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Projection operators on soft inner product spaces

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Abstract. In the present paper projection soft linear operators on soft inner product spaces have been introduced and some basic properties of such operators are investigated.

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1. Introduction

Molodtsov [24] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties while modeling the problems in engineering, physics, computer science, economics, social sciences, and medical sciences. As he argued that soft sets are more general than fuzzy sets, as a mathematical structure and serves as a better tool for processing uncertainty because of its non-restrictive parametrization and easy applicability to various real life problems. Following his work many researchers have worked on different branches of soft sets. Some works are found in [1, 2, 3, 4, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 31]. Recently we have introduced soft real sets, soft real numbers, soft complex sets, soft complex numbers in [5, 6]. Two different concepts of soft metric have been presented in [7, 8]. 'Soft linear (vector) space' and 'soft norm' on an absolute 'soft vector space' have been introduced in [9]. An idea of 'soft inner product' has been introduced in [10]. In [11, 12] we have proposed ideas of 'soft linear operator' and 'soft linear functional' on 'soft linear spaces' and 'soft normed linear spaces'. In [12], four fundamental theorems of functional analysis have also been extended in soft set settings.

In fuzzy settings, metric and norm structures are nicely developed based on the theory of interval analysis. But there are some inherent difficulties in the process of fuzzifying the complex numbers because of its non-comparable order structure on one hand and the lattice ordering of the gradation function of fuzzy sets on the other. As a consequence, fuzzification of inner product space theory has not been developed to that extent as it is required for handling non-probabilistic uncertainties in quantum theory. However, in soft set settings, it has been possible to extend the inner product theory nicely. In the present paper an attempt has been made to extend the operator theory on soft inner product spaces. In [13, 14] we introduced notions of self-adjoint soft linear operators, completely continuous soft linear operators, normal operators, unitary operators, isometric operators, square root of positive operators on soft inner product spaces and studied some of their properties.

In this paper we have further extended the study of operators on soft inner product spaces. In fact, in this paper, projection soft linear operators have been introduced and some basic properties of such operators have been investigated. In section 2, some preliminary results are given. In section 3, a notion of direct sum of soft subspaces is given and some of their basic properties are studied. In section 4, projection soft linear operators over soft inner product spaces are introduced and some fundamental properties of such operators are studied. Section 6 concludes the paper.

2. Preliminaries

Definition 2.1 ([24])**.** Let U be an universe and E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon -$ approximate elements of the soft set (F, A).

Definition 2.2 ([16])**.** For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(1) $A \subseteq B$ and
(2) for all $e \in A$, $F(e) \subseteq G(e)$. We write $(F, A)\widetilde{\subset} (G, B)$.

(F, A) is said to be a soft superset of (G, B), if (G, B) is a soft subset of (F, A). We denote it by$(F, A)\widetilde{\supset} (G, B)$.

Definition 2.3 ([16])**.** Two soft sets (F, A) and (G, B) over a common universe U are said to be equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4 ([22])**.** The union of two soft sets (F, A) and (G, B) over the common universe U is the soft set
(H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cup G(e) & if\ e \in A \cap B. \end{cases}$$

We express it as $(F, A)\,\widetilde{\cup}\,(G, B) = (H, C)$.

The following definition of intersection of two soft sets is given as that of the bi-intersection in [15].

Definition 2.5 ([15]). The intersection of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. We write $(F, A) \widetilde{\cap} (G, B) = (H, C)$.

Let X be an initial universal set and A be the non-empty set of parameters. In the above definitions the set of parameters may vary from soft set to soft set, but in our considerations, throughout this paper all soft sets have the same set of parameters A. The above definitions are also valid for these type of soft sets as a particular case of those definitions.

Definition 2.6 ([16]). The complement of a soft set (F, A) is denoted by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathcal{P}(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$, for all $\alpha \in A$.

Definition 2.7 ([22]). A soft set (F, A) over U is said to be an absolute soft set denoted by $\check{U}$ if for all $\varepsilon \in A$, $F(\varepsilon) = U$.

Definition 2.8 ([22]). A soft set (F, A) over U is said to be a null soft set denoted by Φ if for all $\varepsilon \in A$, $F(\varepsilon) = \varnothing$.

Definition 2.9 ([29]). The difference (H, A) of two soft sets (F, A) and (G, A) over X, denoted by $(F, A) \setminus (G, A)$, is defined by $H(e) = F(e) \setminus G(e)$ for all $e \in A$.

Proposition 2.10 ([29]). *Let (F, A) and (G, A) be two soft sets over X. Then*
(i) $((F, A) \widetilde{\cup} (G, A))^c = (F, A)^c \widetilde{\cap} (G, A)^c$.
(ii) $((F, A) \widetilde{\cap} (G, A))^c = (F, A)^c \widetilde{\cup} (G, A)^c$.

Definition 2.11 ([5]). Let X be a non-empty set and A be a non-empty parameter set. Then a function $\varepsilon : A \to X$ is said to be a soft element of X. A soft element ε of X is said to belongs to a soft set B of X, which is denoted by $\varepsilon \widetilde{\in} B$, if $\varepsilon(e) \in A(e)$, $\forall e \in A$. Thus for a soft set A of X with respect to the index set A, we have $B(e) = \left\{\varepsilon(e), \varepsilon \widetilde{\in} B\right\}$, $e \in A$.

It is to be noted that every singleton soft set (a soft set (F, A) for which $F(e)$ is a singleton set, $\forall e \in A$) can be identified with a soft element by simply identifying the singleton set with the element that it contains $\forall e \in A$.

Definition 2.12 ([5]). Let R be the set of real numbers and $\mathfrak{B}(R)$ the collection of all non-empty bounded subsets of R and A taken as a set of parameters. Then a mapping $F : A \to \mathfrak{B}(R)$ is called a soft real set. It is denoted by (F, A). If specifically (F, A) is a singleton soft set, then after identifying (F, A) with the corresponding soft element, it will be called a soft real number.

The set of all soft real numbers is denoted by $\mathbb{R}(A)$ and the set of all non-negative soft real numbers by $\mathbb{R}(A)^*$.
We use notations $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ to denote soft real numbers whereas $\bar{r}$, $\bar{s}$, $\bar{t}$ will denote a particular type of soft real numbers such that $\bar{r}(\lambda) = r$, for all $\lambda \in A$ etc. For example $\bar{0}$ is the soft real number where $\bar{0}(\lambda) = 0$, for all $\lambda \in A$.

Definition 2.13. For two soft real numbers $\tilde{r}, \tilde{s}$ we define
(i) $\tilde{r} \widetilde{\leq} \tilde{s}$ if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda)$, for all $\lambda \in A$.
(ii) $\tilde{r} \widetilde{\geq} \tilde{s}$ if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda)$, for all $\lambda \in A$.

(iii) $\tilde{r}\tilde{<}\tilde{s}$ if $\tilde{r}(\lambda) < \tilde{s}(\lambda)$, for all $\lambda \in A$.
(iv) $\tilde{r}\tilde{>}\tilde{s}$ if $\tilde{r}(\lambda) > \tilde{s}(\lambda)$, for all $\lambda \in A$.

Definition 2.14 ([6])**.** Let C be the set of all complex numbers and $\wp(C)$ be the collection of all non-empty bounded subsets of the set of complex numbers. Let A be the set of parameters. Then a mapping

$$F : A \to \wp(C)$$

is called a soft complex set. It is denoted by (F, A).
If in particular (F, A) is a singleton soft set , then identifying (F, A) with the corresponding soft element, it will be called a soft complex number.
The set of all soft complex numbers is denoted by $\mathcal{C}(A)$.

Definition 2.15 ([6])**.** Let (F, A), $(G, A) \in \mathbb{C}(A)$. Then the sum, difference, product and division are defined by
$(F + G)(\lambda) = z + w; z \in F(\lambda),\ w \in G(\lambda), \forall\lambda \in A.$
$(F - G)(\lambda) = z - w; z \in F(\lambda),\ w \in G(\lambda), \forall\lambda \in A.$
$(FG)(\lambda) = zw; z \in F(\lambda),\ w \in G(\lambda), \forall\lambda \in A.$
$(F/G)(\lambda) = z/w; z \in F(\lambda),\ w \in G(\lambda),\ \forall\lambda \in A$, provided $G(\lambda) \neq \varnothing$, $\forall\lambda \in A$.

Definition 2.16 ([6])**.** Let (F, A) be a soft complex number. Then the modulus of (F, A) is denoted by $(|F|, A)$and is defined by $|F|(\lambda) = |z|; z \in F(\lambda), \forall\lambda \in A$, where z is an ordinary complex number.
Since the modulus of each ordinary complex number is a non-negative real number and by definition of soft real numbers it follows that $(|F|, A)$is a non-negative soft real number for every soft complex number (F, A).

Let X be a non-empty set. Let $\check{X}$ be the absolute soft set i.e., $F(\lambda) = X$, $\forall\lambda \in A$, where $(F, A) = \check{X}$. Let $\mathcal{S}(\check{X})$ be the collection of the null soft set Φ and those soft sets (F, A) over X for which $F(\lambda) \neq \varnothing$, for all $\lambda \in A$.
Let $(F, A)(\neq \Phi) \in \mathcal{S}(\check{X})$, then the collection of all soft elements of (F, A) will be denoted by $SE(F, A)$. For a collection $\mathfrak{B}$ of soft elements of $\check{X}$, the soft set generated by $\mathfrak{B}$ is denoted by $SS(\mathfrak{B})$.

Definition 2.17 ([7])**.** A mapping $d : SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A)^*$ is said to be a soft metric on the soft set $\check{X}$ if d satisfies the following conditions:
(M1) $d(\tilde{x}, \tilde{y}) \tilde{\geq} \overline{0}$, for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$.
(M2) $d(\tilde{x}, \tilde{y}) = \overline{0}$, if and only if $\tilde{x} = \tilde{y}$.
(M3) $d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x})$for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$.
(M4) For all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \check{X}, d(\tilde{x}, \tilde{z}) \tilde{\leq} d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z})$
The soft set $\check{X}$ with a soft metric d on $\check{X}$ is said to be a *soft metric space* and is denoted by $(\check{X}, d, A)$ or $(\check{X}, d)$.

Theorem 2.18. *([7], Decomposition Theorem) If a soft metric d satisfies the condition:*

(M5) For $(\xi, \eta) \in X \times X$, and $\lambda \in A$, $\{d(\tilde{x}, \tilde{y})(\lambda) : \tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta\}$ is a singleton set, and if for $\lambda \in A$, $d_\lambda : X \times X \to \mathbb{R}^+$ is defined by $d_\lambda(\xi, \eta) = d(\tilde{x}, \tilde{y})(\lambda)$, where $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ with $\tilde{x}(\lambda) = \xi$, $\tilde{y}(\lambda) = \eta$, then d_λ is a metric on X.

Definition 2.19 ([7])**.** Let $(\check{X}, d)$ be a soft metric space, $\tilde{r}$ be a non-negative soft real number and $\tilde{a}\tilde{\in}\check{X}$. By an open ball with centre $\tilde{a}$ and radius $\tilde{r}$, we mean the collection of soft elements of $\check{X}$ satisfying $d(\tilde{x}, \tilde{a})\tilde{<}\tilde{r}$.
The open ball with centre $\tilde{a}$ and radius $\tilde{r}$ is denoted by $B(\tilde{a}, \tilde{r})$.
Thus $B(\tilde{a}, \tilde{r}) = \{\tilde{x}\tilde{\in}\check{X};; d(\tilde{x}, \tilde{a})\tilde{<}\tilde{r}\} \subset SE(\check{X})$.
$SS(B(\tilde{a}, \tilde{r}))$ will be called a soft open ball with centre $\tilde{a}$ and radius $\tilde{r}$.

Definition 2.20 ([7])**.** Let (Y, A) be a soft subset in a soft metric space $(\check{X}, d)$. Then a soft element $\tilde{a}$ is said to be an interior element of (Y, A) if $\exists$ a positive soft real number $\tilde{r}$ such that $\tilde{a} \in B(\tilde{a}, \tilde{r}) \subset SE(Y, A)$.

Definition 2.21 ([7])**.** Let $(\check{X}, d)$ be a soft metric space and $\mathfrak{B}$ be a non-null collection of soft elements of $\check{X}$. Then $\mathfrak{B}$ is said to be 'open in $\check{X}$ with respect to d' or 'open in $(\check{X}, d)$' if all elements of $\mathfrak{B}$ are interior elements of $\mathfrak{B}$.
The empty set $\varnothing$ is separately considered to be open in $(\check{X}, d)$.

Definition 2.22 ([7])**.** Let $(\check{X}, d)$ be a soft metric space and (Y, A) be a non-null soft subset $\in \mathcal{S}(\check{X})$ in $(\check{X}, d)$. Then (Y, A) is said to be 'soft open in $\check{X}$ with respect to d' if there is a collection $\mathfrak{B}$ of soft elements of (Y, A) such that $\mathfrak{B}$ is open with respect to d and $(Y, A) = SS(\mathfrak{B})$.
The null soft set Φ is separately considered to be soft open in $(\check{X}, d)$.

Theorem 2.23 ([7])**.** *In a soft metric space every open ball is an open set and hence every soft open ball is a soft open set.*

Definition 2.24 ([7])**.** Let $(\check{X}, d)$ be a soft metric space. A soft set $(Y, A) \in \mathcal{S}(\check{X})$, is said to be ' soft closed in $\check{X}$ with respect to d' if its complement $(Y, A)^c$ is soft open in $(\check{X}, d)$.

Definition 2.25 ([7])**.** Let $(\check{X}, d)$ be a soft metric space and $\mathfrak{B}$ be a collection of soft elements of $\check{X}$. A soft element $\tilde{a}\tilde{\in}\mathfrak{B}$ is said to be a limit element of $\mathfrak{B}$, if every open ball $B(\tilde{a}, \tilde{r})$ containing $\tilde{a}$ in $(\check{X}, d)$ contains at least one element of $\mathfrak{B}$ different from $\tilde{a}$.
The set of all limit elements of $\mathfrak{B}$ is said to be the derived set of $\mathfrak{B}$ and is denoted by $\mathfrak{B}^d$.

Definition 2.26 ([7])**.** Let $(\check{X}, d)$ be a soft metric space and $(Y, A) \in \mathcal{S}(\check{X})$. A soft element $\tilde{a}\tilde{\in}\check{X}$ is said to be a soft limit element of (Y, A), if every open ball $B(\tilde{a}, \tilde{r})$ containing $\tilde{a}$ in $(\check{X}, d)$ contains at least one soft element of (Y, A) different from $\tilde{a}$.
A soft limit element of a soft set (Y, A) may or may not belong to the soft set (Y, A).
The set of all soft limit elements of (Y, A) is said to be the derived set of (Y, A) and is denoted by $(Y, A)^d$.

Definition 2.27 ([7])**.** Let $(\check{X}, d)$ be a soft metric space and $\mathfrak{B}$ be a collection of soft elements of $\check{X}$. Then the collection of all elements of $\mathfrak{B}$ and limit elements of $\mathfrak{B}$ in $(\check{X}, d)$ is said to be the closure of $\mathfrak{B}$ in $(\check{X}, d)$. It is denoted by $\widetilde{\mathfrak{B}}$.

Definition 2.28 ([7])**.** Let $(\check{X}, d)$ be a soft metric space and (Y, A) be a soft subset $\in \mathcal{S}(\check{X})$. Then the collection of all soft elements of (Y, A) and soft limit elements of (Y, A) in $(\check{X}, d)$ is said to be the soft closure of (Y, A) in $(\check{X}, d)$. It is denoted by $\overline{(Y, A)}$.

Theorem 2.29 ([7]). *Let $\left(\check{X}, d\right)$ be a soft metric space and $\mathfrak{B}_1$, $\mathfrak{B}_2$ be collections of soft elements in $(\check{X}, d)$. Then*
(i) $(\mathfrak{B}_1 \cup \mathfrak{B}_2)^{\mathbf{d}} = {\mathfrak{B}_1}^{\mathbf{d}} \cup {\mathfrak{B}_2}^{\mathbf{d}}$*;*
(ii) $({\mathfrak{B}_1}^{\mathbf{d}})^{\mathbf{d}} \subset {\mathfrak{B}_1}^{\mathbf{d}}$.

Theorem 2.30 ([7]). *Let $\left(\check{X}, d\right)$ be a soft metric space and $(Y, A)\,,\ (Z, A) \in \mathcal{S}(\check{X})$. Then*
(i) $\overline{\Phi} = \varnothing$ *and* $\overline{\check{X}} = SE(\check{X})$;
(ii) $(Y, A)\,\tilde{\subset} SS(\overline{(Y, A)})$*;*
(iii) $\overline{(Y, A)} = \overline{\widetilde{(Y, A)}}$*;*
(iv) $(Y, A)\tilde{\subset}(Z, A) \Longrightarrow \overline{(Y, A)} \subset \overline{(Z, A)}$*;*
(v) $\overline{(Y, A)} \cup \overline{(Z, A)} \subset \overline{(Y, A)\tilde{\cup}(Z, A)}$*;*
(vi) $\overline{(Y, A)\tilde{\cap}(Z, A)} \subset \overline{(Y, A)} \cap \overline{(Z, A)}$.

Definition 2.31 ([9]). Let V be a vector space over a field K and let A be a parameter set. Let G be a soft set over V. Now G is said to be a soft vector space or soft linear space of V over K if $G(\lambda)$ is a vector subspace of V, $\forall \lambda \in A$.

Definition 2.32 ([9]). Let F be a soft vector space of V over K. Let $G : A \to \wp(V)$ be a soft set over (V, A). Then G is said to be a soft vector subspace of F if

(i) for each $\lambda \in A$, $G(\lambda)$ is a vector subspace of V over K and
(ii) $F(\lambda) \supseteq G(\lambda)\,,\ \forall \lambda \in A$.

Definition 2.33 ([9]). Let G be a soft vector space of V over K. Then a soft element of G is said to be a soft vector of G. For example Θ is the null soft vector defined by $\Theta(\lambda) = \theta$ (the null vector of V), for all $\lambda \in A$.
In a similar manner a soft element of the soft set (K, A) is said to be a soft scalar, K being the scalar field.

Definition 2.34 ([9]). Let $\tilde{x}, \tilde{y}$ be soft vectors of G and $\tilde{k}$ be a soft scalar. Then the addition $\tilde{x} + \tilde{y}$ of $\tilde{x}, \tilde{y}$ and scalar multiplication $\tilde{k}.\ \tilde{x}$ of $\tilde{k}$ and $\tilde{x}$ are defined by
$(\tilde{x} + \tilde{y})(\lambda) = \tilde{x}(\lambda) + \tilde{y}(\lambda)\,,\ (\tilde{k}.\tilde{x})(\lambda) = \tilde{k}(\lambda).\tilde{x}(\lambda)\,,\ \forall \lambda \in A$.
Obviously, $\tilde{x} + \tilde{y}$, $\tilde{k}.\tilde{x}$ are soft vectors of G.

Definition 2.35 ([9]). Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X$, $\forall \lambda \in A$. Then a mapping $\|.\| : SE(\check{X}) \to \mathcal{R}(A)^*$ is said to be a soft norm on the soft vector space $\check{X}$ if $\|.\|$ satisfies the following conditions:
(N1) $\|\tilde{x}\| \tilde{\geq} \overline{0}$, for all $\tilde{x} \tilde{\in} \check{X}$;
(N2) $\|\tilde{x}\| = \overline{0}$ if and only if $\tilde{x} = \Theta$;
(N3) $\|\widetilde{\alpha}.\tilde{x}\| = |\widetilde{\alpha}|\,\|\tilde{x}\|$ for all $\tilde{x} \tilde{\in} \check{X}$ and for every soft scalar $\widetilde{\alpha}$, where $|\widetilde{\alpha}|$ denotes the modulus of $\widetilde{\alpha}$;
(N4) For all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$, $\|\tilde{x} + \tilde{y}\| \tilde{\leq} \|\tilde{x}\| + \|\tilde{y}\|$.
The soft vector space $\check{X}$ with a soft norm $\|.\|$ on $\check{X}$ is said to be a soft normed linear space and is denoted by $(\check{X}, \|.\|, A)$ or $(\check{X}, \|.\|)$. (N1), (N2), (N3) and (N4) are said to be soft norm axioms.

Theorem 2.36 ([9, 30]). *Every soft norm $\|.\|$ satisfies the condition*
(A) For $\xi \in X$, and $\lambda \in A$, $\{\|\tilde{x}\|(\lambda) : \tilde{x}(\lambda) = \xi\}$ is a singleton set.

And hence each soft norm $\|.\|$ can be decomposed into a family of crisp norms $\{\|.\|_\lambda,\ \lambda \in A\}$, where $\|.\|_\lambda : X \to \mathbb{R}^+$ is defined by the following:
for each $\xi \in X$, $\|\xi\|_\lambda = \|\tilde{x}\|(\lambda)$, with $\tilde{x} \tilde{\in} \check{X}$ such that $\tilde{x}(\lambda) = \xi$.

Definition 2.37 ([10])**.** Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X$, $\forall \lambda \in A$. Then a mapping $\langle . \rangle : SE(\check{X}) \times SE(\check{X}) \to \mathcal{C}(A)$ is said to be a soft inner product on the soft vector space $\check{X}$ if $\langle . \rangle$ satisfies the following conditions:

(I1) $\langle \tilde{x}, \tilde{x} \rangle \tilde{\geq} \overline{0}$, for all $\tilde{x} \tilde{\in} \check{X}$ and $\langle \tilde{x}, \tilde{x} \rangle = \overline{0}$ if and only if $\tilde{x} = \Theta$;

(I2) $\langle \tilde{x}, \tilde{y} \rangle = \overline{\langle \tilde{y}, \tilde{x} \rangle}$ where bar denotes the complex conjugate of soft complex numbers;

(I3) $\langle \tilde{\alpha}.\tilde{x}, \tilde{y} \rangle = \tilde{\alpha}.\langle \tilde{x}, \tilde{y} \rangle$ for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ and for every soft scalar $\tilde{\alpha}$;

(I4) For all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$, $\langle \tilde{x} + \tilde{y}, \tilde{z} \rangle = \langle \tilde{x}, \tilde{z} \rangle + \langle \tilde{y}, \tilde{z} \rangle$.

The soft vector space $\check{X}$ with a soft inner product $\langle . \rangle$ on $\check{X}$ is said to be a soft inner product space and is denoted by $(\check{X}, \langle . \rangle, A)$ or $(\check{X}, \langle . \rangle)$. (I1), (I2), (I3) and (I4) are said to be soft inner product axioms.

We now state the following result which is a modified version of Decomposition Theorem of [10].

Theorem 2.38 ([30])**.** *Every soft inner product $\langle . \rangle$ satisfies the condition*
(D) For $(\xi,\ \eta) \in X \times X$ and $\lambda \in A$, $\left\{\langle \tilde{x}, \tilde{y} \rangle(\lambda) : \tilde{x}, \tilde{y} \tilde{\in} \check{X} \text{ such that } \tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta\right\}$ is a singleton set.
Hence every soft inner product can be decomposed into a family of crisp inner products $\{\langle . \rangle_\lambda,\ \lambda \in A\}$, where for each $\lambda \in A$, $\langle . \rangle_\lambda : X \times X \to \mathbb{C}$ is defined by for all $(\xi, \eta) \in X \times X, \langle \xi, \eta \rangle_\lambda = \langle \tilde{x}, \tilde{y} \rangle(\lambda)$, with $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ such that $\tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta$.

Theorem 2.39 ([10])**.** *Let $(\check{X}, \langle . \rangle, A)$ be a soft inner product space. Let us define $\|.\| :\ \check{X} \to R(A)^*$ by $\|\tilde{x}\| = \sqrt{\langle \tilde{x},\ \tilde{x} \rangle}$, for all $\tilde{x} \tilde{\in} \check{X}$. Then $\|.\|$ is a soft norm on $\check{X}$.*

Definition 2.40 ([10])**.** A soft inner product space is said to be complete if it is complete with respect to the soft metric defined by soft inner product. A complete soft inner product space is said to be a soft Hilbert space.

Theorem 2.41 ([10])**.** *Let $\tilde{C}$ be a soft closed convex soft subset of a soft Hilbert space $\tilde{H}$ and having a finite set of parameters A. Then $\tilde{C}$ contains a unique soft vector of smallest soft norm.*

Definition 2.42 ([10])**.** Let $\tilde{M}$ be a soft subset of $\tilde{H}$. Then the set of all soft vectors of $\tilde{H}$ orthogonal to $\tilde{M}$ is called the orthogonal complement of $\tilde{M}$. The soft set generated by orthogonal complement of $\tilde{M}$ will be called the soft orthogonal complement of $\tilde{M}$. It will be denoted by $\tilde{M}^\perp$.

Theorem 2.43. *Let $\tilde{M}$ be a soft subset of $\tilde{H}$ and B be the set of all soft vectors of $\tilde{H}$ orthogonal to $\tilde{M}$. Then $SE(\tilde{M}^\perp) = B$, i.e., $SE(SS(B)) = B$.*

Proof. Obviously,

$$B \subset SE(SS(B)). \tag{2.1}$$

Conversely, let $\tilde{x} \tilde{\in} SS(B)$. Then $\tilde{x}(\lambda) \in SS(B)(\lambda)$, for all $\lambda \in A$. Let $\mu \in A$. Then there exists $\tilde{y} \in B$ such that $\tilde{x}(\mu) = \tilde{y}(\mu)$. Let $\tilde{y} \in B$. Then $\tilde{y} \perp \tilde{M}$. Thus $\langle \tilde{y}, \tilde{z} \rangle = \overline{0}$. So $\langle \tilde{y}(\mu), \tilde{z}(\mu) \rangle_\mu = 0$. Hence $\langle \tilde{x}(\mu), \tilde{z}(\mu) \rangle_\mu = 0,\ \forall \tilde{z} \tilde{\in} \tilde{M}$.

Since $\mu \in A$ is arbitrary, we have $\langle \tilde{x}(\lambda), \tilde{z}(\lambda)\rangle_\lambda = 0,\ \forall \tilde{z}\tilde{\in}\tilde{M}$ and for all $\lambda \in A$. Then $\langle \tilde{x}, \tilde{z}\rangle = \overline{0},\ \forall \tilde{z}\tilde{\in}\tilde{M}$. Thus $\tilde{x} \perp \tilde{M}$, i.e., $\tilde{x} \in B$. So

$$SE(SS(B)) \subset B. \tag{2.2}$$

Hence, from (2.1) and (2.2), $SE(SS(B)) = B$, i.e., $SE(\tilde{M}^{\perp}) = B$. □

Theorem 2.44. *Let $\tilde{M}$ be a soft subset of $\tilde{H}$. Then $\tilde{M}^{\perp}$ is a soft closed subspace of $\tilde{H}$.*

Proof. Let $\{\tilde{x}_n\}$ be a sequence of soft elements such that $\tilde{x}_n \to \tilde{x}$ and $\tilde{x}_n\tilde{\in}\tilde{M}^{\perp},\ \forall n \in N$. Then

$\tilde{x}_n, \tilde{y}\rangle = \overline{0}, \forall \tilde{y}\tilde{\in}\tilde{M}$ and $\forall n \in N$
$\Rightarrow lim_{n\to\infty}\langle \tilde{x}_n, \tilde{y}\rangle = \overline{0}, \forall \tilde{y}\tilde{\in}\tilde{M}$
$\Rightarrow \langle \tilde{x}, \tilde{y}\rangle = \overline{0}, \forall \tilde{y}\tilde{\in}\tilde{M}$. (since $lim_{n\to\infty}\tilde{x}_n = \tilde{x}$)
$\Rightarrow \tilde{x} \perp \tilde{M}$
$\Rightarrow \tilde{x}\tilde{\in}\tilde{M}^{\perp}$
$\Rightarrow \tilde{M}^{\perp}$ is a soft closed subspace of $\tilde{H}$. □

Definition 2.45 ([11])**.** Let $T : SE(\check{X}) \to SE(\check{Y})$ be an operator. Then T is said to be soft linear if

(L1) T is additive, i.e., $T(\tilde{x}_1 + \tilde{x}_2) = T(\tilde{x}_1) + T(\tilde{x}_2)$ for every soft elements $\tilde{x}_1, \tilde{x}_2\tilde{\in}\check{X}$,

(L2) T is homogeneous, i.e., for every soft scalar $\tilde{c}$, $T(\tilde{c}\tilde{x}) = \tilde{c}T(\tilde{x})$, for every soft element $\tilde{x}\tilde{\in}\check{X}$.

The properties (L1) and (L2) can be put in a combined form $T(\tilde{c}_1\tilde{x}_1 + \tilde{c}_2\tilde{x}_2) = \tilde{c}_1 T(\tilde{x}_1) + \tilde{c}_2 T(\tilde{x}_2)$ for every soft elements $\tilde{x}_1, \tilde{x}_2\tilde{\in}\check{X}$ and every soft scalars $\tilde{c}_1,\ \tilde{c}_2$.

Definition 2.46 ([11])**.** The operator $T : SE(\check{X}) \to SE(\check{Y})$ is said to be continuous at $\tilde{x}_0\tilde{\in}\check{X}$ if for every sequence $\{\tilde{x}_n\}$ of soft elements of $\check{X}$ with $\tilde{x}_n \to \tilde{x}_0$ as $n \to \infty$, we have $T(\tilde{x}_n) \to T(\tilde{x}_0)$ as $n \to \infty$ i.e., $\|\tilde{x}_n - \tilde{x}_0\| \to \overline{0}$ as $n \to \infty$ implies $\|T(\tilde{x}_n) - T(\tilde{x}_0)\| \to \overline{0}$ as $n \to \infty$. If T is continuous at each soft element of $\check{X}$, then T is said to be a continuous operator.

Definition 2.47 ([11])**.** Let $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator, where $\check{X}, \check{Y}$ are soft normed linear spaces. The operator T is called bounded if there exists some positive soft real number $\tilde{M}$ such that for all $\tilde{x}\tilde{\in}\check{X}$, $\|T(\tilde{x})\| \tilde{\leq} \tilde{M}\,\|\tilde{x}\|$.

Theorem 2.48 ([11])**.** *Let $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator, where $\check{X}, \check{Y}$ are soft normed linear spaces. If T is bounded if and only if T is continuous.*

We now state the following result which is a modified form of the Decomposition Theorem of [11].

Theorem 2.49 ([11])**.** *Every soft linear operator $T : SE(\check{X}) \to SE(\check{Y})$, where $\check{X}, \check{Y}$ are soft vector spaces, satisfies the condition*
(B) For $\xi \in X$, and $\lambda \in A$, $\left\{T(\tilde{x})(\lambda) : \tilde{x}\ \tilde{\in}\check{X} \text{ such that } \tilde{x}(\lambda) = \xi\right\}$ is a singleton set.
And hence every soft linear operator can be decomposed into a family of crisp linear operators $\{T_\lambda,\ \lambda \in A\}$, where for each $\lambda \in A$, the mapping $T_\lambda : X \to Y$ is defined by $T_\lambda(\xi) = T(\tilde{x})(\lambda)$, for all $\xi \in X$ and $\tilde{x}\tilde{\in}\check{X}$ with $\tilde{x}(\lambda) = \xi$.

Theorem 2.50. [11] *Let $\check{X}, \check{Y}$ be soft normed linear spaces and $T : SE(\check{X}) \to SE(\check{Y})$ be a bounded soft linear operator. Then $\|T(\tilde{x})\| \widetilde{\leq} \|T\| \, \|\tilde{x}\|$,* for all $\tilde{x} \widetilde{\in} \check{X}$.

Theorem 2.51 ([11])**.** *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces. Let $T : SE(\check{X}) \to SE(\check{Y})$ be a continuous soft linear operator. Then T_λ is continuous on X for each $\lambda \in A$.*

Theorem 2.52 ([12])**.** *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces. Let $\{T_\lambda; \lambda \in A\}$ be a family of continuous linear operators such that $T_\lambda : X \to Y$ for each λ. Then the soft linear operator $T : SE(\check{X}) \to SE(\check{Y})$ defined by $(T(\tilde{x}))(\lambda) = T_\lambda(\tilde{x}(\lambda))$, $\forall \lambda \in A$, is a continuous soft linear operator.*

Theorem 2.53 ([12])**.** *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces. Let $\{T_\lambda; \lambda \in A\}$ be a family of bounded linear operators such that $T_\lambda : X \to Y$ for each λ. Then the soft linear operator $T : SE(\check{X}) \to SE(\check{Y})$ defined by $(T(\tilde{x}))(\lambda) = T_\lambda(\tilde{x}(\lambda))$, $\forall \lambda \in A$, is a bounded soft linear operator.*

Definition 2.54. ([12], Soft linear space of operators) Let $\check{X}, \check{Y}$ be soft normed linear spaces. Consider the set W of all continuous soft linear operators S, T etc. each mapping $SE(\check{X})$ into $SE(\check{Y})$. Then W can be interpreted as to form an absolute soft vector space. We shall denote this absolute soft linear (vector) space by $L(\check{X}, \check{Y})$.

Proposition 2.55 ([12])**.** *Each element of $SE\left(L\left(\check{X}, \check{Y}\right)\right)$ can be identified uniquely with a member of W i.e., to a continuous soft linear operator $T : SE(\check{X}) \to SE\left(\check{Y}\right)$.*

Definition 2.56 ([12])**.** A soft linear functional f is a soft linear operator such that $f : SE(\check{X}) \to K$ where $\check{X}$ is a soft linear space and $K = \mathbb{R}(A)$ if $\check{X}$ is a real soft linear space and $K = \mathcal{C}(A)$ if $\check{X}$ is a complex soft linear space.

Theorem 2.57 ([12])**.** *Let $\check{X}$ be a soft normed linear space. Let $f : SE(\check{X}) \to K$ be a continuous soft linear functional on $\check{X}$. Then f_λ is continuous linear functional on X for each $\lambda \in A$.*

Theorem 2.58 ([12])**.** *Let $\check{X}$ be a soft normed linear space. Let $\{f_\lambda; \lambda \in A\}$ be a family of continuous linear functionals such that $f_\lambda : X \to \mathbb{R}$ or $\mathcal{C}$ for each λ. Then the functional $f : SE\left(\check{X}\right) \to K(= \mathbb{R}(A)$ or $\mathcal{C}(A))$ defined by $(f(\tilde{x}))(\lambda) = f_\lambda(\tilde{x}(\lambda))$, $\forall \lambda \in A$ and $\forall \tilde{x} \widetilde{\in} \check{X}$, is a continuous soft linear functional.*

Theorem 2.59 ([13])**.** *(Riesz Representation Theorem) Let $\tilde{H}$ be a soft Hilbert space and f be a soft linear functional on $\tilde{H}$. Then f is continuous on $\tilde{H}$ if and only if there exists a unique soft element $\tilde{y}$ in $\tilde{H}$ such that $f(\tilde{x}) = \langle \tilde{x}, \tilde{y} \rangle$ for all $\tilde{x} \widetilde{\in} \tilde{H}$. Also, $\|f\| = \|\tilde{y}\|$.*

Definition 2.60 ([13])**.** Let $\tilde{H}$ be a soft Hilbert space and let T be a continuous soft linear operator such that $T : SE(\tilde{H}) \to SE(\tilde{H})$ and $\tilde{y} \widetilde{\in} \tilde{H}$. We define a functional $f_{\tilde{y}}$ on $\tilde{H}$ by

$$f_{\tilde{y}}(\tilde{x}) = \langle T(\tilde{x}), \tilde{y} \rangle . \tag{2.3}$$

Then $f_{\tilde{y}}$ is a soft linear functional. Moreover, $f_{\tilde{y}}$ is bounded and therefore $f_{\tilde{y}}$ is a continuous soft linear functional defined everywhere on $\tilde{H}$ and $\|f_{\tilde{y}}(\tilde{x})\| \widetilde{\leq} \|T\| \, \|\tilde{y}\|$.

By Theorem 2.59, $f_{\tilde{y}}$ has the form

$$f_{\tilde{y}}(\tilde{x}) = \langle \tilde{x}, \tilde{y}^* \rangle \tag{2.4}$$

for all $\tilde{x}\tilde{\in}\tilde{H}$, where $\tilde{y}^*\tilde{\in}\tilde{H}$ is uniquely determined by $f_{\tilde{y}}$. Thus to each $\tilde{y}\tilde{\in}\tilde{H}$ we get a unique $\tilde{y}^*$ satisfying (2.4). Therefore we obtain an operator T^* such that $\tilde{y}^* = T^*(\tilde{y})$.
This operator $T^* : SE(\tilde{H}) \to SE(\tilde{H})$ is called the adjoint operator to T. From (2,3) and (2.4), we see that the operator T and its adjoint operator T^* are connected by the relation

$$\langle T(\tilde{x}), \tilde{y} \rangle = \langle \tilde{x}, T^*(\tilde{y}) \rangle . \tag{2.5}$$

Definition 2.61 ([13]). A continuous soft linear operator $T : SE(\tilde{H}) \to SE(\tilde{H})$ is called self-adjoint soft linear operator or simply self-adjoint if $T^* = T$.

We now prove the following Theorems which will be used in the next section.

Theorem 2.62. *Let $\tilde{M}$ be a proper soft closed subspace of a soft Hilbert space $\tilde{H}$ having a finite set of parameters A and $\tilde{x}\tilde{\in}\tilde{H}\backslash\tilde{M}$. Let $\tilde{d}$ be the soft real number such that*

$$\tilde{d}(\lambda) = \inf\{\|\tilde{x} - \tilde{y}\|(\lambda) : \ \tilde{y}\tilde{\in}\tilde{M}\}, \text{ for all } \lambda \in A.$$

Then there exists a unique soft element $\tilde{y}_0\tilde{\in}\tilde{M}$ such that $\|\tilde{x} - \tilde{y}_0\| = \tilde{d}$.

Proof. Consider the soft set $\tilde{C}$ defined by

$$\tilde{C}(\lambda) = \{\tilde{x}(\lambda) + \tilde{y}(\lambda) : \ \tilde{y}\tilde{\in}\tilde{M}\}, \text{ for all } \lambda \in A.$$

Since $\tilde{M}(\lambda)$ is closed for all $\lambda \in A$, by the property of crisp inner product spaces, we have that $\tilde{C}(\lambda)$ is a closed convex set for all $\lambda \in A$. Then $\tilde{C}$ is a soft closed convex soft subset of the soft Hilbert space $\tilde{H}$. Thus for all $\lambda \in A$,

$$\begin{aligned}\tilde{d}(\lambda) &= \inf\{\|\tilde{x} - \tilde{y}^*\|(\lambda) : tildey^*\tilde{\in}\tilde{M}\}\\ &= \inf\{\|\tilde{x} + \tilde{y}\|(\lambda) : tildey = -\tilde{y}^*\tilde{\in}\tilde{M}\}\\ &= \inf\{\|\tilde{z}\|(\lambda) : tildez = \tilde{x} + \tilde{y}\tilde{\in}\tilde{C}\}.\end{aligned}$$

By Theorem 2.41, there exists a unique soft element $\tilde{z}_0\tilde{\in}\tilde{C}$ such that $\|\tilde{z}_0\| = \tilde{d}$. Let $\tilde{y}_0 = \tilde{x} - \tilde{z}_0$. Then clearly $\tilde{y}_0\tilde{\in}\tilde{M}$ and $\|\tilde{x} - \tilde{y}_0\| = \|\tilde{z}_0\| = \tilde{d}$.

We now prove the uniqueness of $\tilde{y}_0$.

Let $\tilde{y}_1\tilde{\in}\tilde{M}$ and $\tilde{y}_1 \neq \tilde{y}_0$ and if possible, suppose that $\|\tilde{x} - \tilde{y}_1\| = \tilde{d}$. Then $\tilde{z}_1 = \tilde{x} - \tilde{y}_1\tilde{\in}\tilde{C}$ such that $\tilde{z}_1 \neq \tilde{z}_0$ and moreover $\|\tilde{z}_1\| = \tilde{d}$. This contradicts the uniqueness of $\tilde{z}_0$. Hence the theorem is proved. □

Theorem 2.63. *If $\tilde{M}$ is a proper soft closed subspace of the soft Hilbert space $\tilde{H}$ having a finite set of parameters A. Then there exists a non-zero soft element $\tilde{z}_0\tilde{\in}\tilde{H}$ such that $\tilde{z}_0 \perp \tilde{M}$.*

Proof. Let $\tilde{x}\tilde{\in}\tilde{H}\backslash\tilde{M}$. Let $\tilde{d}$ be the soft real number such that $\tilde{d}(\lambda) = \inf\{\|\tilde{x} - \tilde{y}\|(\lambda) : \tilde{y}\tilde{\in}\tilde{M}\}$, for all $\lambda \in A$. By Theorem 2.62, there exists a unique soft element $\tilde{y}_0\tilde{\in}\tilde{M}$ such that $\|\tilde{x} - \tilde{y}_0\| = \tilde{d}$.

Let $\tilde{z}_0 = \tilde{x} - \tilde{y}_0$. Then $\|\tilde{z}_0\| = \tilde{d} \neq \overline{0}$. Thus $\tilde{z}_0$ is a non-zero soft element.

We now show that $\tilde{z_0} \perp \tilde{M}$ by showing that $\tilde{z_0} \perp \tilde{y}$ for every soft element $\tilde{y}$ of $\tilde{M}$. If $\tilde{\alpha}$ is a soft scalar, then

$$\|\tilde{z_0} - \tilde{\alpha}\tilde{y}\| = \|\tilde{x} - (\tilde{y_0} + \tilde{y})\| \tilde{\geq} \tilde{d} = \|\tilde{z_0}\|.$$

Thus $\|\tilde{z_0} - \tilde{\alpha}\tilde{y}\|^2 - \|\tilde{z_0}\|^2 \tilde{\geq} \overline{0}$,

or $\langle \tilde{z_0} - \tilde{\alpha}\tilde{y}, \tilde{z_0} - \tilde{\alpha}\tilde{y} \rangle - \langle \tilde{z_0}, \tilde{z_0} \rangle \tilde{\geq} \overline{0}$,

or $\langle \tilde{z_0}, \tilde{z_0} \rangle - \overline{\tilde{\alpha}} \langle \tilde{z_0}, \tilde{y} \rangle - \tilde{\alpha} \overline{\langle \tilde{z_0}, \tilde{y} \rangle} + \tilde{\alpha}\overline{\tilde{\alpha}} \langle \tilde{y}, \tilde{y} \rangle - \langle \tilde{z_0}, \tilde{z_0} \rangle \tilde{\geq} \overline{0}$,

or

$$-\overline{\tilde{\alpha}} \langle \tilde{z_0}, \tilde{y} \rangle - \tilde{\alpha} \overline{\langle \tilde{z_0}, \tilde{y} \rangle} + |\tilde{\alpha}|^2 \|\tilde{y}\|^2 \tilde{\geq} \overline{0}. \tag{2.6}$$

We put $\tilde{\alpha} = \langle \tilde{z_0}, \tilde{y} \rangle$. Then inequality (2.5) becomes

$-\overline{2}|\tilde{\alpha}|^2 + |\tilde{\alpha}|^2 \|\tilde{y}\|^2 \tilde{\geq} \overline{0}$

$\Rightarrow |\tilde{\alpha}|^2 (\|\tilde{y}\|^2 - \overline{2}) \tilde{\geq} \overline{0}$

$\Rightarrow |\tilde{\alpha}| = \overline{0}$,

since $\tilde{y}$ is an arbitrary soft element of $\tilde{M}$. So $\langle \tilde{z_0}, \tilde{y} \rangle = \overline{0}$, i.e., $\tilde{z_0} \perp \tilde{y}$.

This proves the theorem. □

3. Direct sum of soft subspaces

Definition 3.1. Let $\tilde{M}$ and $\tilde{N}$ be two soft subspaces of a soft linear space $\tilde{L}$. Consider the soft set $\tilde{M} + \tilde{N}$. If $\tilde{L} = \tilde{M} + \tilde{N}$, then we say that $\tilde{L}$ is the sum of the subspaces $\tilde{M}$ and $\tilde{N}$. If every element $\tilde{z} \tilde{\in} \tilde{L}$ can be expressed uniquely in the form $\tilde{z} = \tilde{x} + \tilde{y}$, where $\tilde{x} \tilde{\in} \tilde{M}$ and $\tilde{y} \tilde{\in} \tilde{N}$, then $\tilde{L}$ is called the direct sum of the subspaces $\tilde{M}$ and $\tilde{N}$ and it is written $\tilde{L} = \tilde{M} \bigoplus \tilde{N}$.

Theorem 3.2. *Let $\tilde{L} = \tilde{M} + \tilde{N}$, then $\tilde{L} = \tilde{M} \bigoplus \tilde{N}$ if and only if $\tilde{M} \tilde{\cap} \tilde{N} = \{\Theta\}$.*

Proof. Suppose that $\tilde{L} = \tilde{M} \bigoplus \tilde{N}$ and if possible let $\tilde{z}$ be a non-zero soft element such that $\tilde{z} \tilde{\in} \tilde{M} \tilde{\cap} \tilde{N}$. Then we can express $\tilde{z}$ in two different ways, as the sum of two soft elements, one from $\tilde{M}$ and another from $\tilde{N}$. Because, we can write

$$\tilde{z} = \tilde{x} + \tilde{y} = \tilde{z} + \Theta, \text{ where } \tilde{x} = \tilde{z} \tilde{\in} \tilde{M} \text{ and } \tilde{y} = \Theta \tilde{\in} \tilde{N}.$$

Also

$$\tilde{z} = \tilde{x} + \tilde{y} = \Theta + \tilde{z}, \text{ where } \tilde{x} = \Theta \tilde{\in} \tilde{M} \text{ and } \tilde{y} = \tilde{z} \tilde{\in} \tilde{N}.$$

This contradicts the fact that every soft element of $\tilde{L}$ can be expressed uniquely as the sum of two elements, one from $\tilde{M}$ and another from $\tilde{N}$. So, $\tilde{M} \tilde{\cap} \tilde{N} = \{\Theta\}$.

Conversely suppose that $\tilde{M} \tilde{\cap} \tilde{N} = \{\Theta\}$. Now $\tilde{L} = \tilde{M} + \tilde{N}$, so each $\tilde{z}$ in $\tilde{L}$ can be expressed like $\tilde{z} = \tilde{x} + \tilde{y}$, where $\tilde{x} \tilde{\in} \tilde{M}$ and $\tilde{y} \tilde{\in} \tilde{N}$. We show that this decomposition is unique and then we have $\tilde{L} = \tilde{M} \bigoplus \tilde{N}$.

If possible, let $\tilde{z}$ be expressed in two ways

$$\tilde{z} = \tilde{x} + \tilde{y} = \tilde{u} + \tilde{v}, \text{ where } \tilde{x}, \tilde{u} \tilde{\in} \tilde{M} \text{ and } \tilde{y}, \tilde{v} \tilde{\in} \tilde{N}.$$

Then

$$\tilde{x} - \tilde{u} = \tilde{v} - \tilde{y} \text{ and } \tilde{x} - \tilde{u} \tilde{\in} \tilde{M} \text{ and } \tilde{v} - \tilde{y} \tilde{\in} \tilde{N} \text{ and they are equal.}$$

Since $\tilde{M}\tilde{\cap}\tilde{N} = \{\Theta\}$, this implies that both sides are zero soft element, i.e., $\tilde{x} = \tilde{u}$ and $\tilde{y} = \tilde{v}$ and the decomposition of $\tilde{z}$ is unique. This completes the proof. □

Definition 3.3. Two non-empty soft subsets $\tilde{M}, \tilde{N}$ of a soft Hilbert space $\tilde{H}$ are said to be *orthogonal* if $\tilde{x} \perp \tilde{y}$ whenever $\tilde{x}\tilde{\in}\tilde{M}$ and $\tilde{y}\tilde{\in}\tilde{N}$. This is written as $\tilde{M} \perp \tilde{N}$.

Theorem 3.4. *Let $\tilde{M}$ and $\tilde{N}$ be two soft closed subspaces of a soft Hilbert space $\tilde{H}$ such that $\tilde{M} \perp \tilde{N}$. Then the soft subspace $\tilde{M} + \tilde{N}$ is also soft closed.*

Proof. Since $\tilde{M} \perp \tilde{N}$, it is clear that $\tilde{M}\tilde{\cap}\tilde{N} = \{\Theta\}$ and it follows by Theorem 3.2, that each soft element $\tilde{z}\tilde{\in}\tilde{M} + \tilde{N}$ can be expressed uniquely in the form $\tilde{z} = \tilde{x} + \tilde{y}$, where $\tilde{x}\tilde{\in}\tilde{M}$ and $\tilde{y}\tilde{\in}\tilde{N}$.

Let $\tilde{z_n} \to \tilde{z}$ be such that $\tilde{z_n}\tilde{\in}\tilde{M} + \tilde{N}$. To prove the theorem we must show that $\tilde{z}\tilde{\in}\tilde{M} + \tilde{N}$. Let $\tilde{z_n} = \tilde{x_n} + \tilde{y_n}$, where $\tilde{x_n}\tilde{\in}\tilde{M}$ and $\tilde{y_n}\tilde{\in}\tilde{N}$.

Now, $\tilde{x_m} - \tilde{x_n}\tilde{\in}\tilde{M}$ and $\tilde{y_m} - \tilde{y_n}\tilde{\in}\tilde{N}$ and so $(\tilde{x_m} - \tilde{x_n}) \perp (\tilde{y_m} - \tilde{y_n})$. By Pythagorean Theorem, we have

$$\|\tilde{x_m} - \tilde{x_n}\|^2 + \|\tilde{y_m} - \tilde{y_n}\|^2 = \|\tilde{x_m} - \tilde{x_n} + \tilde{y_m} - \tilde{y_n}\|^2 = \|\tilde{z_m} - \tilde{z_n}\|^2.$$

Since the right hand side tends to $\overline{0}$ as $m, n \to \infty$, it follows that $\{\tilde{x_n}\}$ in $\tilde{M}$ and $\{\tilde{y_n}\}$ in $\tilde{N}$ are Cauchy sequences. Since $\tilde{M}$, $\tilde{N}$ are both soft closed, they are soft complete. So there exists soft elements $\tilde{x}\tilde{\in}\tilde{M}$ and $\tilde{y}\tilde{\in}\tilde{N}$ such that

$$\tilde{x_n} \to \tilde{x} \text{ and } \tilde{y_n} \to \tilde{y}.$$

Now, $\tilde{x} + \tilde{y}$ is in $\tilde{M} + \tilde{N}$ and $\tilde{z}\tilde{\in}\tilde{M} + \tilde{N}$ follows from

$$\tilde{z} = lim_{n\to\infty} z_n = lim_{n\to\infty}(x_n + y_n) = lim_{n\to\infty} x_n + lim_{n\to\infty} y_n = \tilde{x} + \tilde{y}.$$

This completes the proof. □

Theorem 3.5. *(Projection Theorem) Let $\tilde{M}$ be a soft closed subspace of a soft Hilbert space $\tilde{H}$ having a finite set of parameters A. Then $\tilde{H} = \tilde{M} \bigoplus \tilde{M}^{\perp}$.*

Proof. By Theorem 2.44, $\tilde{M}^{\perp}$ is a soft closed subspace of the soft Hilbert space $\tilde{H}$ and $\tilde{M} \perp \tilde{M}^{\perp}$, Theorem 3.4 implies that $\tilde{M} + \tilde{M}^{\perp}$ is a soft closed subspace of $\tilde{H}$. We now show that $\tilde{H} = \tilde{M} + \tilde{M}^{\perp}$. If this is not so, then by Theorem 2.63, there exists a soft element $\tilde{z_0} \neq \Theta$ such that $\tilde{z_0} \perp \tilde{M} + \tilde{M}^{\perp}$. Thus
$\langle \tilde{z_0}, \tilde{x} + \tilde{y}\rangle = \overline{0}$ where $\tilde{x}\tilde{\in}\tilde{M}$ and $\tilde{y}\tilde{\in}\tilde{M}^{\perp}$. Taking $\tilde{x} = \Theta$ first and then $\tilde{y} = \Theta$, we see that

$$\langle \tilde{z_0}, \tilde{y}\rangle = \overline{0} \text{ and } \langle \tilde{z_0}, \tilde{x}\rangle = \overline{0}, \text{ i.e., } \tilde{z_0} \perp \tilde{M}^{\perp} \text{ and } \tilde{z_0} \perp \tilde{M}, \text{ i.e., } \tilde{z_0}\tilde{\in}\tilde{M}^{\perp} \cap (\tilde{M}^{\perp})^{\perp}.$$

So $\tilde{z_0}\tilde{\in}\tilde{M}^{\perp} \cap \tilde{M}^{\perp\perp}$.

But we know that for any soft set $\tilde{S}$, $\tilde{S} \cap \tilde{S}^{\perp} \subseteq \{\Theta\}$. Hence $\tilde{z_0} \neq \Theta$ cannot be a member of $\tilde{M}^{\perp} \cap \tilde{M}^{\perp\perp}$. this contradiction shows that $\tilde{H} = \tilde{M} + \tilde{M}^{\perp}$.

Now since $\tilde{M} \cap \tilde{M}^{\perp} = \{\Theta\}$, $\tilde{H} = \tilde{M} + \tilde{M}^{\perp}$, by Theorem 3.2, it follows that $\tilde{H} = \tilde{M} \oplus \tilde{M}^{\perp}$. □

Corollary 3.6. *Let $\tilde{M}$ be a soft closed subspace of a soft Hilbert space $\tilde{H}$ having a finite set of parameters A. Then $\tilde{M} = \tilde{M}^{\perp\perp}$.*

Proof. If $\tilde{x}\tilde{\in}\tilde{M}$, then $\tilde{x} \perp \tilde{y}$ for all $\tilde{y}\tilde{\in}\tilde{M}^{\perp}$, i.e., $\tilde{x} \perp \tilde{M}^{\perp}$. Thus $\tilde{x}\tilde{\in}\tilde{M}^{\perp\perp}$. So $\tilde{M} \subset \tilde{M}^{\perp\perp}$.

On the other hand, by Theorem 3.5, we have

$$\tilde{M} \bigoplus \tilde{M}^{\perp} = \tilde{H} = \tilde{M}^{\perp} \bigoplus \tilde{M}^{\perp\perp} = \tilde{M}^{\perp\perp} \bigoplus \tilde{M}^{\perp}.$$

So $\tilde{M}$ cannot be a proper soft subset of $\tilde{M}^{\perp\perp}$. Hence $\tilde{M} = \tilde{M}^{\perp\perp}$. □

The following theorem is now evident from Theorem 3.5.

Theorem 3.7. *Let $\tilde{M}$ be a soft closed subspace of a soft Hilbert space $\tilde{H}$ having a finite set of parameters A and $\tilde{x}\tilde{\in}\tilde{H}$. Then there exists a unique decomposition $\tilde{x} = \tilde{y} + \tilde{z}$, where $\tilde{y}\tilde{\in}\tilde{M}$ and $\tilde{z} \perp \tilde{M}$.*

The soft element $\tilde{y}$ is called the projection of the soft element $\tilde{x}$ in $\tilde{M}$.

4. Projection soft linear operators

From now on we assume that $\tilde{H}$ is a soft Hilbert space having a finite set of parameters A.

Definition 4.1. Let $\tilde{H}$ be a soft Hilbert space and $\tilde{L}$ be a soft closed subspace of $\tilde{H}$. By Theorem 3.7, every soft element $\tilde{x}\tilde{\in}\tilde{H}$ can be represented uniquely in the form

$$\tilde{x} = \tilde{y} + \tilde{z}, \text{ where } \tilde{y}\tilde{\in}\tilde{L} \text{ and } \tilde{z} \perp \tilde{L}.$$

We can define a soft linear operator P by the rule $P(\tilde{x}) = \tilde{y}$. Because this association depends on the soft subspace $\tilde{L}$, we sometimes write $P_{\tilde{L}}$ instead of P to indicate the subspace $\tilde{L}$. This soft linear operator $P_{\tilde{L}}$, whose domain is $SE(\tilde{H})$ and range is $SE(\tilde{L})$, is called a projection soft linear operator or simply projection operator. We say that P is the projection on the soft closed subspace $\tilde{L}$.

Theorem 4.2. *$P_{\tilde{L}}$ is a self-adjoint soft linear operator with $\|P_{\tilde{L}}\| = \overline{1}$ and $P_{\tilde{L}}^2 = P_{\tilde{L}}$.*

Proof. Clearly $P_{\tilde{L}}$ is a soft linear operator. If $\tilde{x} = \tilde{y} + \tilde{z}$, where $\tilde{y}\tilde{\in}\tilde{L}$ and $\tilde{z} \perp \tilde{L}$, then $P_{\tilde{L}}(\tilde{x}) = \tilde{y}$, Since $\tilde{y} \perp \tilde{z}$, we have

$$\|\tilde{x}\|^2 = \|\tilde{y} + \tilde{z}\|^2 = \|\tilde{y}\|^2 + \|\tilde{z}\|^2 \tilde{\geq} \|\tilde{y}\|^2.$$

Thus $\|P_{\tilde{L}}(\tilde{x})\| = \|\tilde{y}\| \tilde{\leq} \|\tilde{x}\|$ for every $\tilde{x}\tilde{\in}\tilde{H}$, i.e., $\|P_L\| \tilde{\leq} \overline{1}$.

But if $\tilde{x}\tilde{\in}\tilde{L}$, then $P_{\tilde{L}}(\tilde{x}) = \tilde{x}$. Thus $\|P_{\tilde{L}}(\tilde{x})\| = \|\tilde{x}\|$. So $\|P_{\tilde{L}}\| = \overline{1}$.

Let $\tilde{x_1},\ \tilde{x_2}\tilde{\in}\tilde{H}$ and let $\tilde{y_1},\ \tilde{y_2}$ be their projections on $\tilde{L}$, i.e.,

$$\tilde{x_1} = \tilde{y_1} + \tilde{z_1},\ \ \tilde{x_2} = \tilde{y_2} + \tilde{z_2},\ \text{ where } \tilde{y_1}, \tilde{y_2}\tilde{\in}\tilde{L} \text{ and } \tilde{z_1}, \tilde{z_2} \perp \tilde{L}.$$

Then

$$\langle P_{\tilde{L}}(\tilde{x_1}), \tilde{x_2}\rangle = \langle \tilde{y_1}, \tilde{x_2}\rangle = \langle \tilde{y_1}, \tilde{y_2}\rangle \text{ and } \langle \tilde{x_1}, P_{\tilde{L}}(\tilde{x_2})\rangle = \langle \tilde{x_1}, \tilde{y_2}\rangle = \langle \tilde{y_1}, \tilde{y_2}\rangle.$$

Thus

$$\langle \tilde{x_1}, P_{\tilde{L}}(\tilde{x_2})\rangle = \langle P_{\tilde{L}}(\tilde{x_1}), \tilde{x_2}\rangle = \langle \tilde{x_1}, P_{\tilde{L}}^*(\tilde{x_2})\rangle, \text{ for all } \tilde{x_1},\ \tilde{x_2}\tilde{\in}\tilde{H}.$$

Hence $P_{\tilde{L}} = P_{\tilde{L}}^*$ and so $P_{\tilde{L}}$ is a self-adjoint soft linear operator.

Now for all $\tilde{x}\tilde{\in}\tilde{H}$, $P_{\tilde{L}}(\tilde{x})\tilde{\in}\tilde{L}$ and for $\tilde{x}'\tilde{\in}\tilde{L}$, $P_{\tilde{L}}(\tilde{x}') = \tilde{x}'$,

$$P_{\tilde{L}}^2(\tilde{x}) = P_{\tilde{L}}(P_{\tilde{L}}(\tilde{x})) = P_{\tilde{L}}(\tilde{x}) \text{ for every } \tilde{x}\tilde{\in}\tilde{H}.$$

Then $P_{\tilde{L}}^2 = P_{\tilde{L}}$. □

The converse of the above theorem is also true as it is evident from the following theorem.

Theorem 4.3. *Every self-adjoint soft linear operator P with $P^2 = P$ is a projection soft linear operator on some soft closed subspace.*

Proof. Let $\tilde{L}$ denote the soft set containing all the soft elements $\tilde{y}\tilde{\in}\tilde{H}$ of the form $\tilde{y} = P(\tilde{x})$ for all $\tilde{x}\tilde{\in}\tilde{H}$. Since P is soft linear it is clear that $\tilde{L}$ is a soft subspace. We now show that $\tilde{L}$ is soft closed.

Suppose that $\tilde{y}_n \to \tilde{y}$ where $\tilde{y}_n\tilde{\in}\tilde{L}$. Then we can assume that $\tilde{y}_n = P(\tilde{x}_n)$ with $\tilde{x}_n\tilde{\in}\tilde{H}$. Thus $P(\tilde{y}_n) = P^2(\tilde{x}_n) = P(\tilde{x}_n) = \tilde{y}_n$ and the soft continuity of P implies that $P(\tilde{y}_n) \to P(\tilde{y})$, i.e., $\tilde{y}_n \to P(\tilde{y})$. So $\tilde{y} = P(\tilde{y})$ and $\tilde{y}\tilde{\in}\tilde{L}$. Hence $\tilde{L}$ is soft closed.

Now, for $\tilde{x}, \tilde{x}'\tilde{\in}\tilde{H}$,

$\langle \tilde{x} - P(\tilde{x}), P(\tilde{x}')\rangle = \langle P^*(\tilde{x} - P(\tilde{x})), \tilde{x}'\rangle = \langle P(\tilde{x} - P^2(\tilde{x})), \tilde{x}'\rangle = \langle \Theta, \tilde{x}'\rangle = \overline{0}$.

Then $(\tilde{x} - P(\tilde{x})) \perp P(\tilde{x}')$. Since $\tilde{x}'\tilde{\in}\tilde{H}$ is arbitrary, we have $(\tilde{x} - P(\tilde{x})) \perp \tilde{L}$.

Now, $\tilde{x} = P(\tilde{x}) + (\tilde{x} - P(\tilde{x}))$ where $P(\tilde{x})\tilde{\in}\tilde{L}$ and $(\tilde{x} - P(\tilde{x})) \perp \tilde{L}$. Hence P is a projection soft linear operator on $\tilde{L}$. □

Definition 4.4. Two projection soft linear operators P_1, P_2 are called orthogonal if $P_1P_2 = O$, the zero soft linear operator.

If $P_1P_2 = O$ then $P_2P_1 = O$, because $O = (P_1P_2)^* = P_2^*P_1^* = P_2P_1$.

Theorem 4.5. *The projection soft linear operators P_1, P_2 are orthogonal if and only if the corresponding soft subspaces $\tilde{L}_1$, $\tilde{L}_2$ are orthogonal, i.e., $\tilde{L}_1 \perp \tilde{L}_2$.*

Proof. If $P_1P_2 = O$, $\tilde{x}_1\tilde{\in}\tilde{L}_1$ and $\tilde{x}_2\tilde{\in}\tilde{L}_2$, then we have

$$\langle \tilde{x}_1, \tilde{x}_2\rangle = \langle P_1(\tilde{x}_1), P_2(\tilde{x}_2)\rangle = \langle \tilde{x}_1, P_1^*P_2(\tilde{x}_2)\rangle = \langle \tilde{x}_1, P_1P_2(\tilde{x}_2)\rangle = \langle \tilde{x}_1, \Theta\rangle = \overline{0}.$$

Thus $\tilde{L}_1 \perp \tilde{L}_2$.

Conversely, if $\tilde{L}_1 \perp \tilde{L}_2$, then for all $\tilde{x}\tilde{\in}\tilde{H}$, $P_2(\tilde{x})\tilde{\in}\tilde{L}_2$, it follows that $P_2(\tilde{x}) \perp \tilde{L}_1$. Thus $P_2(\tilde{x}) = \Theta + P_2(\tilde{x})$ where $\Theta\tilde{\in}\tilde{L}_1$ and $P_2(\tilde{x}) \perp \tilde{L}_1$. So $P_1P_2(\tilde{x}) = \Theta$ for all $\tilde{x}\tilde{\in}\tilde{H}$, i.e., $P_1P_2 = O$. Hence P_1, P_2 are orthogonal. □

Theorem 4.6. *The sum of two projection soft linear operators $P_{\tilde{L}}$ and $P_{\tilde{L}'}$ is a projection soft linear operator if and only if these are orthogonal. If $P_{\tilde{L}}$ is orthogonal to $P_{\tilde{L}'}$, then $P_{\tilde{L}} + P_{\tilde{L}'} = P_{\tilde{L}\bigoplus\tilde{L}'}$.*

Proof. First we suppose that $P_{\tilde{L}} + P_{\tilde{L}'}$ is a projection soft linear operator. Then

$$(P_{\tilde{L}} + P_{\tilde{L}'})^2 = P_{\tilde{L}} + P_{\tilde{L}'} \text{ or } P_{\tilde{L}}P_{\tilde{L}'} + P_{\tilde{L}'}P_{\tilde{L}} = O.$$

We operate by $P_{\tilde{L}}$ on the left and obtain

$$P_{\tilde{L}}P_{\tilde{L}'} + P_{\tilde{L}}P_{\tilde{L}'}P_{\tilde{L}} = O \tag{4.1}$$

If we now operate by $P_{\tilde{L}}$ on the right and obtain $P_{\tilde{L}}P_{\tilde{L}'}P_{\tilde{L}} = O$ and so equation (4.1) becomes $P_{\tilde{L}}P_{\tilde{L}'} = O$.

Conversely, suppose that $P_{\tilde{L}}P_{\tilde{L}'} = P_{\tilde{L}'}P_{\tilde{L}} = O$. Then

$$(P_{\tilde{L}} + P_{\tilde{L}'})^2 = P_{\tilde{L}}^2 + P_{\tilde{L}} P_{\tilde{L}'} + P_{\tilde{L}'} P_{\tilde{L}} + P_{\tilde{L}'}^2 = P_{\tilde{L}} + P_{\tilde{L}'}$$

and

$$(P_{\tilde{L}} + P_{\tilde{L}'})^* = P_{\tilde{L}} + P_{\tilde{L}'}.$$

By Theorem 4.3, $P_{\tilde{L}} + P_{\tilde{L}'}$ is a projection soft linear operator. Now suppose that $P_{\tilde{L}} P_{\tilde{L}'} = O$. Then, by Theorem 4.5, $\tilde{L}_1 \perp \tilde{L}_2$. If $P = P_{\tilde{L}} + P_{\tilde{L}'}$ and $\tilde{x} \tilde{\in} \tilde{H}$, then

$$P(\tilde{x}) = P_{\tilde{L}}(\tilde{x}) + P_{\tilde{L}'}(\tilde{x}) \tilde{\in} \tilde{L} \bigoplus \tilde{L}^{'}.$$

Also,

$$\begin{aligned}\langle \tilde{x} - P(\tilde{x}), P(\tilde{x}) \rangle &= \langle P^*(\tilde{x} - P(\tilde{x})), \tilde{x} \rangle \\ &= \langle P(\tilde{x}) - P^2(\tilde{x}), \tilde{x} \rangle \\ &= \langle P(\tilde{x}) - P(\tilde{x}), \tilde{x} \rangle \\ &= \langle \Theta, \tilde{x} \rangle = \overline{0},\end{aligned}$$

i.e., $\tilde{x} - P(\tilde{x}) \perp P(\tilde{x})$. Thus $\tilde{x} = P(\tilde{x}) + (\tilde{x} - P(\tilde{x}))$ where $P(\tilde{x}) \tilde{\in} \tilde{L} \bigoplus \tilde{L}^{'}$ and $\tilde{x} - P(\tilde{x}) \perp (\tilde{L} \bigoplus \tilde{L}^{'})$. So P is a projection soft linear operator on $\tilde{L} \bigoplus \tilde{L}^{'}$. □

Theorem 4.7. *The product of two projection soft linear operators $P_{\tilde{L}}$ and $P_{\tilde{L}'}$ is a projection soft linear operator if and only if these are permutable. If $P_{\tilde{L}}$ and $P_{\tilde{L}'}$ are permutable, then $P_{\tilde{L}} P_{\tilde{L}'} = P_{\tilde{L} \tilde{\cap} \tilde{L}'}$.*

Proof. Suppose that $P_{\tilde{L}} P_{\tilde{L}'}$ is a projection soft linear operator. Then $P_{\tilde{L}} P_{\tilde{L}'}$ is self-adjoint. Thus $P_{\tilde{L}} P_{\tilde{L}'} = (P_{\tilde{L}} P_{\tilde{L}'})^* = P_{\tilde{L}'}^* P_{\tilde{L}}^* = P_{\tilde{L}'} P_{\tilde{L}}$ and the permutability is obtained.

Conversely, suppose that $P_{\tilde{L}} P_{\tilde{L}'} = P_{\tilde{L}'} P_{\tilde{L}}$. Then

$$(P_{\tilde{L}} P_{\tilde{L}'})^* = P_{\tilde{L}'}^* P_{\tilde{L}}^* = P_{\tilde{L}'} P_{\tilde{L}} = P_{\tilde{L}} P_{\tilde{L}'}.$$

Thus $P_{\tilde{L}} P_{\tilde{L}'}$ is self-adjoint. Also,

$$(P_{\tilde{L}} P_{\tilde{L}'})^2 = P_{\tilde{L}} P_{\tilde{L}'} P_{\tilde{L}} P_{\tilde{L}'} = P_{\tilde{L}}^2 P_{\tilde{L}'}^2 = P_{\tilde{L}} P_{\tilde{L}'}.$$

Thus $P = P_{\tilde{L}} P_{\tilde{L}'}$ is a projection soft linear operator, by Theorem 4.3.

Now suppose that $P_{\tilde{L}} P_{\tilde{L}'} = P_{\tilde{L}'} P_{\tilde{L}}$ and let $\tilde{x} \tilde{\in} \tilde{H}$ be arbitrary soft element.

If $P = P_{\tilde{L}} P_{\tilde{L}'}$, then $P(\tilde{x}) = P_{\tilde{L}} P_{\tilde{L}'}(\tilde{x}) = P_{\tilde{L}'} P_{\tilde{L}}(\tilde{x})$ lies in $\tilde{L}$ and also in $\tilde{L}^{'}$ and also lies in $\tilde{L} \tilde{\cap} \tilde{L}^{'}$. If $\tilde{x} \tilde{\in} \tilde{L} \tilde{\cap} \tilde{L}^{'}$, then $P(\tilde{y}) = P_{\tilde{L}} P_{\tilde{L}'}(\tilde{y}) = P_{\tilde{L}}(\tilde{y}) = \tilde{y}$.

If now $\tilde{x} \tilde{\in} \tilde{H}$ and $\tilde{y} \tilde{\in} \tilde{L} \cap \tilde{L}^{'}$, then

$$\begin{aligned}\langle \tilde{x} - P(\tilde{x}), \tilde{y} \rangle &= \langle \tilde{x} - P(\tilde{x}), P(\tilde{y}) \rangle \\ &= \langle P(\tilde{x}) - P^2(\tilde{x}), \tilde{y} \rangle \\ &= \langle P(\tilde{x}) - P(\tilde{x}), \tilde{y} \rangle \\ &= \langle \Theta, \tilde{y} \rangle = \overline{0}.\end{aligned}$$

Thus $\tilde{x} - P(\tilde{x}) \perp \tilde{L} \cap \tilde{L}^{'}$. So any $\tilde{x} \tilde{\in} \tilde{H}$ has a representation

$$\tilde{x} = P(\tilde{x}) + (\tilde{x} - P(\tilde{x})) \text{ where } P(\tilde{x}) \tilde{\in} \tilde{L} \cap \tilde{L}^{'} \text{ and } \tilde{x} - P(\tilde{x}) \perp \tilde{L} \cap \tilde{L}^{'}.$$

Hence P is a projection soft linear operator on $\tilde{L} \tilde{\cap} \tilde{L}^{'}$. □

Theorem 4.8. *The difference $P_1 - P_2$ of two projection soft linear operators is a projection soft linear operator if and only if $P_1P_2 = P_2$.*

Proof. Suppose that $P_1 - P_2$ is a projection soft linear operator. Then

$$P_1 - P_2 = (P_1 - P_2)^2 = P_1^2 - P_1P_2 - P_2P_1 + P_2^2 = P_1 - P_1P_2 - P_2P_1 + P_2.$$

Thus

$$P_2P_1 + P_1P_2 = 2P_2. \tag{4.2}$$

Operating by P_1 from the left and from the right, we get

$$P_1P_2P_1 + P_1P_2 = 2P_1P_2 \text{ and } P_2P_1 + P_1P_2P_1 = 2P_2P_1.$$

So $P_1P_2P_1 = P_1P_2,\ P_1P_2P_1 = P_2P_1$ and by equation (4.2), $P_1P_2 = P_2P_1 = P_2$.

Conversely, suppose that $P_1P_2 = P_2$. Then $P_2P_1 = P_2$. If $P = P_1 - P_2$, then

$$\begin{aligned} P^2 &= (P_1 - P_2)^2 \\ &= \langle P_1^2 - P_1P_2 - P_2P_1 + P_2^2 \\ &= P_1 - P_1P_2 - P_2P_1 + P_2 \\ &= P_1 - P_2P_1 \\ &= P_1 - P_2 = P. \end{aligned}$$

and

$$P^* = (P_1 - P_2)^* = P_1 - P_2 = P.$$

By Theorem 4.3, P is a projection soft linear operator. □

Theorem 4.9. *Suppose that $P_1, P_2, P_3, ..., P_n$ be the projections on soft closed subspaces $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, ..., \tilde{M}_n$ respectively of $\tilde{H}$. Then $P = P_1 + P_2 + P_3 + ... + P_n$ is a projection if and only if the P_i's are pairwise orthogonal. If this condition is satisfied the P is the projection on $\tilde{M} = \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3 + ... + \tilde{M}_n$.*

Proof. The operator P is clearly self-adjoint. Then, P is a projection if and only if $P^2 = P$.

Suppose that P_i's are pairwise orthogonal. Then

$$\begin{aligned} P^2 &= P_1^2 + P_2^2 + ... + P_n^2 + \sum_{i,j=1}^{n} P_iP_j, i \neq j \\ &= P_1^2 + P_2^2 + ... + P_n^2 \\ &= P_1 + P_2 + ... + P_n = P. \end{aligned}$$

Thus P is a projection soft linear operator.

Conversely, suppose that P is a projection soft linear operator. Let $\tilde{x}\tilde{\in}\tilde{M}_i$. Then $P_i(\tilde{x}) = \tilde{x}$. Thus

$$\begin{aligned}\|\tilde{x}\|^2 &= \|P_i(\tilde{x})\|^2 \\ &\tilde{\leq} \sum_{j=1}^{n} \|P_j(\tilde{x})\|^2 \\ &= \sum_{j=1}^{n} \langle P_j(\tilde{x}), P_j(\tilde{x})\rangle \\ &= \sum_{j=1}^{n} \langle \tilde{x}, P_j^* P_j(\tilde{x})\rangle \\ &= \sum_{j=1}^{n} \langle \tilde{x}, P_j(\tilde{x})\rangle \\ &= \langle \tilde{x}, \sum_{j=1}^{n} P_j(\tilde{x})\rangle \\ &= \langle \tilde{x}, P(\tilde{x})\rangle \tilde{\leq} \|\tilde{x}\|^2 \|P\| \tilde{\leq} \|\tilde{x}\|^2,\end{aligned}$$

because $\|P\| = \overline{1}$.
So, all the steps are equal and thus $\sum_{j=1}^{n} \|P_j(\tilde{x})\|^2 = \|P_i(\tilde{x})\|^2$. Hence $\|P_j(\tilde{x})\|^2 = \overline{0}$ for $j \neq i$, i.e., $P_j(\tilde{x}) = \Theta$ for $j \neq i$. This indicates that the range $\tilde{M}_i$ of P_i is contained in $\tilde{N}_j$, the null space of P_j, i.e., $\tilde{M}_i \tilde{\subset} \tilde{N}_j$ for $j \neq i$.

On the other hand, if $\tilde{y}\tilde{\in}\tilde{M}_j$, $j \neq i$ and $\tilde{x}\tilde{\in}\tilde{N}_j$, then

$$\langle \tilde{y}, \tilde{x}\rangle = \langle P_j(\tilde{y}), \tilde{x}\rangle = \langle \tilde{y}, P_j^*(\tilde{x})\rangle = \langle \tilde{y}, P_j(\tilde{x})\rangle = \langle \tilde{y}, \Theta\rangle = \overline{0}.$$

Thus $\tilde{x} \perp \tilde{y}$, i.e., $\tilde{x}\tilde{\in}\tilde{M}_j^{\perp}$. So $\tilde{N}_j \tilde{\subset} \tilde{M}_j^{\perp}$. Hence from above, we have $\tilde{M}_i \tilde{\subset} \tilde{M}_j^{\perp}$ for $j \neq i$. This means that $\tilde{M}_i \perp \tilde{M}_j,\ i \neq j$ and by Theorem 4.5, the P_i's are pairwise orthogonal.

Now, we prove the last statement. We have already seen that if $\tilde{x}\tilde{\in}\tilde{M}_i$ then $P_j(\tilde{x}) = \Theta$ for $j \neq i$. Let $\tilde{x}\tilde{\in}\tilde{M}_i$. Then

$$P(\tilde{x}) = P_1(\tilde{x}) + P_2(\tilde{x}) + ... + P_n(\tilde{x}) = P_i(\tilde{x}) = \tilde{x}.$$

Thus each $\tilde{M}_i$ is contained in the range of P. Clearly, the range of P is a soft subspace. So $\tilde{M} = \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3 + ... + \tilde{M}_n$ is contained in the range of P.

Now let $\tilde{x}$ be a soft element in the range of P. Then

$$\tilde{x} = P(\tilde{x}^{'}) = P_1(\tilde{x}^{'}) + P_2(\tilde{x}^{'}) + ... + P_n(\tilde{x}^{'})$$

is evidently in $\tilde{M}$, i.e., the range of P is contained in $\tilde{M}$. Thus $\tilde{M}$ is the same as the range of P. So P is the projection on $\tilde{M}$. This completes the proof. □

5. Conclusions

The concept of operator plays a very important role in many aspects of linear algebra and functional analysis. In the dynamics of quantum theory, one must study operators on infinite dimensional Hilbert spaces. On the other hand, the usual uncertainty principle of Heisenberg ultimates generalized uncertainty principle, this has been motivated by string theory and non-commutative geometry. In string quantum gravity regime space-time points are determined in a fuzzy manner. Thus Hilbert spaces and operators on Hilbert spaces involving the uncertainties need to be developed. In this regard it is to be noted that the study of operator theory on fuzzy inner product spaces is limited since in fuzzy setting complex valued inner product space is not so developed. In soft set settings it has been possible to develop the concept of soft inner product nicely. Operator theory on soft inner product spaces has been introduced in [13, 14]. In this paper we have further extended the operator theory on soft inner product spaces. This concept can be extended to spectral theory of bounded self-adjoint operators and unbounded linear operators on soft Hilbert spaces. A generalization of quantum mechanics can be done by using the generalized unbounded linear operators on soft inner product spaces. There is an ample scope for further research on operator theory in soft inner product spaces.

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