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An extension of the concept of n-level sets to multisets

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Abstract. In this paper the concept of n-level sets is extended to n-upper level sets, n-level multisets and n-upper level multisets, and some related results are obtained.

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1. Introduction

An important generalization of set known as Multiset, emerged as a result of violating a basic underlying set condition. Multiset is an unordered collection of elements in which elements can occur more than once. The term multiset (mset in short) as Knuth [11] notes was first suggested by N.G. de Bruijn in a private communication to him. In literature, variety of terms viz. list, heap, bunch, bag, sample, weighted set, occurrence set and fire set (finitely repeated element set) are used in different contexts but conveying the same meaning with mset. A comprehensive account of fundamentals of multiset and its applications in various forms can be found in [1, 8, 9, 11, 13, 14] and applications of cardinality bounded multisets can be found in [2, 3, 4, 5, 6].
In [12], concept of n-level sets was introduced and some results were obtained. We extend this n-level sets to n-upper level sets, n-level multisets, and n-upper level multisets, and prove some related results.

2. Preliminaries

Definition 2.1. Let X be a set. A multiset (mset) M drawn from X is represented by a count function C_M, defined as

$$C_M \quad : X \to N \cup \{0\}.$$

For each $x \in X$, $C_M(x)$ denotes the number of occurrences of the element x in the mset M. The representation of the mset M drawn from $X = \{x_1, x_2, \ldots, x_n\}$ will be as $M = [{x_1}^{m_1}, {x_2}^{m_2}, \ldots, {x_n}^{m_n}]$ where m_i is the number of occurrences of the element x_i, $i = 1, 2, \ldots, n$ in the mset M.

Also, for any positive integer w, $[X]^w$ is the set of all msets drawn from X such that no element in the mset occurs more than w times and $[X]^\infty$ is the set of all msets drawn from X such that there is no limit on the number of occurrences of an object in an mset. $[X]^w$ and $[X]^\infty$ are referred to as mset spaces. [9, 10]

Definition 2.2. Let M_1 and M_2 be two msets drawn from a set X. An mset M_1 is a submset of M_2 ($M_1 \subseteq M_2$) if $C_{M_1}(x) \leq C_{M_2}(x)$, $\forall\ x \in X$. Whereas, $M_1$is said to be equal to M_2 ($M_1 = M_2$) if $C_{M_1}(x) = C_{M_2}(x), \forall\ x \in X$.

Definition 2.3. Let $\{M_i : i \in I\}$ be a nonempty family of msets drawn from a set X. Then

(i) Their Intersection, denoted by $\cap_{i \in I} M_i$, is defined as

$$C_{\cap_{i \in I} M_i}(x) = \ \wedge_{i \in I} C_{M_i}(x), \forall\ x \in X,$$

where $\wedge$ is the minimum operation.

(ii) Their Union, denoted by $\cup_{i \in I} M_i$ is defined as

$$C_{\cup_{i \in I} M_i}(x) = \vee_{i \in I} C_{M_i}(x),\ \forall\ x \in X,$$

where $\vee$ is the maximum operation.

(iii) The Complement of any mset $M_i \in [X]^w$, denoted by M_i^c is defined as

$$C_{M_i^c}(x) = w - C_{M_i}(x), \forall\ x \in X.$$

Definition 2.4 ([12])**.** Let M be an mset over a set X. Then the sets of the form

$$M_n \ = \{x \in X \ : C_M(x) \geq n, \quad n \in \mathbb{N}\}$$

are called n-level sets of M.

Proposition 2.5 ([12])**.** *Let A, B be msets over X and $m, n \in \mathbb{N}$.*

(1) *If $A \subseteq B$, then $A_n \subseteq B_n$.*
(2) *If $m \leq n$, then $A_m \supseteq A_n$.*
(3) $(A \cap B)_n = A_n \cap B_n$.
(4) $(A \cup B)_n = A_n \cup B_n$.
(5) $A = B$ *iff* $A_n = B_n$ $, \forall\ n \in \mathbb{N}$.

Theorem 2.6 (First Decomposition Theorem [12])**.** *If A_n , $n \in \mathbb{N}$ be the level sets of an mset A over X, then $C_A(x) = \sum_{n \in \mathbb{N}} \chi_{A_n}(x)$, where χ_{A_n} is the characteristics function of A_n .*

Theorem 2.7 (Second Decomposition Theorem [12])**.** *If A_n, $n \in \mathbb{N}$ be the level sets of an mset A over X, then $A = \cup_{n \in \mathbb{N}} nA_n$, where $\cup$ denotes the standard mset union.*

3. n-upper level set

In this section, we define n-upper level set of an mset and some related properties are investigated.

Definition 3.1. Let M be an mset over a set X. Then we define a set

$$M^n = \{x \in X : C_M(x) \leq n, \quad n \in \mathbb{N}\},$$

called n-upper level set of M.

Definition 3.2. Let $A^n \subseteq X$. Then for each $n \in \mathbb{N}$, the mset nA^n is defined over X such that $C_{nA^n}(x) = n$, $\forall x \in X$.

Proposition 3.3. *Let A, B be msets over X and $m, n \in \mathbb{N}$.*
(1) *If $A \subseteq B$, then $B^n \subseteq A^n$.*
(2) *If $m \leq n$, then $A^m \subseteq A^n$.*
(3) $A^n \cap B^n \subseteq (A \cap B)^n$.
(4) $(A \cup B)^n \subseteq A^n \cup B^n$.
(5) $A = B$ *iff* $A^n = B^n$, $\forall\ n \in \mathbb{N}$.

Proof. The proofs are straightforward. □

Proposition 3.4. *The first and second decomposition theorems do not hold for n-upper level set of A over X.*

Example 3.5. Counter example (First Decomposition Theorem)
Let $X = \{a, b, c, d\}$ with $A = [a, b^4, d^3]$ and $\chi_{A^n}(x) = \begin{cases} 1\ if\ x \in A^n \\ 0\ if\ x \notin A^n \end{cases}$
Then $A^1 = \{a\}, A^2 = \{a\}$. Thus

$$\textstyle\sum_{n=1}^{2} \chi_{A^n}(x) = \chi_{A^1}(a) + \chi_{A^2}(a) = 1 + 1 = 2 \neq 1 = C_A(a).$$

Example 3.6. Counter example (Second Decomposition Theorem)
Let $X = \{a, b, c, d\}$ and $A = \left[a, b^4, d^3\right]$. Then

$$A^1 = \{a\},\ A^2 = \{a\},\ A^3 = \{a, d\},\ A^n = \{a, b, d\},\ n \geq 4.$$

Thus
$1A^1 = \{a\}, 2A^2 = \{a, a\}, 3A^3 = \{a, a, a, d, d, d\}, 4A^4 = \{a, a, a, a, b, b, b, b, d, d, d, d\}$.
So $1A^1 \vee 2A^2 \vee 3A^3 \vee 4A^4 \vee \cdots \neq A$.

Definition 3.7. (Difference of multisets) If $B \subseteq A$, the difference $A - B$ is defined by

$$C_{A-B}(x) = C_A(x) - C_B(x),\ \forall\ x \in X.$$

Definition 3.8. (Symmetric Difference)
Let A and B be two multisets, then the symmetric difference of A and B is the multiset $A \triangle B$ such that

$$C_{A\ \triangle\ B}(x) = C_{A \cup B}(x) - C_{A \cap B}(x) = |C_A(x) - C_B(x)|, \forall\ x \in X.$$

Definition 3.9. (Symmetric Difference of n-level sets of A and B)
Let A and B be multisets over X. Then

$$A_n \triangle B_n = (A_n \backslash B_n) \cup (B_n \backslash A_n) = (A \cup B)_n - (A \cap B)_n = (A \cup B)_n \backslash (A \cap B)_n.$$

Proposition 3.10. *Let A and B be msets over X. Then*
(1) $(A \cup B)_n \backslash A_n = B_n \backslash A_n$.
(2) $(A \cup B)_n \backslash B_n = A_n \backslash B_n$.
(3) $A_n \bigtriangleup B_n = A_n \cup B_n$ *iff* $A \cap B = \varnothing$.

Proof. (1) Let $x \in (A \cup B)_n \backslash A_n$. Then

$$\begin{aligned}
& x \in (A \cup B)_n \ \wedge x \notin A_n \\
& \Leftrightarrow \ (x \in A_n \vee x \in B_n) \ \wedge x \notin A_n \\
& \Leftrightarrow [(C_A(x) \geq n, \forall\, x \in X) \vee (C_B(x) \geq n, \forall\, x \in X)] \wedge (C_A(x) < n, \forall\, x \in X) \\
& \Leftrightarrow \ [(C_A(x) \geq n, \forall\, x \in X) \wedge (C_A(x) < n, \forall\, x \in X)] \vee \\
& \qquad [(C_B(x) \geq n, \forall\, x \in X) \wedge (C_A(x) < \ n, \forall\, x \in X)] \\
& \Leftrightarrow \varnothing \vee [(C_B(x) \geq n, \forall\, x \in X) \wedge (C_A(x) < n, \forall\, x \in X)] \\
& \Leftrightarrow [(C_B(x) \geq n, \ \forall\, x \in X) \wedge (C_A(x) < n, \forall\, x \in X)] \\
& \Leftrightarrow x \in B_n \ \wedge x \notin A_n \\
& \Leftrightarrow x \in B_n \backslash A_n.
\end{aligned}$$

Thus $(A \cup B)_n \backslash A_n = B_n \backslash A_n$.

(2) Let $x \in (A \cup B)_n \backslash B_n$. Then

$$\begin{aligned}
& x \in (A \cup B)_n \ \wedge x \notin B_n \\
& \Leftrightarrow (x \in A_n \vee x \in B_n) \wedge x \notin B_n \\
& \Leftrightarrow [(C_A(x) \geq n, \forall\, x \in X) \vee (C_B(x) \geq n, \forall\, x \in X)] \wedge (C_B(x) < n, \forall\, x \in X) \\
& \Leftrightarrow [(C_A(x) \geq n, \forall\, x \in X) \wedge (C_B(x) < n, \forall\, x \in X)] \vee \\
& \qquad [(C_B(x) \geq n, \forall\, x \in X) \wedge (C_B(x) < n, \forall\, x \in X)] \\
& \Leftrightarrow [(C_A(x) \geq n, \forall\, x \in X) \wedge (C_B(x) < n, \forall\, x \in X)] \vee \varnothing \\
& \Leftrightarrow [(C_A(x) \geq n, \forall\, x \in X) \wedge (C_B(x) < n, \forall\, x \in X)] \\
& \Leftrightarrow x \in A_n \ \wedge x \notin B_n \\
& \Leftrightarrow x \in A_n \backslash B_n.
\end{aligned}$$

Thus $(A \cup B)_n \backslash B_n = A_n \backslash B_n$.

(3) By definition,

$$\begin{aligned}
A_n \bigtriangleup B_n &= (A \cup B)_n - (A \cap B)_n \\
&= \{x \in X : C_{A \cup B}(x) \geq n\} - \varnothing \text{ iff } A \cap B = \varnothing \\
&= \{x \in X : C_A(x) \vee C_B(x) \geq n\} \\
&= \{x \in X : C_A(x) \geq n\} \vee \{x \in X : C_B(x) \geq n\} \\
&= \ A_n \cup B_n.
\end{aligned}$$

□

4. n-level multiset and n-upper level multiset

In this section, we introduce the concepts of n-level multiset and n-upper level multiset, and some of their properties are discussed.

Definition 4.1. Let X be a set, then the n-level multisets of $\mathcal{A} \subseteq [X]^w$, denoted $\mathcal{A}_n$, is defined as $\mathcal{A}_n = \{A \in [X]^w : C_A(x) \geq n, \quad n \in \mathbb{N}\} \ \forall\, x \in X$.

Example 4.2. Let $X = \{a, b\}$ and

$$\mathcal{A} = \big\{[a^3, b^3], [a^3, b], [a^2, b^2], [a, b^2], [a^3], [b^3]\big\} \subseteq [X]^3.$$

$$\begin{aligned}
\mathcal{A}_1 &= \left\{[a^3, b^3], [a^3, b], [a^2, b^2], [a, b^2], [a^3], [b^3]\right\}, \\
\mathcal{A}_2 &= \left\{[a^3, b^3], [a^2, b^2], [a^3], [b^3]\right\}, \\
\mathcal{A}_3 &= \left\{[a^3, b^3], [a^2], [a^3], [b^3]\right\}, \\
\mathcal{A}_n &= \{\varnothing\}, \quad n \geq 4.
\end{aligned}$$

Definition 4.3. Let $X = \{a, b\}$ and $\mathcal{A},\ \mathcal{B} \subseteq [X]^w$. Then

(i) $\mathcal{A} \cup \mathcal{B} = \{[a_i, max\ C(b_j)], [maxC((a)_i), b_j] : [a_i, b_j] \in \mathcal{A}\ or\ [a_i, b_j] \in \mathcal{B}\}$, $i,\ j \in \mathbb{N}$, where $\cup$ is called multiset space union.

(ii) $\mathcal{A} \cap \mathcal{B} = \{[a_i, min\ C(b_j)], [\min C(a_i)\ , b_j] : [a_i, b_j] \in \mathcal{A}, [a_i, b_j] \in \mathcal{B}\}$, $i,\ j \in \mathbb{N}$, where $\cap$ is called multiset space intersection.

(iii) $\mathcal{A} \subseteq \mathcal{B}$ if $\forall\ A \in \mathcal{A}\ \exists\ B \in \mathcal{B}$ such that $A \subseteq B$.

Remark 4.4. Let $\mathcal{A},\ \mathcal{B} \subseteq [X]^w$. Then

(1) $\mathcal{A} \subseteq \mathcal{B} \Longrightarrow C_A(x) \leq C_B(x),\ \forall\ x \in X$.

(2) $\mathcal{A} = \mathcal{B} \Longrightarrow C_A(x) = C_B(x),\ \forall\ x \in X$.

(3) $\mathcal{A}^c = w -\ C_A(x),\ \forall\ x \in X$.

Example 4.5. Let $X = \{a, b\}$ and $\mathcal{A}, \mathcal{B} \subseteq [X]^3$, given by

$$\mathcal{A} = \{[a^3, b^3], [a^3, b], [a^2, b^2]\}$$

and

$$\mathcal{B} = \{[a, b^3], [a^3], [b^3]\}.$$

By Definition 4.3(i) and 4.3(ii)

$$\mathcal{A} \cup \mathcal{B} = \{[a^3, b^3], [a^2, b^2], [a, b^3], [a^3, b], [a^3], [b^3]\}$$

and

$$\mathcal{A} \cap \mathcal{B} = \{[a, b^3]\}.$$

Note: In (i) and (ii) mset as an element of multiset space is not allowed to repeat.

Proposition 4.6. *Let* $\mathcal{A},\ \mathcal{B}\ \subseteq [X]^w$ *and* $m, n \in \mathbb{N}$.

(1) $\mathcal{A} \subset \mathcal{B}$ *iff* $\mathcal{A}_n \subset \mathcal{B}_n$.

(2) *If* $m \leq n$, *then* $\mathcal{A}_m \supseteq \mathcal{A}_n$.

(3) $(\mathcal{A} \cap \mathcal{B})_n \subseteq \mathcal{A}_n \cap \mathcal{B}_n$.

(4) $(\mathcal{A} \cup \mathcal{B})_n = \mathcal{A}_n \cup \mathcal{B}_n$.

(5) $\mathcal{A} = \mathcal{B}$ *iff* $\mathcal{A}_n = \mathcal{B}_n,\ \forall\ n \in \mathbb{N}$.

Proof. (1) Let $A \in \mathcal{A}_n$. Then $C_A(x) \geq n,\ \forall\ x \in X$. Thus $n \leq C_A(x) \leq C_B(x)$, since $\mathcal{A} \subset \mathcal{B}$. So $A \in \mathcal{B}_n$, i.e., $\mathcal{A}_n \subset \mathcal{B}_n,\ \forall\ x \in X$.

Conversely, let $A \in \mathcal{A}_n \subset \mathcal{B}_n$. Then $C_A(x) \geq n$. Thus $C_B(x) \geq n, \forall x \in X$. Since $\mathcal{A}, \mathcal{B} \subset X^w$ and $\mathcal{A}_n$ and $\mathcal{B}_n$ are the level multisets of $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \subset \mathcal{B}$.

(2) By definition, if $A \in \mathcal{A}_n$, then $C_A(x) \geq n, \forall x \in X$ and if $A \in \mathcal{A}_m$, then $C_A(x) \geq m,\ \forall\ x \in X$. Since $m, n \in \mathbb{N}$ and $m \leq n$, $\{C_A(x) \geq n\} \leq \{C_A(x) \geq m\}$, $\forall\ x \in X$. Thus $\mathcal{A}_m \supseteq \mathcal{A}_n$.

(3) By the definition of mset space intersection, $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B}$. Then $(\mathcal{A} \cap \mathcal{B})_n \subseteq \mathcal{A}_n$ and $(\mathcal{A} \cap \mathcal{B})_n \subseteq \mathcal{B}_n$. Thus $(\mathcal{A} \cap \mathcal{B})_n \subseteq \mathcal{A}_n \cap \mathcal{B}_n$.

The converse is not true in general, we consider the following counter example: Let $X = \{a, b\}$, then $\mathcal{A} = \{[a^3, b^3], [a^3, b], [a^2, b^2]\}$ and

$\mathcal{B} = \{[a^3, b^2], [a^2, b^3], [a, b^2]\}$.
By definition, $\mathcal{A}_2 = \{[a^3, b^3], [a^2, b^2]\}$ and $\mathcal{B}_2 = \{[a^3, b^2], [a^2, b^3]\}$,
$\mathcal{A}_2 \cap \mathcal{B}_2 = \{[a^3, b^2], [a^2, b^2], [a^2, b^3]\}$,
$\mathcal{A} \cap \mathcal{B} = \{[a^3, b], [a^2, b^2], [a^2, b^3], [a, b^2]\}$,
$(\mathcal{A} \cap \mathcal{B})_2 = \{[a^2, b^2], [a^2, b^3]\}$.
Thus $\mathcal{A}_2 \cap \mathcal{B}_2 \nsubseteq (\mathcal{A} \cap \mathcal{B})_2$. So $(\mathcal{A} \cap \mathcal{B})_n \subseteq \mathcal{A}_n \cap \mathcal{B}_n$.

(4) Since $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B}$, by definition of mset space union, $\mathcal{A}_n \subseteq (\mathcal{A} \cup \mathcal{B})_n$ and $\mathcal{B}_n \subseteq (\mathcal{A} \cup \mathcal{B})_n$. Then $\mathcal{A}_n \cup \mathcal{B}_n \subseteq (\mathcal{A} \cup \mathcal{B})_n$.

Conversely, let $A \in (\mathcal{A} \cup \mathcal{B})_n$. Then $A \in \mathcal{A}_n$ or $A \in \mathcal{B}_n$.

If $C_A(x) \geq n$, then $A \in \mathcal{A}_n$. Also, if $C_B(x) \geq n$, then $A \in \mathcal{B}_n$. Thus $A \in \mathcal{A}_n \cup \mathcal{B}_n$ [By Definition 4.3(i)]. So $(\mathcal{A} \cup \mathcal{B})_n \subseteq \mathcal{A}_n \cup \mathcal{B}_n$. Hence $(\mathcal{A} \cup \mathcal{B})_n = \mathcal{A}_n \cup \mathcal{B}_n$.

(5) Suppose $\mathcal{A}_n = \mathcal{B}_n$. Then $\mathcal{A}_n \subseteq \mathcal{B}_n$ and $\mathcal{B}_n \subseteq \mathcal{A}_n$. Thus $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$. So $\mathcal{A} = \mathcal{B}$. □

Remark 4.7. If $\mathcal{A}_n$, $n \in \mathbb{N}$ be the class of multisets $\mathcal{A} \subseteq [X]^w$, then $\mathcal{A} = \cup_{n \in \mathbb{N}} \mathcal{A}_n$, where $\cup$ denotes mset space union.

Example 4.8. Let $\mathcal{A}$ be any subclass of $[X]^3$, given by

$$\mathcal{A} = \left\{ \left[a^3, b^3\right], \left[a^3, b\right], \left[a^2, b^2\right], \left[a, b^2\right], \left[a^3\right], \left[b^3\right] \right\},$$

where $\mathcal{A}_n = \{A \in [X]^w : C_A(x) \geq n,\ n \in \mathbb{N}\}$, $\forall\, x \in X$.

Now
$$\begin{aligned} \mathcal{A}_1 &= \left\{ \left[a^3, b^3\right], \left[a^3, b\right], \left[a^2, b^2\right], \left[a, b^2\right], \left[a^3\right], \left[b^3\right] \right\}, \\ \mathcal{A}_2 &= \left\{ \left[a^3, b^3\right], \left[a^2, b^2\right], \left[a^3\right], \left[b^3\right] \right\}, \\ \mathcal{A}_3 &= \left\{ \left[a^3, b^3\right], \left[a^3\right], \left[b^3\right] \right\}, \\ \mathcal{A}_n &= \{\varnothing\},\ n \geq 4. \end{aligned}$$

Thus $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \{\varnothing\} \cup \{\varnothing\} \cup \ldots$ So $\mathcal{A} = \cup_{n=1}^{3} \mathcal{A}_n$ [By Definition 4.3(i)].
In general, $\mathcal{A} = \cup_{n \in \mathbb{N}} \mathcal{A}_n$.

Definition 4.9. Let X be a set, then the n- upper level multisets of $\mathcal{A} \subseteq [X]^w$, denoted $\mathcal{A}^n$, is defined as $\mathcal{A}^n = \{A \in [X]^w : C_A(x) \leq n, \quad n \in \mathbb{N}\}\ \forall\, x \in X$.

Example 4.10. Let $X = \{a, b\}$ and $\mathcal{A} = \left\{[a^3, b^3], [a^3, b], [a^2, b^2], [a, b^2], [a^3], [b^3]\right\} \subseteq X^3$.

$$\mathcal{A}^1 = \{\varnothing\},$$

$$\mathcal{A}^2 = \left\{[a^2, b^2], [a, b^2]\right\},$$

$$\mathcal{A}^n = \left\{[a^3, b^3], [a^3, b], [a^2, b^2], [a, b^2], [a^3], [b^3]\right\}, \quad n \geq 3.$$

Proposition 4.11. *Let $\mathcal{A}$, $\mathcal{B} \subseteq [X]^w$ and $m, n \in \mathbb{N}$.*
(1) *If $m \leq n$, then $\mathcal{A}^m \subseteq \mathcal{A}^n$.*
(2) $\mathcal{A}^n \cap \mathcal{B}^n \subseteq (\mathcal{A} \cap \mathcal{B})^n$.
(3) $(\mathcal{A} \cup \mathcal{B})^n \subseteq \mathcal{A}^n \cup \mathcal{B}^n$.

Proof. (1) By definition, $A \in \mathcal{A}^n$. Then $C_A(x) \leq n, \forall\, x \in X$ and
$A \in \mathcal{A}^m$. Thus $C_A(x) \leq m$, $\forall\, x \in X$. Since $m, n \in \mathbb{N}$, if $m \leq n$,, then

$$\{C_A(x) \leq m\} \leq \{C_A(x) \leq n\},\ \forall\, x \in X.$$

So $\mathcal{A}^m \subseteq \mathcal{A}^n$.

(2) Let $A \in \mathcal{A}^n \cap \mathcal{B}^n$. Then, by the definition of intersection of mset space, $A \in \mathcal{A}^n$, $A \in \mathcal{B}^n$. Thus $C_A(x) \leq n$ and $C_B(x) \leq n$. So $C_{A\cap B}(x) \leq n$, i.e., $A \in (\mathcal{A} \cap \mathcal{B})^n$. Hence $\mathcal{A}^n \cap \mathcal{B}^n \subseteq (\mathcal{A} \cap \mathcal{B})^n$.

The converse is not true. Let $X = \{a, b\}$, $\mathcal{A} = \{[a^3, b^3], [a^3, b], [a^2, b^2]\}$ and $\mathcal{B} = \{[a^3, b^2], [a^2, b^3], [a, b^2]\}$. Then, by definition, $\mathcal{A}^2 = \{[a^2, b^2]\}$ and $\mathcal{B}^2 = \{[a, b^2]\}$. Thus $\mathcal{A}^2 \cap \mathcal{B}^2 = \{[a, b^2]\}$ [By Definition 4.3(i)],

$$\mathcal{A} \cap \mathcal{B} = \{[a^3, b], [a^2, b^2], [a^2, b^3], [a, b^2]\},\ (\mathcal{A} \cap \mathcal{B})^2 = \{[a^2, b^2], [a, b^2]\}.$$

So $(\mathcal{A} \cap \mathcal{B})^2 \nsubseteq \mathcal{A}^2 \cap \mathcal{B}^2$. Hence $\mathcal{A}^n \cap \mathcal{B}^n \subseteq (\mathcal{A} \cap \mathcal{B})^n$.

(3) Let $A \in (\mathcal{A} \cup \mathcal{B})^n$. Then $A \in \mathcal{A}^n$ or $A \in \mathcal{B}^n$.

If $C_A(x) \leq n$, then $A \in \mathcal{A}^n$. Also, if $C_B(x) \leq n$, then $A \in \mathcal{B}^n$. Thus $A \in \mathcal{A}^n \cup \mathcal{B}^n$ [By Definition 4.3(i)]. So $(\mathcal{A} \cup \mathcal{B})^n \subseteq \mathcal{A}^n \cup \mathcal{B}^n$.

The converse is not true. Let $X = \{a, b\}$, $\mathcal{A} = \{[a^3, b^3], [a^3, b], [a^2, b^2]\}$ and $\mathcal{B} = \{[a^3, b^2], [a^2, b^3], [a, b^2]\}$. Then, by definition, $\mathcal{A}^2 = \{[a^2, b^2]\}$ and $\mathcal{B}^2 = \{[a, b^2]\}$. Thus $\mathcal{A}^2 \cup \mathcal{B}^2 = \{[a^2, b^2], [a, b^2]\}$ [By Definition 4.3(i)],

$$\mathcal{A} \cup \mathcal{B} = \{[a^3, b^3], [a^2, b^3], [a, b^2], [a^3, b^2], [a^3, b]\},\ (\mathcal{A} \cup \mathcal{B})^2 = \{[a, b^2]\}.$$

So $\mathcal{A}^2 \cup \mathcal{B}^2 \nsubseteq (\mathcal{A} \cup \mathcal{B})^2$. Hence $(\mathcal{A} \cup \mathcal{B})^n \subseteq \mathcal{A}^n \cup \mathcal{B}^n$. □

Remark 4.12. If $\mathcal{A}^n$, $n \in \mathbb{N}$ be the class of multisets $\mathcal{A} \subseteq [X]^w$, then $\mathcal{A} = \cup_{n\in\mathbb{N}} \mathcal{A}^n$, where $\cup$ denotes mset space union.

Example 4.13. Let $\mathcal{A}$ be any subclass of $[X]^3$, given by

$\mathcal{A} = \{[a^3, b^3], [a^3, b], [a^2, b^2], [a, b^2], [a^3], [b^3]\}$,

$\mathcal{A}^n = \{A \in [X]^w : C_A(x) \leq n,\ n \in \mathbb{N}\}$.

Now $\mathcal{A}^1 = \{\varnothing\}$.

$\mathcal{A}^2 = \{[a^2, b^2], [a, b^2]\}$

$\mathcal{A}^n = \{[a^3, b^3], [a^3, b], [a^2, b^2], [a, b^2], [a^3], [b^3]\}$, $n \geq 3$.

Then $\mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \mathcal{A}^3 \cup \ldots$ Thus $\mathcal{A} = \cup_{n=1}^{3} \mathcal{A}^n$ [By Definition 4.3(i)]. In general, $\mathcal{A} = \cup_{n\in\mathbb{N}} \mathcal{A}^n$.

5. Conclusions

In this paper, the concept of n-level multiset was introduced, some of its properties described and some related results outlined. In addition, n-upper level sets and n-upper level multisets were introduced and some related results were obtained.

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