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Fuzzy weak bi-ideals of near-rings

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Abstract. In this paper, we define new notion of fuzzy weak bi-ideals of near-rings, which is a generalized concept of fuzzy bi-ideals of near-rings. We also investigate some of its properties with examples.

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1. Introduction

Zadeh[10] initiated the concept of fuzzy sets in 1965. Abou-Zaid[1] first made the study of fuzzy subnear-rings and ideals of near-rings. The concept of bi-ideals was applied to near-rings in [8]. The idea of fuzzy ideals of near-rings was first proposed by Kim et al.[3]. Jun et al.[4] defined the concept of fuzzy R-subgroups of near-rings. Moreover, Manikantan[5] introduced the notion of fuzzy bi-ideals of near-rings and discussed some of its properties. Yong Uk Cho et al.[9] introduced the concept of weak bi-ideals applied to near-rings. Narayanan[6, 7] has discussed the idea of $(\in, \in \vee q)$-fuzzy subnear-rings and $(\in, \in \vee q)$-fuzzy ideals which is a generalization of fuzzy subnear-rings and fuzzy ideals. In this paper, we define a new notion of fuzzy weak bi-ideals of near-rings, which is a generalized concept of fuzzy bi-ideals of near-rings. We also investigate some of its properties with examples.

2. Preliminaries

In this section, we listed some basic definitions. Throughout this paper R denotes a left near-ring.

Definition 2.1 ([2])**.** A near-ring is an algebraic system $(R, +, \cdot)$ consisting of a non empty set R together with two binary operations called $+$ and $\cdot$ such that $(R, +)$ is a group not necessarily abelian and $(R, \cdot)$ is a semigroup connected by the following distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$ valid for all $x, y, z \in R$. We use the word

'near-ring 'to mean 'left near-ring '. We denote xy instead of $x \cdot y$.
(1) An ideal I of a near-ring R is a subset of R such that
(i) $(I, +)$ is a normal subgroup of $(R, +)$,
(ii) $RI \subseteq I$,
(iii) $(x + a)y - xy \in I$, for any $a \in I$ and $x, y \in R$.
(2) A R-subgroup H of a near-ring R is the subset of R such that
(i) $(H, +)$ is a subgroup of $(R, +)$,
(ii) $RH \subseteq H$,
(iii) $HR \subseteq H$.
Note that H is a left R-subgroup of R if H satisfies (i) and (ii) and a right R-subgroup of R if H satisfies (i) and (iii).

Definition 2.2 ([5])**.** Let R be a near-ring. Given two subsets A and B of R, we define the following products

$$AB = \{ab | a \in A, b \in B\}$$

and

$$A * B = \{(a' + b)a - a'a | a,\ a' \in A,\ b \in B\}.$$

Definition 2.3 ([8])**.** A subgroup B of $(R, +)$ is said to be a bi-ideal of R if $BRB \cap B * RB \subseteq B$.

Definition 2.4 ([9])**.** A subgroup B of $(R, +)$ is said to be a weak bi-ideal of R if $BBB \subseteq B$.

Through out this paper, f_I is the characteristic function of the subset I of R and the characteristic function of R is denoted by $\mathbf{R}$, that means, $\mathbf{R} : R \to [0, 1]$ mapping every element of R to 1.

Definition 2.5 ([6])**.** A function μ from a nonempty set R to the unit interval $[0, 1]$ is called a fuzzy subset of R. Let μ be any fuzzy subset of R, for $t \in [0, 1]$ the set $\mu_t = \{x \in R | \mu(x) \geq t\}$ is called a level subset of μ.

Definition 2.6 ([6])**.** Let μ and λ be any two fuzzy subsets of R. Then $\mu \cap \lambda,\ \mu \cup \lambda,\ \mu + \lambda,\ \mu\lambda$ and $\mu * \lambda$ are fuzzy subsets of R defined by:

$$(\mu \cap \lambda)(x) = \min\{\mu(x),\ \lambda(x)\}.$$

$$(\mu \cup \lambda)(x) = \max\{\mu(x),\ \lambda(x)\}.$$

$$(\mu + \lambda)(x) = \begin{cases} \sup\limits_{x=y+z} \min\{\mu(y),\ \lambda(z)\} & \text{if } x \text{ can be expressed as } x = y + z \\ 0 & \text{otherwise.} \end{cases}$$

$$(\mu\lambda)(x) = \begin{cases} \sup\limits_{x=yz} \min\{\mu(y),\ \lambda(z)\} & \text{if } x \text{ can be expressed as } x = yz \\ 0 & \text{otherwise.} \end{cases}$$

$$(\mu * \lambda)(x) = \begin{cases} \sup\limits_{x=(a+c)b-ab} \min\{\mu(c),\ \lambda(b)\} & \text{if } x \text{ can be expressed as} \\ & x = (a + c)b - ab. \\ 0 & \text{otherwise.} \end{cases}$$

A fuzzy subset μ of R is called a fuzzy subgroup of R, if $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

Definition 2.7 ([5]). A fuzzy subgroup μ of $(R, +)$ is said to be a fuzzy bi-ideal of R if $\mu\mathbf{R}\mu \cap \mu * \mathbf{R}\mu \subseteq \mu$.

Definition 2.8 ([1]). Let R be a near-ring and μ be a fuzzy subset of R. Then μ is a fuzzy ideal of R if

(i) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in R$,
(ii) $\mu(y + x - y) \geq \mu(x)$, for all $x, y \in R$,
(iii) $\mu(xy) \geq \mu(y)$, for all $x, y \in R$,
(iv) $\mu((x + z)y - xy) \geq \mu(z)$, for all $x, y, z \in R$.

A fuzzy subset with (i)-(iii) is called a fuzzy left ideal of R, whereas a fuzzy subset with (i), (ii) and (iv) is called a fuzzy right ideal of R.

Definition 2.9 ([4]). A fuzzy subset μ of a near-ring R is called a fuzzy R-subgroup of R if

(i) μ is a fuzzy subgroup of $(R, +)$,
(ii) $\mu(xy) \geq \mu(y)$,
(iii) $\mu(xy) \geq \mu(x)$, for all $x, y \in R$.

A fuzzy subset with (i) and (ii) is called a fuzzy left R-subgroup of R, whereas a fuzzy subset with (i) and (iii) is called a fuzzy right R-subgroup of R.

Definition 2.10. Let I be a subset of a near-ring R. Define a function $f_I : R \to [0, 1]$ by

$$f_I(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in R$. Clearly f_I is a fuzzy subset of R and is called the characteristic function of I.

3. Fuzzy weak bi-ideals of near-rings

In this section, we introduce the notion of fuzzy weak bi-ideal of R and discuss some of its properties.

Definition 3.1. A fuzzy subgroup μ of R is called a fuzzy weak bi-ideal of R, if

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}.$$

Example 3.2. Let $R = \{0, a, b, c\}$ be near-ring with two binary operations $+$ and $\cdot$ is defined as follws:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	a	0	a
a	0	a	0	a
b	0	a	b	c
c	0	a	b	c

Let $\mu : R \to [0, 1]$ be a fuzzy subset defined by $\mu(0) = 0.7, \mu(a) = 0.2 = \mu(b)$ and $\mu(c) = 0.6$. Then μ is a fuzzy weak bi-ideal of R.

Proposition 3.3. *Let μ be a fuzzy subgroup of R. Then μ is a fuzzy weak bi-ideal of R if and only if $\mu\mu\mu \subseteq \mu$.*

Proof. Assume that μ is a fuzzy weak bi-ideal of R. Let $x, y, z, y_1, y_2 \in R$ such that $x = yz$ and $y = y_1y_2$. Then

$$\begin{aligned}
(\mu\mu\mu)(x) &= \sup_{x=yz}\{\min\{(\mu\mu)(y), \mu(z)\}\} \\
&= \sup_{x=yz}\{\min\{\sup_{y=y_1y_2}\min\{\mu(y_1), \mu(y_2)\}, \mu(z)\}\} \\
&= \sup_{x=yz}\sup_{y=y_1y_2}\{\min\{\min\{\mu(y_1), \mu(y_2)\}, \mu(z)\}\} \\
&= \sup_{x=y_1y_2z}\{\min\{\mu(y_1), \mu(y_2), \mu(z)\}\} \\
&\qquad \text{Since } \mu \text{ is a fuzzy weak bi-ideal of } X, \mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\} \\
&\leq \sup_{x=y_1y_2z}\mu(y_1y_2z) \\
&= \mu(x).
\end{aligned}$$

If x can not be expressed as $x = yz$, then $(\mu\mu\mu)(x) = (0) \leq \mu(x)$. In both cases $\mu\mu\mu \subseteq \mu$.

Conversely, assume that $\mu\mu\mu \subseteq \mu$. For $x', x, y, z \in R$, let x' be such that $x' = xyz$. Then

$$\begin{aligned}
\mu(xyz) = \mu(x') &\geq (\mu\mu\mu)(x') \\
&= \sup_{x'=pq}\{\min\{(\mu\mu)(p), \mu(q)\}\} \\
&= \sup_{x'=pq}\{\min\{\sup_{p=p_1p_2}\min\{\mu(p_1), \mu(p_2)\}, \mu(q)\}\} \\
&= \sup_{x'=p_1p_2q}\{\min\{\mu(p_1), \mu(p_2), \mu(q)\}\} \\
&\geq \min\{\mu(x), \mu(y), \mu(z)\}.
\end{aligned}$$

Hence $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$. □

Lemma 3.4. *Let μ and λ be fuzzy weak bi-ideals of R. Then the products $\mu\lambda$ and $\lambda\mu$ are also fuzzy weak bi-ideals of R.*

Proof. Let μ and λ be fuzzy weak bi-ideals of R. Then

$$\begin{aligned}
(\mu\lambda)(x - y) &= \sup_{x-y=ab}\min\{\mu(a), \lambda(b)\} \\
&\geq \sup_{x-y=a_1b_1-a_2b_2<(a_1-a_2)(b_1-b_2)}\min\{\mu(a_1 - a_2), \lambda(b_1 - b_2)\} \\
&\geq \sup\min\{\min\{\mu(a_1), \mu(a_2)\}, \min\{\lambda(b_1), \lambda(b_2)\}\} \\
&= \sup\min\{\min\{\mu(a_1), \lambda(b_1)\}, \min\{\mu(a_2), \lambda(b_2)\}\} \\
&\geq \min\{\sup_{x=a_1b_1}\min\{\mu(a_1), \lambda(b_1)\}, \sup_{y=a_2b_2}\min\{\mu(a_2), \lambda(b_2)\}\} \\
&= \min\{(\mu\lambda)(x), (\mu\lambda)(y)\}.
\end{aligned}$$

It follows that $\mu\lambda$ is a fuzzy subgroup of R. Further,

$$\begin{aligned}(\mu\lambda)(\mu\lambda)(\mu\lambda) &= \mu\lambda(\mu\lambda\mu)\lambda \\ &\subseteq \mu\lambda(\lambda\lambda\lambda)\lambda, \text{ since } \lambda \text{ is a fuzzy weak bi-ideal of } R \\ &\subseteq \mu(\lambda\lambda\lambda), \text{ since } \lambda \text{ is a fuzzy weak bi-ideal of } R \\ &\subseteq \mu\lambda.\end{aligned}$$

Therefore $\mu\lambda$ is a fuzzy weak bi-ideal of R. Similarly $\lambda\mu$ is a fuzzy weak bi-ideal of R. □

Theorem 3.5. *Every fuzzy bi-ideal of R is a fuzzy weak bi-ideal of R.*

Proof. Assume that μ is a fuzzy bi-ideal of R. Then $\mu\mathbf{R}\mu \cap \mu * \mathbf{R}\mu \subseteq \mu$. We have $\mu\mu\mu \subseteq \mu\mathbf{R}\mu$ and $\mu\mu\mu \subseteq \mu * \mathbf{R}\mu$. This implies that $\mu\mu\mu \subseteq \mu\mathbf{R}\mu \cap \mu * \mathbf{R}\mu \subseteq \mu$. Therefore μ is a fuzzy weak bi-ideal of R. □

However the converse of the Theorem 3.5 is not true in general which is demonstrated by the following Example.

Example 3.6. Let $R = \{0, a, b, c\}$ be near-ring with two binary operations $+$ and $\cdot$ is defined as follws:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	a	c	b
c	0	a	b	c

Let $\mu : R \to [0, 1]$ be a fuzzy subset defined by $\mu(0) = 0.8, \mu(a) = 0.5 = \mu(b)$ and $\mu(c) = 0.7$. Then μ is a fuzzy weak bi-ideal of R. But μ is not a fuzzy bi-ideal of R, since $\min\{(\mu\mathbf{R}\mu)(b), (\mu * \mathbf{R}\mu)(b)\} = 0.7 \nleq \mu(b)$.

Theorem 3.7. *Every fuzzy ideal of R is a fuzzy weak bi-ideal of R.*

Proof. We know that every fuzzy ideal of R is a fuzzy bi-ideal of R by Theorem 3.14[5]. By Theorem 3.5, every fuzzy bi-ideal of R is a fuzzy weak bi-ideal of R. Thus μ is a fuzzy weak bi-ideal of R. □

Theorem 3.8. *Every fuzzy R-subgroup of R is a fuzzy weak bi-ideal of R.*

Proof. We know that every fuzzy R-subgroup of R is a fuzzy bi-ideal of R by Theorem 3.11[5]. By Theorem 3.5, every fuzzy bi-ideal of R is a fuzzy weak bi-ideal of R. Thus μ is a fuzzy weak bi-ideal of R. □

The converse of the Theorem 3.7 and 3.8 are not true in general as shown in following Example.

Example 3.9. In Example 3.2, μ is a fuzzy weak bi-ideal of R. But μ is not a fuzzy ideal of R, because $\mu(0c) = \mu(a) = 0.2 \ngeq 0.6 = \mu(c)$. Also μ is not a fuzzy R-subgroup of R, because $\mu(0c) = \mu(a) = 0.2 \ngeq 0.6 = \mu(c)$ and $\mu(ca) = \mu(a) = 0.2 \ngeq 0.6 = \mu(c)$.

Example 3.10. In Example 3.6, μ is a fuzzy weak bi-ideal of R. But μ is not a fuzzy ideal of R, μ is not a fuzzy R-subgroup of R and μ is not a fuzzy bi-ideal of R.

Theorem 3.11. *Let $\{\mu_i | i \in \Omega\}$ be family of fuzzy weak bi-ideals of a near-ring R, then $\bigcap_{i\in\Omega} \mu_i$ is also a fuzzy weak bi-ideal of R, where Ω is any index set.*

Proof. Let $\{\mu_i\}_{i\in\Omega}$ be a family of fuzzy weak bi-ideals of R. Let $x, y, z \in R$ and $\mu = \bigcap_{i\in\Omega} \mu_i$. Then, $\mu(x) = \bigcap_{i\in\Omega} \mu_i(x) = \left(\inf_{i\in\Omega} \mu_i\right)(x) = \inf_{i\in\Omega} \mu_i(x)$.

$$\begin{aligned}
\mu(x-y) &= \inf_{i\in\Omega}\ \mu_i(x-y) \\
&\geq \inf_{i\in\Omega}\ \min\{\mu_i(x), \mu_i(y)\} \\
&= \min\left\{\inf_{i\in\Omega} \mu_i(x), \inf_{i\in\Omega} \mu_i(y)\right\} \\
&= \min\left\{\bigcap_{i\in\Omega} \mu_i(x), \bigcap_{i\in\Omega} \mu_i(y)\right\} \\
&= \min\{\mu(x), \mu(y)\}.
\end{aligned}$$

And

$$\begin{aligned}
\mu(xyz) &= \inf_{i\in\Omega}\ \mu_i(xyz) \\
&\geq \inf_{i\in\Omega}\ \min\{\mu_i(x), \mu_i(y), \mu_i(z)\} \\
&= \min\left\{\inf_{i\in\Omega} \mu_i(x), \inf_{i\in\Omega} \mu_i(y), \inf_{i\in\Omega} \mu_i(z)\right\} \\
&= \min\left\{\bigcap_{i\in\Omega} \mu_i(x), \bigcap_{i\in\Omega} \mu_i(y), \bigcap_{i\in\Omega} \mu_i(z)\right\} \\
&= \min\{\mu(x), \mu(y), \mu(z)\}.
\end{aligned}$$

□

Theorem 3.12. *Let μ be a fuzzy subset of R. Then μ is a fuzzy weak bi-ideal of R if and only if μ_t is a weak bi-ideal of R, for all $t \in [0, 1]$.*

Proof. Assume that μ is a fuzzy weak bi-ideal of R. Let $t \in [0, 1]$ such that $x, y \in \mu_t$. Then $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} \geq \min\{t, t\} = t$. Thus $x - y \in \mu_t$. Let $x, y, z \in \mu_t$. This implies that $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\} \geq \min\{t, t, t\} = t$. Thus $xyz \in \mu_t$. So μ_t is a weak bi-ideal of R.

Conversely, assume that μ_t is a weak bi-ideal of R, for all $t \in [0, 1]$. Let $x, y \in R$. Suppose $\mu(x - y) < \min\{\mu(x), \mu(y)\}$. Choose t such that $\mu(x - y) < t < \min\{\mu(x), \mu(y)\}$. This implies that $\mu(x) > t, \mu(y) > t$ and $\mu(x - y) < t$. Then we have $x, y \in \mu_t$ but $x - y \notin \mu_t$ a contradiction. Thus $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$. If there exist $x, y, z \in R$ such that $\mu(xyz) < \min\{\mu(x), \mu(y), \mu(z)\}$. Choose t such that $\mu(xyz) < t < \min\{\mu(x), \mu(y), \mu(z)\}$. So $\mu(x) > t, \mu(y) > t, \mu(z) > t$ and $\mu(xyz) < t$. Thus $x, y, z \in \mu_t$ but $xyz \notin \mu_t$, which is a contradiction. Hence $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$. Therefore μ is a fuzzy weak bi-ideal of R. □

Theorem 3.13. *Let W be a nonempty subset of R and μ be a fuzzy subset of R defined by*

$$\mu(x) = \begin{cases} s & \text{if } x \in W \\ t & \text{otherwise} \end{cases}$$

for some $x \in R$, $s, t \in [0, 1]$ and $s > t$. Then W is a weak bi-ideal of R if and only if μ is a fuzzy weak bi-ideal of R.

Proof. Assume that W is a weak bi-ideal of R. Let $x, y \in R$. We consider four cases :

(1) $x \in W$ and $y \in W$.
(2) $x \in W$ and $y \notin W$.
(3) $x \notin W$ and $y \in W$.
(4) $x \notin W$ and $y \notin W$.

Case (1): If $x \in W$ and $y \in W$, then $\mu(x) = s = \mu(y)$. Since W is a weak bi-ideal of R, $x - y \in W$. Thus $\mu(x - y) = s = \min\{s, s\} = \min\{\mu(x), \mu(y)\}$.

Case (2): If $x \in W$ and $y \notin W$, then $\mu(x) = s$ and $\mu(y) = t$. Thus $\min\{\mu(x), \mu(y)\} = t$. Now, $\mu(x - y) = s$ or t according as $x - y \in W$ or $x - y \notin W$. By assumption, $s > t$, we have $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$.

Similarly, we prove case (3).

Case (4): If $x, y \notin W$, then $\mu(x) = t = \mu(y)$. Thus $\min\{\mu(x), \mu(y)\} = t$. Next, $\mu(x-y) = s$ or t according as $x-y \in W$ or $x-y \notin W$. So $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$.
Now let $x, y, z \in R$. Then we have the following eight cases :

(1) $x \in W, y \in W$ and $z \in W$.
(2) $x \notin W, y \in W$ and $z \in W$.
(3) $x \in W, y \notin W$ and $z \in W$.
(4) $x \in W, y \in W$ and $z \notin W$.
(5) $x \notin W, y \notin W$ and $z \in W$.
(6) $x \in W, y \notin W$ and $z \notin W$.
(7) $x \notin W, y \in W$ and $z \notin W$.
(8) $x \notin W, y \notin W$ and $z \notin W$.

These cases can be proved by arguments similar to the fuzzy cases above. Hence $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$. Therefore μ is a fuzzy weak bi-ideal of R.

Conversely, assume that μ is a fuzzy weak bi-ideal of R. Let $x, y, z \in W$ be such that $\mu(x) = \mu(y) = \mu(z) = s$. Since μ is a fuzzy weak bi-ideal of R, we have

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\} = s$$

and

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\} = s.$$

Thus $x - y, xyz \in W$. So W is a weak bi-ideal of R. □

Theorem 3.14. *A nonempty subset W of R is a weak bi-ideal of R if and only if the characteristic function f_W is a fuzzy weak bi-ideal of R.*

Proof. Straightforward from Theorem 3.13. □

Theorem 3.15. *Let μ be fuzzy weak bi-ideal of R, then the set $R_\mu = \{x \in R|\ \mu(x) = \mu(0)\}$ is a weak bi-ideal of R.*

Proof. Let μ be fuzzy weak bi-ideal of R. Let $x, y \in R_\mu$. Then

$$\mu(x) = \mu(0), \mu(y) = \mu(0)$$

and

$$\begin{aligned}\mu(x-y) &\geq \min\{\mu(x), \mu(y)\} \\ &= \min\{\mu(0), \mu(0)\} = \mu(0).\end{aligned}$$

Thus $\mu(x-y) = \mu(0)$. So $x - y \in R_\mu$. For every $x, y, z \in R_\mu$, we have

$$\begin{aligned}\mu(xyz) &\geq \min\{\mu(x), \mu(y), \mu(z)\} \\ &= \min\{\mu(0), \mu(0), \mu(0)\} = \mu(0).\end{aligned}$$

Hence $xyz \in R_\mu$. Therefore R_μ is a weak bi-ideal of R. □

4. Homomorphism of fuzzy weak bi-ideals of near-rings

In this section, we characterize fuzzy weak bi-ideals using homomorphism.

Definition 4.1 ([3])**.** Let f be a mapping from a set R to a set S. Let μ and ν be fuzzy subsets of R and S, respectively. Then $f(\mu)$, the image of μ under f, is a fuzzy subset of S defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

and the pre-image of ν under f is a fuzzy subset of R defined by
$f^{-1}(\nu(x)) = \nu(f(x))$, for all $x \in R$ and $f^{-1}(y) = \{x \in R | f(x) = y\}$.

Definition 4.2 ([3])**.** Let R and S be near-rings. A map $\theta : R \to S$ is called a (near-ring)homomorphism if $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in R$.

Theorem 4.3. *Let $f : R \to S$ be a homomorphism between near-rings R and S. If ν is a fuzzy weak bi-ideal of S, then $f^{-1}(\nu)$ is a fuzzy weak bi-ideal of R.*

Proof. Let ν be a fuzzy weak bi-ideal of S. Let $x, y, z \in R$. Then

$$\begin{aligned}f^{-1}(\nu)(x-y) &= \nu(f(x-y)) \\ &= \nu(f(x) - f(y)) \\ &\geq \min\{\nu(f(x)), \nu(f(y))\} \\ &= \min\{f^{-1}(\nu(x)), f^{-1}(\nu(y))\}. \\ f^{-1}(\nu)(xyz) &= \nu(f(xyz)) \\ &= \nu(f(x)f(y)f(z)) \\ &\geq \min\{\nu(f(x)), \nu(f(y)), \nu(f(z))\} \\ &= \min\{f^{-1}(\nu(x)), f^{-1}(\nu(y)), f^{-1}(\nu(z))\}.\end{aligned}$$

Therefore $f^{-1}(\nu)$ is a fuzzy weak bi-ideal of R. □

We can also state the converse of the Theorem 4.3 by strengthening the condition on f as follows.

Theorem 4.4. *Let $f : R \to S$ be an onto homomorphism of near-rings R and S. Let ν be a fuzzy subset of S. If $f^{-1}(\nu)$ is a fuzzy weak bi-ideal of R, then ν is a fuzzy weak bi-ideal of S.*

Proof. Let $x, y, z \in S$. Then $f(a) = x, f(b) = y$ and $f(c) = z$ for some $a, b, c \in R$. It follows that

$$\begin{aligned}\nu(x-y) &= \nu(f(a)-f(b)) \\ &= \nu(f(a-b)) \\ &= f^{-1}(\nu)(a-b) \\ &\geq \min\{f^{-1}(\nu)(a), f^{-1}(\nu)(b)\} \\ &= \min\{\nu(f(a)), \nu(f(b))\} \\ &= \min\{\nu(x), \nu(y)\}.\end{aligned}$$

And

$$\begin{aligned}\nu(xyz) &= \nu(f(a)f(b)f(c)) \\ &= \nu(f(abc)) \\ &= f^{-1}(\nu)(abc) \\ &\geq \min\{f^{-1}(\nu)(a), f^{-1}(\nu)(b), f^{-1}(\nu)(c)\} \\ &= \min\{\nu(f(a)), \nu(f(b), \nu(f(c))\} \\ &= \min\{\nu(x), \nu(y), \nu(z)\}.\end{aligned}$$

Hence ν is a fuzzy weak bi-ideal of R. □

Theorem 4.5. *Let $f : R \to S$ be an onto near-ring homomorphism. If μ is a fuzzy weak bi-ideal of R, then $f(\mu)$ is a fuzzy weak bi-ideal of S.*

Proof. Let μ be a fuzzy weak bi-ideal of R. Since $f(\mu)(x') = \sup\limits_{f(x)=x'} (\mu(x))$, for $x' \in S$, $f(\mu)$ is nonempty. Let $x', y' \in S$. Then we have
$\{x | x \in f^{-1}(x'-y')\} \supseteq \{x-y | x \in f^{-1}(x') \text{ and } y \in f^{-1}(y')\}$
and $\{x | x \in f^{-1}(x'y')\} \supseteq \{xy | x \in f^{-1}(x') \text{ and } y \in f^{-1}(y')\}$

$$\begin{aligned}f(\mu)(x'-y') &= \sup_{f(z)=x'-y'} \{\mu(z)\} \\ &\geq \sup_{f(x)=x', f(y)=y'} \{\mu(x-y)\} \\ &\geq \sup_{f(x)=x', f(y)=y'} \{\min\{\mu(x), \mu(y)\}\} \\ &= \min\{\sup_{f(x)=x'} \{\mu(x)\}, \sup_{f(y)=y'} \{\mu(y)\}\} \\ &= \min\{f(\mu)(x'), f(\mu)(y')\}.\end{aligned}$$

Next,

$$\begin{aligned} f(\mu)(x'y'z') &= \sup_{f(w)=x'y'z'} \{\mu(w)\} \\ &\geq \sup_{f(x)=x',f(y)=y',f(z)=z'} \{\mu(xyz)\} \\ &\geq \sup_{f(x)=x',f(y)=y',f(z)=z'} \{\min\{\mu(x),\mu(y),\mu(z)\}\} \\ &= \min\{\sup_{f(x)=x'}\{\mu(x)\}, \sup_{f(y)=y'}\{\mu(y)\}, \sup_{f(z)=z'}\{\mu(z)\}\} \\ &= \min\{f(\mu)(x'), f(\mu)(y'), f(\mu)(z')\}. \end{aligned}$$

Therefore $f(\mu)$ is a fuzzy weak bi-ideal of S. □

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