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Coupled fixed point theorems in partially ordered non-Archimedean complete fuzzy metric spaces

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Abstract. In this paper, we derive new coupled fixed point theorems for mapping having the mixed monotone property in partially ordered non-Archimedean complete fuzzy metric spaces. We give an example to support our result.

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1. Introduction

Fixed point of functions and operators are important in many part of classical mathematics like mathematical analysis, dynamical systems, geometry etc. The Banach contraction principle is the most celebrated fixed point theorem. Afterward many authors obtained many important extensions of this principle under different contractive conditions. Recently Bhaskar and Lakshmikantham [2], Nieto and Lopez [12, 13], Ran and Reurings [15] and Agarwal et al. [1] presented some new results for contractions in partially ordered metric spaces. Bhaskar and Lakshmikantham [2] noted that their theorem can be used to investigate a large class of problems and have discussed the existence and uniqueness of solution for a periodic boundary value problem. A few interesting results have been developed in the papers [9, 10] and that have been generalized in this paper.
Fuzzy set theory, a generalization of crisp set theory, was first introduced by Zadeh [22] in 1965 to describe situations in which data are imprecise or vague or uncertain. Consequently, the last three decades remained productive for various authors [3, 4, 6, 17, 18] etc. have extensively developed the theory of fuzzy sets due to a wide range of application in the field of population dynamics , chaos control, computer programming , medicine , etc. Kramosil and Michalek [8] introduced the concept of fuzzy metric spaces (briefly, FM-spaces) in 1975, which opened an avenue for

further development of analysis in such spaces. Later on it is modified that a few concepts of mathematical analysis have been generalized by George and Veeramani [4, 5] and also they have generalized the fixed point theorem in fuzzy metric space [20]. Afterwards many articles have been published on fixed point theorems under different contractive condition in fuzzy metric spaces.
In this paper, we derive new coupled fixed point theorems for mapping having the mixed monotone property in partially ordered non-Archimedean complete fuzzy metric spaces.

2. Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequel.

Definition 2.1 ([19])**.** A binary operation $* : [0\,,\,1] \times [0\,,\,1] \longrightarrow [0\,,\,1]$ is continuous t - norm if $*$ satisfies the following conditions :
(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $a * 1 = a \ \forall \ a \ \varepsilon \ [0\,,\,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c\,,\, b \leq d$ and $a\,,\,b\,,\,c\,,\,d \ \varepsilon \ [0\,,\,1]$.

Result 2.2 ([7])**.** (a) For any $r_1\,,\,r_2 \in (0\,,1)$ with $r_1 > r_2$, there exist $r_3 \in (0\,,1)$ such that $r_1 * r_3 > r_2$.
(b) For any $r_5 \in (0\,,1)$, there exist $r_6 \in (0\,,1)$ such that $r_6 * r_6 \geq r_5$.

Definition 2.3 ([11])**.** The 3-tuple $(X\,,\mu\,,*)$ is called a non-archimedean fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and μ is a fuzzy set in $X^2 \times (0,\infty)$ satisfying the following conditions :
(i) $\mu(x\,,y\,,t) > 0$,
(ii) $\mu(x\,,y\,,t) = 1$ if and only if $x = y$,
(iii) $\mu(x\,,y\,,t) = \mu(y\,,x\,,t)$,
(iv) $\mu(x\,,y\,,s) * \mu(y\,,z\,,t) \leq \mu(x\,,z\,,max\{s\,,t\})$,
(v) $\mu(x\,,y,\cdot) : (0,\ \infty) \rightarrow (0,\ 1]$ is continuous, for all x , y , z $\in X$ and $t,\,s > 0$.
Note that $\mu(x\,,y\,,t)$ can be thought of as the degree of nearness between x and y with respect to t.

Remark 2.4. $\mu(x\,,y\,,.)$ is non-decreasing for all $x,\,y \in X$ and $\mu(x\,,y\,,t) \geq \mu(x\,,z\,,t) * \mu(z\,,y\,,t)$ for all $x,\,y,\,z \in X$ and $t > 0$.
Every non-Archimedean fuzzy metric space is also a fuzzy metric space.

Definition 2.5. A partially ordered set is a set P and a binary relation $\preceq$, denoted by $(X \preceq)$ such that for all $a\,,b\,,c \in P$,
(i) $a \preceq a\,(reflexivity)$,
(ii) $a \preceq b \ and \ b \preceq c \ implies \ a \preceq c\,(transitivity)$,
(iii) $a \preceq b \ and \ b \preceq a \ implies \ a = b\,(anti - symmetry)$.

Definition 2.6. An element $(x\,,y) \in X \times X$ is a called a coupled fixed point of the mapping $F : X \times X \longrightarrow X$ if $x = F(x\,,y)$ and $y = F(y\,,x)$.

Definition 2.7. Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \longrightarrow X$. The mapping F is said to has the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is , for any $x, y \in X$

$$x_1, x_2 \in X, x_1 \preceq x_2 \Longrightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Longrightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 2.8 ([16]). Let $(X, \mu, *)$ be a non-archimedean fuzzy metric space. A sequence $\{x_n\}_n$ in X is said to converge to $x \in X$ if and only if

$$\lim_{n \to \infty} \mu(x_n, x, t) = 1 \quad \text{for each t ¿ 0}.$$

A sequence $\{x_n\}_n$ in X is called Cauchy sequence if and only if

$$\lim_{n \to \infty} \mu(x_n, x_{n+p}, t) = 1 \text{ for each t ¿ 0 and p } = 1, 2, 3, \cdots.$$

A non-archimedean fuzzy metric space $(X, \mu, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent in X.

Lemma 2.9 ([21]). *Let $\{x_n\}$ be a sequence in fuzzy metric space X with $t * t \geq t$. If there exist a number $k \in (0, 1)$ such that*

$$\mu(x_{n+2}, x_{n+1}, kt) \geq \mu(x_{n+1}, x_n, t)$$

for all $x, y \in X, t > 0$ and $n = 1, 2, \cdots$, then $\{x_n\}$ is a Cauchy sequence in X.

3. Results and discussion

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and $(X, \mu, *)$ is a complete non-archimedean fuzzy metric space. Let $F : X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ and*

$$(3.1)\quad \begin{aligned} &\mu(F(x, y), F(u, v), kt) \\ &\geq \max\{\mu(F(x, y), x, t), \mu(F(u, v), x, t)\} * \max\{\mu(F(x, y), u, t), \\ &\quad \mu(F(u, v), u, t)\} * \max\{\mu(F(x, y), u, t), \mu(F(u, v), x, t)\}, \end{aligned}$$

for all $x, y, u, v \in X$ with $x \succeq u$, $y \preceq v$ and $k \in (0, 1)$. Then F has a coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0). \tag{3.2}$$

Define the sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n), \tag{3.3}$$

for all $n = 0, 1, 2, \cdots$.
We claim that $\{x_n\}$ is monotonic increasing and $\{y_n\}$ is monotonic decreasing, i.e.,

$$x_n \preceq x_{n+1} \text{ and } y_n \succeq y_{n+1}, \tag{3.4}$$

for all $n = 0, 1, 2, \cdots$. From (3.2) and (3.3), we have

$x_0 \preceq F(x_0, y_0), y_0 \succeq F(y_0, x_0)$ *and* $x_1 = F(x_0, y_0), y_1 = F(y_0, x_0)$.

Thus $x_0 \preceq x_1$, $y_0 \succeq y_1$, i.e., equation (3.4) is true for $n = 0$. Now suppose that equation (3.4) hold for some n.

$$i.e., \qquad x_n \preceq x_{n+1} \text{ and } y_n \succeq y_{n+1}.$$

We shall prove that (3.4) is true for $n + 1$. Now $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$ then by mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = x_{n+1}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_{n+1}) \preceq F(y_n, x_n) = y_{n+1}.$$

Thus by the mathematical induction principle (4) holds for all n in N. So,

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$$

and

$$y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_n \succeq y_{n+1} \succeq \cdots.$$

Since $x_{n-1} \preceq x_n$ and $y_{n-1} \succeq y_n$, from (1) we have

$$\begin{aligned}
&\mu(F(x_n, y_n), F(x_{n-1}, y_{n-1}), kt) \\
&\geq \max\{\mu(F(x_n, y_n), x_n, t), \mu(F(x_{n-1}, y_{n-1}), x_n, t)\} * \\
&\quad \max\{\mu(F(x_n, y_n), x_{n-1}, t), \mu(F(x_{n-1}, y_{n-1}), x_{n-1}, t)\} * \\
&\quad \max\{\mu(F(x_n, y_n), x_{n-1}, t), \mu(F(x_{n-1}, y_{n-1}), x_n, t)\} \\
&\Longrightarrow \mu(x_{n+1}, x_n, kt) \geq \max\{\mu(x_{n+1}, x_n, t), \mu(x_n, x_n, t)\} * \\
&\quad \max\{\mu(x_{n+1}, x_{n-1}, t), \mu(x_n, x_{n-1}, t)\} * \max\{\mu(x_{n+1}, x_{n-1}, t), \\
&\qquad \mu(x_n, x_n, t)\} \\
&\Longrightarrow \mu(x_{n+1}, x_n, kt) \geq \max\{\mu(x_{n+1}, x_n, t), 1\} * \\
&\quad \max\{\mu(x_{n+1}, x_{n-1}, t), \mu(x_n, x_{n-1}, t)\} * \max\{\mu(x_{n+1}, x_{n-1}, t), 1\} \\
&\Longrightarrow \mu(x_{n+1}, x_n, kt) \geq \max\{\mu(x_{n+1}, x_{n-1}, t), \mu(x_n, x_{n-1}, t)\} \\
&\geq \max\{\mu(x_{n+1}, x_n, t) * \mu(x_n, x_{n-1}, t), \mu(x_n, x_{n-1}, t)\} \\
&\Longrightarrow \mu(x_{n+1}, x_n, t) \geq \mu\left(x_n, x_{n-1}, \frac{t}{k}\right) \\
&\qquad\qquad \vdots \\
&\qquad\qquad \geq \mu\left(x_1, x_0, \frac{t}{k^n}\right) \\
&\Longrightarrow \lim_{n \to \infty} \mu(x_{n+1}, x_n, t) = 1.
\end{aligned}$$

We now verify that $\{x_n\}_n$ is a Cauchy sequence. So,

$$\mu(x_n, x_{n+p}, t) \geq \mu(x_n, x_{n+1}, t) * \cdots * \mu(x_{n+p-1}, x_{n+p}, t)$$

$$\Longrightarrow \lim_{n \to \infty} \mu(x_n, x_{n+p}, t) \geq 1 * \cdots * 1 = 1$$

$\Longrightarrow \{x_n\}_n$ is a Cauchy sequence.
Similarly, since $y_{n-1} \succeq y_n$ and $x_{n-1} \preceq x_n$ and from (3.1), we have

$$\begin{aligned}
&\mu(F(y_{n-1}, x_{n-1}), F(y_n, x_n), kt) \\
&\geq \max\{\mu(F(y_{n-1}, x_{n-1}), y_{n-1}, t), \mu(F(y_n, x_n), y_{n-1}, t)\} * \\
&\quad \max\{\mu(F(y_{n-1}, x_{n-1}), y_n, t), \mu(F(y_n, x_n), y_n, t)\} * \\
&\quad \max\{\mu(F(y_{n-1}, x_{n-1}), y_n, t), \mu(F(y_n, x_n), y_{n-1}, t)\} \\
&\Longrightarrow \mu(y_n, y_{n+1}, kt) \geq \max\{\mu(y_n, y_{n-1}, t), \mu(y_{n+1}, y_{n-1}, t)\} * \\
&\quad \max\{\mu(y_n, y_n, t), \mu(y_{n+1}, y_n, t)\} * \max\{\mu(y_n, y_n, t), \\
&\qquad\qquad \mu(, y_{n+1}, y_{n-1}, t)\} \\
&\Longrightarrow \mu(y_n, y_{n+1}, kt) \geq \max\{\mu(y_n, y_{n-1}, t), \mu(y_{n+1}, y_{n-1}, t)\} \\
&\qquad\qquad \geq \max\{\mu(y_n, y_{n-1}, t), \mu(y_{n+1}, y_n, t) * \\
&\qquad\qquad\qquad \mu(y_n, y_{n-1}, t)\} \\
&\Longrightarrow \mu(y_n, y_{n+1}, t) \geq \mu\left(y_n, y_{n-1}, \frac{t}{k}\right) \geq \cdots \geq \mu\left(y_1, y_0, \frac{t}{k^n}\right) \\
&\Longrightarrow \lim_{n \to \infty} \mu(y_n, y_{n+1}, t) = 1 \\
&\Longrightarrow \lim_{n \to \infty} \mu(y_n, y_{n+p}, t) = 1.
\end{aligned}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is a complete fuzzy metric space, there exist $x, y \in X$ such that

$$\lim_{n \to \infty} x_n = x \quad and \quad \lim_{n \to \infty} y_n = y.$$

Thus by taking limit as $n \longrightarrow \infty$ in equation (3), we get

$$x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} F(x_{n-1}, y_{n-1}) =$$
$$F\left(\lim_{n \to \infty} x_{n-1}, \lim_{n \to \infty} y_{n-1}\right) = F(x, y)$$

and

$$y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} F(y_{n-1}, x_{n-1}) =$$
$$F\left(\lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} x_{n-1}\right) = F(y, x).$$

Therefore $x = F(x, y)$ and $y = F(y, x)$. Thus F has a coupled fixed point. Hence the theorem. □

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and $(X, \mu, *)$ is a complete non-archimedean fuzzy metric space. Let $F : X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \ and \ y_0 \succeq F(y_0, x_0).$$

Suppose that

$$(3.5) \qquad \mu(F(x, y), F(u, v), kt) \geq \max\{\mu(x, u, t), \mu(y, v, t)\}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$, $k \in (0, 1)$. Also suppose either
(a) F is a continuous or
(b) X has the following property :

(i) if a non-decreasing sequence $\{x_n\} \longrightarrow x$ *then* $x_n \preceq x$ *for all* n.

(ii) if a non-increasing sequence $\{y_n\} \longrightarrow y$ *then* $y \preceq y_n$ *for all* n.

Then there exist $x, y \in X$ *such that*

$$x = F(x, y) \text{ and } y = F(y, x),$$

that is, F *has a coupled fixed point in* X.

Proof. Let $x_0, y_0 \in X$ be such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0). \tag{3.6}$$

Define the sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n) \tag{3.7}$$

for all $n \geq 0$.
We claim that $\{x_n\}$ is monotonic increasing and $\{y_n\}$ is monotonic decreasing, i.e.,

$$x_n \preceq x_{n+1} \text{ and } y_n \succeq y_{n+1} \text{ for all } n = 0, 1, 2, \cdots. \tag{3.8}$$

From (3.6) and (3.7),we have

$$x_0 \preceq F(x_0, y_0),\ y_0 \succeq F(y_0, x_0) \text{ and } x_1 = F(x_0, y_0),\ y_1 = F(y_0, x_0).$$

Thus $x_0 \preceq x_1$, $y_0 \succeq y_1$, i.e., (3.8) is true for $n = 0$. Now suppose that (3.8) holds for some n. i.e.,

$$x_n \preceq x_{n+1} \quad \text{and} \quad y_n \succeq y_{n+1}.$$

We shall prove that (3.8) is true for $n+1$. Since $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$, by mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = x_{n+1}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_{n+1}) \preceq F(y_n, x_n) = y_{n+1}.$$

Thus by the mathematical induction we conclude that (3.8) holds for all $n \geq 0$. So,

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$$

and

$$y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_n \succeq y_{n+1} \succeq \cdots.$$

Since $x_{n-1} \preceq x_n$ and $y_{n-1} \succeq y_n$, from (3.5) we have,

$$\mu(F(x_n, y_n), F(x_{n-1}, y_{n-1}), kt) \geq \max\{\mu(x_n, x_{n-1}, t), \mu(y_n, y_{n-1}, t)\}$$

$$\Longrightarrow \mu(x_{n+1}, x_n, kt) \geq \max\{\mu(x_n, x_{n-1}, t), \mu(y_n, y_{n-1}, t)\}$$

$$\begin{aligned} \Longrightarrow \mu(x_{n+1}, x_n, t) &\geq \max\left\{\mu\left(x_n, x_{n-1}, \frac{t}{k}\right), \mu\left(y_n, y_{n-1}, \frac{t}{k}\right)\right\} \\ &\vdots \\ &\geq \max\left\{\mu\left(x_1, x_0, \frac{t}{k^n}\right), \mu\left(y_1, y_0, \frac{t}{k^n}\right)\right\} \end{aligned}$$

$$\Longrightarrow \lim_{n \to \infty} \mu(x_{n+1}, x_n, t) = 1$$

$\Longrightarrow \lim_{n \to \infty} \mu(x_n, x_{n+p}, t) = 1$.

Again, since $y_{n-1} \preceq y_n$ and $x_{n-1} \preceq x_n$, from (3.5) we have

$\mu(F(y_{n-1}, x_{n-1}), F(y_n, x_n), kt) \geq \max\{\mu(y_{n-1}, y_n, t), \mu(x_{n-1}, x_n, t)\}$

$\Longrightarrow \mu(y_n, y_{n+1}, kt) \geq \max\{\mu(y_{n-1}, y_n, t), \mu(x_{n-1}, x_n, t)\}$

$$\Longrightarrow \mu(y_n, y_{n+1}, t) \geq \max\left\{\mu\left(y_{n-1}, y_n, \frac{t}{k}\right), \mu\left(x_{n-1}, x_n, \frac{t}{k}\right)\right\}$$

$$\vdots$$

$$\geq \max\left\{\mu\left(y_0, y_1, \frac{t}{k^n}\right), \mu\left(x_0, x_1, \frac{t}{k^n}\right)\right\}$$

$\Longrightarrow \lim_{n \to \infty} \mu(y_n, y_{n+1}, t) = 1$

$\Longrightarrow \lim_{n \to \infty} \mu(y_n, y_{n+p}, t) = 1$.

Therefore, $\{x_n\}$ and $\{y_n\}$ are cauchy sequences in X. Since X is a complete fuzzy metric space, there exist $x, y \in X$ such that

$$\lim_{n \to \infty} x_n = x \textit{ and } \lim_{n \to \infty} y_n = y.$$

Now, suppose that assumption (a) holds. Thus by taking limit as $n \longrightarrow \infty$ in equation (3.7), we get

$$x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} F(x_{n-1}, y_{n-1}) =$$

$$F\left(\lim_{n \to \infty} x_{n-1}, \lim_{n \to \infty} y_{n-1}\right) = F(x, y)$$

and

$$y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} F(y_{n-1}, x_{n-1}) =$$

$$F\left(\lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} x_{n-1}\right) = F(y, x).$$

Hence, $x = F(x, y)$ and $y = F(y, x)$.

Finally, suppose that (b) holds. Since $\{x_n\}$ is non-decreasing sequence and $\{x_n\} \longrightarrow x$ and as $\{y_n\}$ is non-increasing sequence and $\{y_n\} \longrightarrow y$, by assumption (b), we have $x_n \preceq x$ and $y \preceq y_n$ for all n. We have

$$\mu(F(x_n, y_n), F(x, y), kt) \geq \max\{\mu(x_n, x, t), \mu(y_n, y, t)\} \tag{3.9}$$

Taking $n \longrightarrow \infty$ in (3.9), we get $\mu(x, F(x, y), kt) = 1 \implies F(x, y) = x$. Similarly, we can show that $F(y, x) = y$. Therefore, it is proved that F has a coupled fixed point. □

Theorem 3.3. *In addition to hypotheses of Theorem 3.2, suppose that for every $(x, y), (z, l) \in X \times X$, there exists a $(u, v) \in X \times X$ that is a comparable to (x, y) and (z, l), then F has a unique coupled fixed point.*

Proof. From Theorem 3.2, the set of coupled fixed points of F is non-empty. Suppose (x, y) and (z, l) are coupled fixed point of F, that is, $x = F(x, y)$, $y = F(y, x)$, $z = F(z, l)$ and $l = F(l, z)$. We shall show that $x = z$ and $y = l$. By assumption, there exists $(u, v) \in X \times X$ that is comparable to (x, y) and (z, l).
We define sequences $\{u_n\}$, $\{v_n\}$ as follows:

$$u_0 = u, v_0 = v, u_{n+1} = F(u_n, v_n) \textit{ and } v_{n+1} = F(v_n, u_n).$$

Since (u, v) is comparable with (x, y), we may assume that $(x, y) \succeq (u, v) = (u_0, v_0)$. By using the mathematical induction, it is easy to prove that

$$(x, y) \succeq (u_n, v_n). \tag{3.10}$$

From (3.5) and (3.10), we have

$$\mu(F(x, y), F(u_n, v_n), kt) \geq \max\{\mu(x, u_n, t), \mu(y, v_n, t)\}$$

$$\implies \mu(x, u_{n+1}, t) \geq \max\left\{\mu\left(x, u_n, \frac{t}{k}\right), \mu\left(y, v_n, \frac{t}{k}\right)\right\}$$

$$\vdots$$

$$\geq \max\left\{\mu\left(x, u_0, \frac{t}{k^n}\right), \mu\left(y, v_0, \frac{t}{k^n}\right)\right\}$$

$$\implies \lim_{n \to \infty} \mu(x, u_{n+1}, t) = 1. \tag{3.11}$$

Similarly, we also have

$$\mu(F(v_n, u_n), F(y, x), kt) \geq \max\{\mu(v_n, y, t), \mu(u_n, x, t)\}$$

$$\implies \mu(v_{n+1}, y, t) \geq \max\left\{\mu\left(v_n, y, \frac{t}{k}\right), \mu\left(u_n, x, \frac{t}{k}\right)\right\}$$

$$\vdots$$

$$\geq \max\left\{\mu\left(v_0, y, \frac{t}{k^n}\right), \mu\left(u_0, x, \frac{t}{k^n}\right)\right\}$$

$$\implies \lim_{n \to \infty} \mu(v_{n+1}, y, t) = 1. \tag{3.12}$$

Similarly, one can show that

$$\lim_{n \to \infty} \mu(z, u_{n+1}, t) = \lim_{n \to \infty} \mu(l, v_{n+1}, t) = 1. \tag{3.13}$$

From (3.11), (3.12) and (3.13), we obtain $x = z$ and $y = l$. □

Theorem 3.4. *In addition to hypotheses of Theorem 3.2, if x_0 and y_0 are comparable then F has a fixed point, that is, there exists $x \in X$ such that $x = F(x, x)$.*

Proof. Following the proof of Theorem 3.2, F has a coupled fixed point (x, y). We only have to show that $x = y$. Since x_0 and y_0 are comparable, we may assume that $x_0 \succeq y_0$. By using the mathematical induction, one can show that

(3.14) $$x_n \succeq y_n \qquad \forall\, n \geq 0$$

where $\{x_n\}$ and $\{y_n\}$ be defined by (3.7). From (3.5) and (3.14), we have

$$\mu(F(x_n, y_n), F(y_n, x_n), kt) \geq \max\{\mu(x_n, y_n, t), \mu(y_n, x_n, t)\}$$

$$\Longrightarrow \mu(x_{n+1}, y_{n+1}, kt) \geq \mu(x_n, y_n, t)$$

$$\Longrightarrow \mu(x_{n+1}, y_{n+1}, t) \geq \mu\left(x_n, y_n, \frac{t}{k}\right)$$

$$\Longrightarrow \mu(x_{n+1}, y_{n+1}, t) \geq \mu\left(x_n, y_n, \frac{t}{k}\right) \geq \cdots \geq \mu\left(x_0, y_0, \frac{t}{k^{n+1}}\right).$$

Thus by taking limit as $n \longrightarrow \infty$ in the above inequality, we have

$$\Longrightarrow \lim_{n \to \infty} \mu(x, y, t) = 1.$$

Therefore $x = y$, that is, F has a fixed point. □

Example 3.5. Let $X = [0, 1]$ with fuzzy metric $\mu(x, y, kt) = \frac{kt}{kt + |x - y|}$ for all $x, y \in X$ and $k \in \left[\frac{1}{2}, 1\right)$. On the set X, we consider the following relation: for $x, y \in X$, $x \preceq y \iff x, y \in [0, 1]$ and $x \leq y$ where $\leq$ is the usual ordering of real numbers, then $(X, \preceq)$ is a partially ordered set. Also $(X, \mu, *)$ is a complete fuzzy metric. Moreover, X has the property :

(i) if a non-decreasing sequence $\{x_n\} \longrightarrow x$ then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \longrightarrow y$ then $y \preceq y_n$ for all n.

Let $F : X \times X \longrightarrow X$ as $F(x, y) = \frac{x - y}{2}$ if $x, y \in [0, 1], x \geq y$ and $F(x, y) = 0$ if $x < y$.

Then clearly F is continuous and has the mixed monotone property. Also, there are $x_0 = 0$, $y_0 = 0$ in X such that $x_0 = 0 \preceq F(0, 0) = F(x_0, y_0)$ and $y_0 = 0 \succeq F(0, 0) = F(y_0, x_0)$. Then it is obvious that $(0, 0)$ is a coupled fixed point of F. We show that the mapping F satisfies the conditions of Theorem 3.2.

We take $x, y, u, v \in X$ such that $x \succeq u$, $y \preceq v$ or $(x, y) \succeq (u, v)$. We have the following cases :

Case 1. If $(x, y) = (u, v)$ or $(x, y) = (0, 0)$, $(u, v) = (0, 1)$ or $(x, y) = (1, 1)$, $(u, v) = (0, 1)$, then clearly $\mu(F(x, y), F(u, v), kt) = 1$.

Thus (3.5) holds.

Case 2. Now, suppose $(x, y) = (1, 0)$, $(u, v) = (0, 0)$. Then

$$\begin{aligned} LHS\ of\ (3.5) &= \mu(F(x, y), F(u, v), kt) \\ &= \mu(F(1, 0), F(0, 0), kt) \\ &= \mu\left(\frac{1}{2}, 0, kt\right) = \frac{kt}{kt + \frac{1}{2}} \end{aligned}$$

and

$$RHS\ of\ (3.5) = \min\{\mu(1,0,t), \mu(0,0,t)$$
$$= \min\left\{\frac{t}{t+1}, 1\right\} = \frac{t}{t+1}.$$

Thus (3.5)holds.

Case 3. If $(x, y) = (1, 0)$, $(u, v) = (0, 1)$, Then

$$LHS\ of\ (3.5) = \mu(F(x,y), F(u,v), kt)$$
$$= \mu(F(1,0), F(0,1), kt)$$
$$= \mu\left(\frac{1}{2}, 0, kt\right) = \frac{kt}{kt + \frac{1}{2}}$$

and

$$RHS\ of\ (3.5) = \min\{\mu(1,0,t), \mu(0,1,t)\}$$
$$= \min\left\{\frac{t}{t+1}, \frac{t}{t+1}\right\} = \frac{t}{t+1}.$$

Thus (3.5) holds.

Case 4. $(x, y) = (1, 0)$, $(u, v) = (1, 1)$, then

$$LHS\ of\ (3.5) = \mu(F(x,y), F(u,v), kt)$$
$$= \mu(F(1,0), F(1,1), kt)$$
$$= \mu\left(\frac{1}{2}, 0, kt\right) = \frac{kt}{kt + \frac{1}{2}}$$

and

$$RHS\ of\ (3.5) = \min\{\mu(1,1,t), \mu(0,1,t)\}$$
$$= \min\left\{1, \frac{t}{t+1}\right\} = \frac{t}{t+1}.$$

Thus (3.5) holds and hence all the conditions of theorem (3.2) are satisfied. Applying theorem (3.2) we can conclude that F has a coupled fixed point in X.

4. Conclusion

In the paper [14], Author Jong Seo Park has used a citation of our paper, entitled "Coupled fixed point theorems in partially ordered non-Archimedean complete fuzzy metric spaces", T. K. Samanta, S. Mohinta, Punjab Univ. J., (2014), To appear at serial no.10 in the reference list, but this paper has not been published in Punjab University Journal. Later on, only changing the order of the author's name, it is going to be published in this esteemed journal.

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