) is a normal fuzzy soft N-ideal over ${}_N\Gamma$. Moreover $(f^+, A) \subseteq (f, A)$.*

Proof. (1) Consider

$$\begin{aligned} f_a^+(\gamma-\delta) &= f_a(\gamma-\delta) + 1 - f_a(0_\Gamma) \\ &\geq \min\{f_a(\gamma), f_a(\delta)\} + 1 - f_a(0_\Gamma) \\ &\geq \min\{f_a(\gamma) + 1 - f_a(0_\Gamma), f_a(\delta) + 1 - f_a(0_\Gamma)\} \\ &= \min\{f_a^+(\gamma), f_a^+(\delta)\}. \end{aligned}$$

Now, $f_a^+([x\ y\ \gamma]_\Gamma) = f_a([x\ y\ \gamma]_\Gamma) + 1 - f_a(0_\Gamma) \geq f_a(\gamma) + 1 - f_a(0_\Gamma) = f_a^+(\gamma)$, as (f, A) be a fuzzy soft right N-subgroup over N. Also, $f_a^+(0_\Gamma) = 1$. Thus (f^+, A) is a normal fuzzy soft N-subgroup over ${}_N\Gamma$. This completes the proof of (1).

(2) Consider

$$\begin{aligned} f_a^+(\gamma+\delta-\gamma) &= f_a(\gamma+\delta-\gamma) + 1 - f_a(0_\Gamma) \\ &\geq f_a(\delta) + 1 - f_a(0_\Gamma) = f_a^+(\delta). \end{aligned}$$

Now for $x, y \in N$, $\gamma, \delta \in {}_N\Gamma$,

$$\begin{aligned} f_a^+([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) &= f_a([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) + 1 - f_a(0_\Gamma) \\ &\geq f_a(\delta) + 1 - f_a(0_\Gamma) \\ &= f_a^+(\delta). \end{aligned}$$

Thus (f^+, A) is a normal fuzzy soft N-ideal over ${}_N\Gamma$. Moreover, since $f_a^+(\gamma) \leq f_a(\gamma)$ for all $a \in A$, $(f^+, A) \subseteq (f, A)$. □

Theorem 3.22. *Let (f, A) be a fuzzy soft N-subgroup (N-ideal) over ${}_N\Gamma$. Let (f^*, A) be a fuzzy soft set over ${}_N\Gamma$ where $f_a^*(\gamma) = f_a(\gamma)/f_a(0_\Gamma)$. Then (f^*, A) is a normal fuzzy soft N-subgroup (N-ideal) over ${}_N\Gamma$. Moreover $(f^*, A) = (f, A)$ iff (f, A) is a normal fuzzy soft N-subgroup (N-ideal) over ${}_N\Gamma$.*

Proof. Consider

$$\begin{aligned} f_a^*(\gamma-\delta) &= f_a(\gamma-\delta)/f_a(0_\Gamma) \\ &\geq \min\{f_a(\gamma), f_a(\delta)\}/f_a(0_\Gamma) \\ &= \min\{f_a(\gamma)/f_a(0_\Gamma), f_a(\delta)/f_a(0_\Gamma)\} \\ &= \min\{f_a^*(\gamma), f_a^*(\delta)\}. \end{aligned}$$

Now

$$\begin{aligned} f_a^*([x\ y\ \gamma]_\Gamma) &= f_a([x\ y\ \gamma]_\Gamma)/f_a(0_\Gamma) \\ &\geq f_a(\gamma)/f_a(0_\Gamma) \\ &= f_a^*(\gamma). \end{aligned}$$

Also $f_a^*(0_\Gamma) = 1$. Thus (f^*, A) is a normal fuzzy soft N-subgroup over ${}_N\Gamma$.
Consider for $\gamma, \delta \in \Gamma$,

$$\begin{aligned} f_a^*(\gamma+\delta-\gamma) &= f_a(\gamma+\delta-\gamma)/f_a(0_\Gamma) \\ &\geq f_a(\delta)/f_a(0_\Gamma) \\ &= f_a^*(\delta). \end{aligned}$$

Now for $x, y \in N$ and $\gamma, \delta \in \Gamma$,

$$\begin{aligned} f_a^*([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) &= f_a([x\ y\ (\gamma+\delta)]_\Gamma - [x\ y\ \gamma]_\Gamma)/f_a(0_\Gamma) \\ &\geq f_a(\delta)/f_a(0_\Gamma) \\ &= f_a^*(\delta). \end{aligned}$$

Also $f_a^*(0_\Gamma) = 1$. Thus (f^*, A) is a normal fuzzy soft N-ideal over ${}_N\Gamma$.
Moreover it is obvious that $(f^*, A) = (f, A)$ if (f, A) is a normal fuzzy soft N-subgroup (N-ideal). Conversely if $f_a^*(\gamma) = f_a(\gamma)$ then $f_a(0_\Gamma) = 1$ which implies that (f, A) is a normal fuzzy soft N-subgroup (N-ideal) over ${}_N\Gamma$. □

4. Fuzzy soft congruences over an RTNG

In this section congruences and fuzzy soft congruences over an RTNG are defined. The set of all fuzzy soft congruence relations is proved to be a modular lattice. A fuzzy soft set defined via fuzzy soft congruence relation is shown to be a normal fuzzy soft N-ideal.

Definition 4.1. Let ${}_N\Gamma$ be an RTNG and R be an equivalence relation on ${}_N\Gamma$. Then R is a congruence relation if

(i) (γ, δ), $(\lambda, \theta) \in R$ implies $(\gamma+\lambda, \delta+\theta) \in R$ and
(ii) $(\gamma, \delta) \in R$ implies $([x\ y\ \gamma]_\Gamma, [x\ y\ \delta]_\Gamma) \in R$ for every $x, y \in N$.

Definition 4.2. Let ${}_N\Gamma$ be an RTNG. Let E be a set of parameters of ${}_N\Gamma \times {}_N\Gamma$ and $C \subseteq E$. If (ρ, C) is a fuzzy soft similarity relation over ${}_N\Gamma \times {}_N\Gamma$, then (ρ, C) is fuzzy soft congruence if

(i) $\rho_c(\gamma+\lambda, \delta+\theta) \geq \min\{\rho_c(\gamma, \delta), \rho_c(\lambda, \theta)\}$,
(ii) $\rho_c([x\ y\ \gamma]_\Gamma,\ [x\ y\ \delta]_\Gamma) \geq \rho_c(\gamma, \delta)$ where $c \in C$, $x, y \in N$ and $\gamma, \delta \in \Gamma$.

Example 4.3. Let $({}_N\Gamma, +, [\ ]_\Gamma)$ be the right ternary N-group as in Example 3.5. Now let $E = \{0, x, y, z\}$ and $C = E$. Define

$$(\rho_c)((u,v)) = \begin{cases} 1 & \text{if } (u,v) \in \Gamma \times \Gamma \text{ and } u = v \\ 0.4 & \text{if } (u,v) \in \Gamma \times \Gamma \text{ and } u \neq v. \end{cases}$$

Then (ρ, C) is a fuzzy soft congruence relation over ${}_N\Gamma \times {}_N\Gamma$.

Proposition 4.4. *Let ${}_N\Gamma$ be an right ternary N-group and A and B be subsets of set of parameters E of ${}_N\Gamma \times {}_N\Gamma$. Let $(\rho, A)\tilde{\cap}(\gamma, B) = (h, C)$ where $C = A \cap B$. If (ρ, A) and (γ, B) are fuzzy soft congruences over ${}_N\Gamma \times {}_N\Gamma$ then (h, C) is also a fuzzy soft congruence over ${}_N\Gamma \times {}_N\Gamma$.*

Proof. By Proposition 2.18 (1) (h, C) is a fuzzy soft similarity relation. Now for $c \in C$, $\gamma, \delta, \lambda, \theta \in \Gamma$. Consider

$$\begin{aligned} \min\{h_c(\gamma,\delta), h_c(\lambda,\theta)\} &= \min\{\{\rho_a(\gamma,\delta) \wedge \gamma_b(\gamma,\delta)\}, \{\rho_a(\lambda,\theta) \wedge \gamma_b(\lambda,\theta)\}\} \\ &= \min\{\rho_a(\gamma,\delta), \rho_a(\lambda,\theta)\} \wedge \min\{\gamma_b(\gamma,\delta), \gamma_b(\lambda,\theta)\} \\ &\leq \rho_a(\gamma+\lambda, \delta+\theta) \wedge \gamma_b(\gamma+\lambda, \delta+\theta) \\ &= h_c(\gamma+\lambda, \delta+\theta). \end{aligned}$$

Now for $x, y \in N$ and $\gamma, \delta \in \Gamma$, consider

$$\begin{aligned} h_c(\gamma,\delta) &= \rho_a(\gamma,\delta) \wedge \gamma_b(\gamma,\delta) \\ &\leq \rho_a([x\ y\ \gamma]_\Gamma, [x\ y\ \delta]_\Gamma) \wedge \gamma_b([x\ y\ \gamma]_\Gamma, [x\ y\ \delta]_\Gamma) \\ &= h_c([x\ y\ \gamma]_\Gamma, [x\ y\ \delta]_\Gamma). \end{aligned}$$

Thus (h, C) is fuzzy soft congruence over ${}_N\Gamma \times {}_N\Gamma$. □

Proposition 4.5. *Let ${}_N\Gamma$ be an RTNG and A and B be subsets of set of parameters E of ${}_N\Gamma \times {}_N\Gamma$. Let $(\rho, A)\tilde{\circ}(\gamma, B) = (h, C)$ where $C = A \cap B$. If (ρ, A) and (γ, B) are fuzzy soft congruences over ${}_N\Gamma \times {}_N\Gamma$. Then (h, C) is also a fuzzy soft congruence over ${}_N\Gamma \times {}_N\Gamma$ if $(\rho, A)\tilde{\circ}(\gamma, B) = (\gamma, B)\tilde{\circ}(\rho, A)$.*

Proof. By Proposition 2.18 (2), (h, C) is a fuzzy soft similarity relation. For $c \in C$ $\gamma, \delta, \lambda, \theta, \xi \in \Gamma$, consider

$$\begin{aligned} h_c(\gamma+\lambda, \delta+\theta) &= \bigvee_{\xi \in \Gamma} (\rho_a(\gamma+\lambda, \xi) \wedge \gamma_b(\xi, \delta+\theta)) \\ &\geq \rho_a(\gamma+\lambda, \delta+\theta) \wedge \gamma_b(\delta+\theta, \delta+\theta) \\ &= \rho_a(\gamma+\lambda, \delta+\theta) \\ &\geq \min\{\rho_a(\gamma,\delta), \rho_a(\lambda,\theta)\}. \end{aligned}$$

Also

$$\begin{aligned} h_c(\gamma+\lambda, \delta+\theta) &= \bigvee_{\xi \in \Gamma} (\rho_a(\gamma+\lambda, \xi) \wedge \gamma_b(\xi, \delta+\theta)) \\ &\geq \rho_a(\gamma+\lambda, \gamma+\lambda) \wedge \gamma_b(\gamma+\lambda, \delta+\theta) \\ &= \gamma_b(\gamma+\lambda, \delta+\theta) \\ &\geq \min\{\gamma_b(\gamma,\delta), \gamma_b(\lambda,\theta)\}. \end{aligned}$$

This implies that

$$\begin{aligned} h_c(\gamma+\lambda,\delta+\theta) &\geq \max\{\min\{\rho_a(\gamma,\delta),\rho_a(\lambda,\theta)\},\min\{\gamma_b(\gamma,\delta),\gamma_b(\lambda,\theta)\} \\ &\geq \min\{\min\{\rho_a(\gamma,\delta),\rho_a(\lambda,\theta)\},\min\{\gamma_b(\gamma,\delta),\gamma_b(\lambda,\theta)\}. \end{aligned}$$

Thus

$$\begin{aligned} h_c(\gamma+\lambda,\delta+\theta) &\geq \min\{\rho_a(\gamma,\delta)\wedge\gamma_b(\gamma,\delta),\rho_a(\lambda,\theta)\wedge\gamma_b(\lambda,\theta)\} \\ &= \min\{h_c(\gamma,\delta),h_c(\lambda,\theta)\}. \end{aligned}$$

Now for $x, y \in N$ and $\gamma, \delta \in \Gamma$, consider

$$\begin{aligned} h_c([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma) &= \bigvee_{\xi\in\Gamma}(\rho_a([x\ y\ \gamma]_\Gamma,\xi)\wedge(\xi,[x\ y\ \delta]_\Gamma)) \\ &\geq \rho_a([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma)\wedge\gamma_b([x\ y\ \delta]_\Gamma,[x\ y\ \delta]_\Gamma) \\ &= \rho_a([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma) \\ &\geq \rho_a(\gamma,\delta). \end{aligned}$$

Similarly it can be proved that $h_c([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma) \geq \gamma_b(\gamma,\delta)$.
Thus $h_c([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma) \geq \rho_a(\gamma,\delta)\wedge\gamma_b(\gamma,\delta) = h_c(\gamma,\delta)$. So (h, C) is a fuzzy soft congruence over ${}_N\Gamma \times {}_N\Gamma$. □

Theorem 4.6. *If $FSCR({}_N\Gamma)$ is the set of all fuzzy soft congruence relations such that $(\rho,A)\tilde{\circ}(\gamma,B) = (\gamma,B)\tilde{\circ}(\rho,A)$ for all (ρ,A), $(\gamma,B) \in FSCR({}_N\Gamma)$ then $(FSCR({}_N\Gamma),+,.)$ is a lattice where $+$ and $.$ are defined by $(\rho,A)+(\gamma,B) = (\rho,A)\tilde{\circ}(\gamma,B)$ and $(\rho,A).(\gamma,B) = (\rho,A)\tilde{\cap}(\gamma,B)$.*

Proof. The proof of the theorem follows from Proposition 4.4, Proposition 4.5 and Lemma 2.23. □

Corollary 4.7. *If $(FSCR({}_N\Gamma),+,.)$ is such that $(\rho,A)\tilde{\circ}(\gamma,B) = (\gamma,B)\tilde{\circ}(\rho,A)$ for all $(\rho,A)(\gamma,B) \in FSCR({}_N\Gamma)$ then $FSCR({}_N\Gamma)$ is a modular lattice.*

Proof. The proof follows from Theorem 4.6 and Theorem 2.25. □

Theorem 4.8. *Let ${}_N\Gamma$ be an RTNG. Let E be a set of parameters of ${}_N\Gamma \times {}_N\Gamma$. Let $C \subseteq E$. Then (ρ,C) is a fuzzy soft congruence relation over ${}_N\Gamma \times {}_N\Gamma$ iff $(\rho_c)_\alpha$ is a congruence relation on ${}_N\Gamma$, for each $c \in C$ and $\alpha \in (0,1]$.*

Proof. Let (ρ,C) be a fuzzy soft congruence relation over ${}_N\Gamma\times{}_N\Gamma$. Let $(\gamma,\delta),(\lambda,\theta) \in (\rho_c)_\alpha$. Then $\rho_c(\gamma+\lambda,\delta+\theta) \geq \alpha$, by hypothesis. Thus $(\gamma+\lambda,\delta+\theta) \in (\rho_c)_\alpha$.

Similarly, $\rho_c([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma) \geq \alpha$, if $(\gamma,\delta) \in (\rho_c)_\alpha$. Thus $([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma) \in (\rho_c)_\alpha$. So $(\rho_c)_\alpha$ is a congruence relation on ${}_N\Gamma$.

Conversely, let $(\rho_c)_\alpha$ be a congruence relation on ${}_N\Gamma$. Then $(\gamma+\lambda,\delta+\theta) \in (\rho_c)_\alpha$, whenever $(\gamma,\delta),(\lambda,\theta) \in (\rho_c)_\alpha$. Choosing $\alpha = \min\{\rho_c(\gamma,\delta),\rho_c(\lambda,\theta)\}$. Since $\rho_c(\gamma,\delta) \geq \alpha$ and $\rho_c(\lambda,\theta) \geq \alpha$, $\rho_c(\gamma+\lambda,\delta+\theta) \geq \min\{\rho_c(\gamma,\delta),\rho_c(\lambda,\theta)\}$. Similarly it can be easily proved that $\rho_c([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma) \geq \rho_c(\gamma,\delta)$. Thus (ρ,C) is a fuzzy soft congruence relation over ${}_N\Gamma \times {}_N\Gamma$. □

Theorem 4.9. *Let ${}_N\Gamma$ be an RTNG. Let E be a set of parameters of ${}_N\Gamma \times {}_N\Gamma$. Then R is a congruence relation on ${}_N\Gamma$ iff (Ψ_R,E) is fuzzy soft congruence over ${}_N\Gamma \times {}_N\Gamma$.*

Proof. Let R be a congruence relation on ${}_N\Gamma$. Let $(\gamma,\delta),(\lambda,\theta)\in R$. Then $(\gamma+\lambda,\delta+\theta)\in R$. Thus $(\Psi_R)_e((\gamma+\lambda,\delta+\theta))=1=\min\{(\Psi_R)_e(\gamma,\delta),(\Psi_R)_e(\lambda,\delta)\}$. If $(\gamma,\delta)\in R$, then $([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma)\in R$. So $(\Psi_R)_e((\gamma,\delta))=1=(\Psi_R)_e(([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma))$, for each $e\in E$. Hence (Ψ_R,E) is a fuzzy soft congruence relation.

Conversely let (Ψ_R,E) be fuzzy soft congruence relation over ${}_N\Gamma\times{}_N\Gamma$. Then $(\Psi_R)_e((\gamma+\lambda,\delta+\theta))\geq\min\{(\Psi_R)_e(\gamma,\delta),(\Psi_R)_e(\lambda,\theta)\}$. Now if $(\gamma,\delta),(\lambda,\theta)\in R$, then $(\Psi_R)_e((\gamma+\lambda,\delta+\theta))=1$ and thus $(\gamma+\lambda,\delta+\theta)\in R$. Also $(\Psi_R)_e(([x\,y\,\gamma]_\Gamma,[x\,y\,\delta]_\Gamma))\geq(\Psi_R)_e(\gamma,\delta)\}$, for each $e\in E$ and $x,y\in N$. Now if $(\gamma,\delta)\in R$, then $(\Psi_R)_e(([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma))=1$. So $([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma)\in R$. Hence R is a congruence relation on ${}_N\Gamma$. □

Lemma 4.10. *Let* $\rho_c^{-1}(1)=\{(\gamma,\delta)\in\Gamma\times\Gamma|\rho_c(\gamma,\delta)=1\}$. *Then* $\rho_c^{-1}(1)$ *is a congruence relation on* ${}_N\Gamma$ *if* (ρ,C) *is fuzzy soft congruence over* ${}_N\Gamma\times{}_N\Gamma$.

Proof. Let $(\gamma,\delta),(\lambda,\theta)\in\rho_c^{-1}(1)$. Then $\rho_c(\gamma+\lambda,\delta+\theta)=1$, as $\rho_c(\gamma+\lambda,\delta+\theta)\geq\min\{\rho_c(\gamma,\delta),\rho_c(\lambda,\theta)\}$. Thus $(\gamma+\lambda,\delta+\theta)\in\rho_c^{-1}(1)$. Now let $(\gamma,\delta)\in\rho_c^{-1}(1)$. Then $\rho_c([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma)\geq\rho_c(\gamma,\delta)=1$. This implies that $([x\ y\ \gamma]_\Gamma,[x\ y\ \delta]_\Gamma)\in\rho_c^{-1}(1)$ So $\rho_c^{-1}(1)$ is a congruence relation on ${}_N\Gamma$. □

Proposition 4.11. *If* N *is an RTNR,* E *is a set of parameters of* ${}_N\Gamma\times{}_N\Gamma$, $C\subseteq E$ *and if* (ρ,C) *is a fuzzy soft congruence over* ${}_N\Gamma\times{}_N\Gamma$ *then for each* $c\in C$ *and* $\gamma,\delta\in\Gamma$,

(1) $\rho_c(\gamma-\delta,0)=\rho_c(\gamma,\delta)=\rho_c(0,\gamma-\delta)$,
(2) $\rho_c(-\gamma,-\delta)=\rho_c(\gamma,\delta)$.

Proof. (1) Since

$$\begin{aligned}\rho_c(\gamma-\delta,0_\Gamma)&=\rho_c(\gamma-\delta,\delta-\delta)\\&\geq\rho_c(\gamma,\delta)\\&=\rho_c(\gamma-\delta+\delta,0_\Gamma+\delta)\\&\geq\rho_c(\gamma-\delta,0_\Gamma),\end{aligned}$$

we have $\rho_c(\gamma-\delta,0_\Gamma)=\rho_c(\gamma,\delta)$.

(2) Since (ρ,C) is fuzzy soft symmetric, by (1),

$$\rho_c(-\gamma,-\delta)\leq\rho_c(\gamma-\gamma,\gamma-\delta)=\rho_c(0_\Gamma,\gamma-\delta)=\rho_c(\gamma,\delta).$$

Also, by (1),

$$\rho_c(\gamma,\delta)\leq\rho_c(-\gamma+(-\gamma),-\gamma+\delta)=\rho_c(0_\Gamma,-\gamma-(-\delta))=\rho_c(-\gamma,-\delta)$$

Thus $\rho_c(-\gamma,-\delta)=\rho_c(\gamma,\delta)$. □

Lemma 4.12. *If* ${}_N\Gamma$ *is an RTNG,* E *is a set of parameters of* ${}_N\Gamma\times{}_N\Gamma$ *as well as* ${}_N\Gamma$, $C\subseteq E$ *and if* (ρ,C) *is a fuzzy soft congruence over* ${}_N\Gamma\times{}_N\Gamma$ *then* (ρ_{0_Γ},C) *where* $(\rho_{0_\Gamma})_c(\gamma)=\rho_c(0_\Gamma,\gamma)$ *for every* $\gamma\in\Gamma$ *and* $c\in C$ *is a normal fuzzy soft* N*-ideal over* ${}_N\Gamma$

Proof. Consider for every $\gamma, \delta \in \Gamma$ and $c \in C$,

$$\begin{aligned}(\rho_{0_\Gamma})_c(\gamma - \delta) &= \rho_c(0_\Gamma, \gamma - \delta)\\ &= \rho_c(\gamma, \delta) \geq (\rho_c \circ \rho_c)(\gamma, \delta)\\ &= \bigvee_{\lambda \in \Gamma} (\rho_c(\gamma, \lambda) \wedge \rho_c(\lambda, \delta))\\ &\geq \rho_c(\gamma, 0_\Gamma) \wedge \rho_c(0_\Gamma, \delta)\\ &= \rho_c(0_\Gamma, \gamma) \wedge (0_\Gamma, \delta)\\ &= \min\{(\rho_{0_\Gamma})_c(\gamma), (\rho_{0_\Gamma})_c(\delta)\}.\end{aligned}$$

Consider

$$\begin{aligned}(\rho_{0_\Gamma})_c(\gamma + \delta - \gamma) &= \rho_c(0_\Gamma, \gamma + \delta - \gamma)\\ &= \rho_c(\gamma - \gamma, \gamma + \delta - \gamma)\\ &\geq \rho_c(\gamma, \gamma + \delta)\\ &\geq \rho_c(0_\Gamma, \delta)\\ &= (\rho_{0_\Gamma})_c(\delta).\end{aligned}$$

Now for $x, y \in N$ and $(\gamma, \delta) \in \Gamma$,

$$\begin{aligned}(\rho_{0_\Gamma})_c([x\ y\ (\gamma + \delta)]_\Gamma - [x\ y\ \gamma]_\Gamma) &= \rho_c([x\ y\ \gamma]_\Gamma - [x\ y\ \gamma]_\Gamma, [x\ y\ (\gamma + \delta)]_\Gamma - [x\ y\ \gamma]_\Gamma)\\ &\geq \rho_c[x\ y\ \gamma]_\Gamma, ([x\ y\ \gamma + \delta]_\Gamma))\\ &\geq \rho_c(\gamma, \gamma + \delta)\\ &\geq \rho_c(0_\Gamma, \delta)\\ &= (\rho_{0_\Gamma})_c(\delta).\end{aligned}$$

Also $(\rho_{0_\Gamma})_c(0_\Gamma) = \rho_c(0_\Gamma, 0_\Gamma) = 1$. Hence (ρ_{0_Γ}, C) is a normal fuzzy soft ideal over ${}_N\Gamma$. □

Proposition 4.13. *Let (f, A) be a normal fuzzy soft normal N-subgroup over an RTNG. Then for every $\gamma, \delta \in \Gamma$ and $a \in A$,*
(1) $f_a(\gamma + \delta) = \min\{f_a(\gamma), f_a(\delta)\}$, *if* $f_a(\gamma) \neq f_a(\delta)$.
(2) $f_a(\gamma + \delta) = f_a(\delta + \gamma)$.

Proof. (1) Suppose $f_a(\gamma) < f_a(\delta)$. Then $f_a(\gamma + \delta) \geq \min\{f_a(\gamma), f_a(\delta)\} = f_a(\gamma)$, as (f, A) is a fuzzy soft additive subgroup. Also $\min\{f_a(\gamma), f_a(\delta)\} = f_a(\gamma) = f_a(\gamma+\delta-\delta) \geq \min\{f_a(\gamma + \delta), f_a(\delta)\} = f_a(\gamma + \delta)$, otherwise it will contradict the assumption. Thus $f_a(\gamma + \delta) = \min\{f_a(\gamma), f_a(\delta)\}$. The case that $f_a(\delta) < f_a(\gamma)$ will also lead to the same conclusion.

(2) $f_a(\gamma + \delta) = f_a(-\delta + \delta + \gamma + \delta) \geq f_a(\delta + \gamma) = f_a(-\gamma + \gamma + \delta + \gamma) \geq f_a(\gamma + \delta)$. Then $f_a(\gamma + \delta) = f_a(\delta + \gamma)$. □

Lemma 4.14. *If ${}_N\Gamma$ is an RTNG, E is a set of parameters of ${}_N\Gamma \times {}_N\Gamma$ as well as ${}_N\Gamma$, $C \subseteq E$ and (f, C) is a normal fuzzy soft N-ideal over ${}_N\Gamma$ then a fuzzy soft relation (ρ, C) over ${}_N\Gamma \times {}_N\Gamma$ is a fuzzy soft congruence relation where $\rho_c(\gamma, \delta) = f_c(\gamma - \delta)$ for every $\gamma, \delta \in \Gamma$ and $c \in C$.*

Proof. Consider $\rho_c(\gamma,\gamma) = f_c(0_\Gamma) = 1$, proving that (ρ, C) is fuzzy soft reflexive. Since for any $\gamma \in \Gamma$, $f_c(-\gamma) = f_c(\gamma)$ it follows that (ρ, C) is fuzzy soft symmetric. Consider $(\rho_c \circ \rho_c)(\gamma,\xi) = \bigvee_{\delta\in\Gamma} (\rho_c(\gamma,\delta) \wedge \rho_c(\delta,\xi)) = \bigvee_{\delta\in\Gamma} (f_c(\gamma-\delta) \wedge f_c(\delta-\xi))$. Since $f_c(\gamma-\xi) = f_c(\gamma-\delta+\delta-\xi) \geq f_c(\gamma-\delta) \wedge f_c(\delta-\xi)$ for each $\delta \in \Gamma$, $f_c(\gamma-\xi) \geq (\rho_c \circ \rho_c)(\gamma,\xi)$. Then $\rho_c \circ \rho_c \subseteq \rho_c$. Thus (ρ, C) is a fuzzy soft similarity relation over ${}_N\Gamma \times {}_N\Gamma$.

We now proceed to show that (ρ, C) is a fuzzy soft congruence relation. Consider,

$$\begin{aligned}
\rho_c(\gamma+\lambda, \delta+\nu) &= f_c(\gamma+\lambda-(\delta+\nu)) \\
&= f_c(\gamma+\lambda-\nu-\delta) \\
&= f_c((\gamma+\lambda-\nu)-\delta) \\
&= f_c(-\delta+\gamma+\lambda-\nu), \text{ using Proposition 4.13} \\
&\geq \min\{f_c(-\delta+\gamma), f_c(\lambda-\nu)\} \\
&= \min\{f_c(\gamma-\delta), f_c(\lambda-\nu)\} \\
&= \min\{\rho_c(\gamma,\delta), \rho_c(\lambda,\nu)\}
\end{aligned}$$

Now

$$\begin{aligned}
\rho_c([x\ y\ \gamma]_\Gamma, [x\ y\ \delta]_\Gamma) &= f_c([x\ y\ \gamma]_\Gamma - [x\ y\ \delta]_\Gamma) \\
&= f_c([x\ y\ (\delta-\delta+\gamma)]_\Gamma - [x\ y\ \delta]_\Gamma) \\
&\geq f_c(-\delta+\gamma), \text{ as } (f,C) \text{ is a fuzzy soft } N\text{-ideal} \\
&= f_c(\gamma-\delta) \\
&= \rho_c(\gamma,\delta).
\end{aligned}$$

Thus (ρ, C) is a fuzzy soft congruence relation over ${}_N\Gamma \times {}_N\Gamma$. □

Lemma 4.15. *If ${}_N\Gamma$ is an abelian RTNG, E is a set of parameters of ${}_N\Gamma \times {}_N\Gamma$ as well as ${}_N\Gamma$, $C \subseteq E$ and (f, C) is a normal fuzzy soft N-ideal over ${}_N\Gamma$ then a fuzzy soft relation (ρ, C) over ${}_N\Gamma \times {}_N\Gamma$ is a fuzzy soft congruence relation where $\rho_c(\gamma,\delta) = \underset{\gamma+\lambda=\delta+\nu}{Sup} f_c(\lambda) \wedge f_c(\nu)$ for every $\gamma, \delta, \lambda, \nu \in \Gamma$ and $c \in C$.*

Proof. Consider $\rho_c(\gamma,\gamma) = \underset{\gamma+\lambda=\gamma+\nu}{Sup} f_c(\lambda) \wedge f_c(\nu) \geq f_c(0_\Gamma) \wedge f_c(0_\Gamma) = 1$. Then $\rho_c(\gamma,\gamma) = 1$. Obviously $\rho_c(\gamma,\delta) = \rho_c(\delta,\gamma)$.
Also

$$\begin{aligned}
\rho_c(\gamma,\delta) &= \underset{\gamma+\lambda=\delta+\nu}{Sup} f_c(\lambda) \wedge f_c(v)) \\
&\geq \underset{\gamma+\lambda=\theta+\xi}{Sup} \left(\underset{\theta+\xi=\delta+\nu}{Sup} \{\{f_c(\lambda) \wedge f_c(\xi)\} \wedge \{f_c(\xi) \wedge f_c(\nu)\}\} \right) \\
&\geq \min\{ \underset{\gamma+\lambda=\theta+\xi}{Sup} \{f_c(\lambda) \wedge f_c(\xi)\}, \underset{\theta+\xi=\delta+\nu}{Sup} \{f_c(\xi) \wedge f_c(\nu)\} \\
&= \min\{\rho_c(\gamma,\theta), \rho_c(\theta,\delta)\}.
\end{aligned}$$

Thus $\rho_c \circ \rho_c \subseteq \rho_c$. So (ρ, C) is a fuzzy soft similarity relation.
Consider

$$\begin{aligned}\rho_c(\gamma+\lambda, \gamma+\delta) &= \underset{\gamma+\lambda+\xi=\gamma+\delta+\theta}{Sup}(f_c(\xi) \wedge f_c(\theta)) \\ &= \underset{\lambda+\xi=\delta+\theta}{Sup}(f_c(\xi) \wedge f_c(\theta)) \\ &= \rho_c(\lambda, \delta).\end{aligned}$$

As Γ is abelian, $\rho_c(\lambda+\gamma, \delta+\gamma) = \rho_c(\lambda, \delta)$.
Since (ρ, C) is fuzzy soft transitive,

$$\begin{aligned}\rho_c(\gamma+\lambda, \delta+\nu) &\geq \rho_c(\gamma+\lambda, \gamma+\nu) \wedge \rho_c(\gamma+\nu, \delta+\nu) \\ &= \rho_c(\lambda, \nu) \wedge \rho_c(\gamma, \delta).\end{aligned}$$

Given γ and δ there exists ξ and θ such that $\gamma - \delta = \xi - \theta$.
Consider

$$\begin{aligned}\rho_c([x\ y\ \gamma]_\Gamma, [x\ y\ \delta]_\Gamma) &= \underset{[x\ y\ \gamma]_\Gamma+\lambda=[x\ y\ \delta]_\Gamma+\nu}{Sup} f_c(\lambda) \wedge f_c(\nu) \\ &\geq f_c([x\ y\ (\gamma+\theta)]_\Gamma - [x\ y\ \gamma]_\Gamma) \wedge f_c([x\ y\ (\delta+\xi)]_\Gamma - [x\ y\ \delta]_\Gamma) \\ &\geq f_c(\theta) \wedge f_c(\xi).\end{aligned}$$

Then $\rho_c([x\ y\ \gamma]_\Gamma, [x\ y\ \delta]_\Gamma) \geq \underset{\gamma+\theta=\delta+\xi}{Sup}(f_c(\theta) \wedge f_c(\xi)) = \rho_c(\gamma, \delta)$. Thus (ρ, C) is a fuzzy soft congruence relation over ${}_N\Gamma \times {}_N\Gamma$. □

5. Conclusion

In this paper fuzzy soft N-subgroups and fuzzy soft N-ideals over right ternary N-groups were defined and their basic algebraic properties were studied. A normal fuzzy soft ideal was obtained from a fuzzy soft congruence relation. Lattice structure of the set of all fuzzy soft congruence relations on a right ternary N-group was given. A fuzzy soft quotient RTNG over its fuzzy soft N-ideals may further be defined and its structural properties may be studied. Fuzzy soft congruences on fuzzy soft quotient RTNG may also be explored.

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