Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 5, (May 2016), pp. 783–798
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Study on interval valued generalized fuzzy ideals of ordered semigroups

Jian Tang, Noor Mohammad Khan, Mohammad Aasim Khan

Received 9 October 2015; Revised 16 November 2015; Accepted 30 December 2015

Abstract. In the present paper, the notions of interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideals and interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideals of ordered semigroups are introduced, and their related properties are investigated. Furthermore, we characterize different classes of ordered semigroups by using the properties of these interval valued generalized fuzzy ideals.

2010 AMS Classification: 06F05, 08A72

Keywords: Ordered semigroup, Interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideal, Interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal.

Corresponding Author: Jian Tang (tangjian0901@126.com)

1. Introduction

The theory of fuzzy sets was introduced by Zadeh [28] in 1965. This theory has provided a useful mathematical tool for describing the behaviour of systems that are too complex or ill-defined to admits precise mathematical analysis by classical methods and tools. Extensive applications of the fuzzy set theory have been found in various field. Since Rosenfeld [20] applied the notion of fuzzy sets to algebra and introduced the notion of fuzzy subgroups, the literature of various fuzzy algebraic concepts has been growing very rapidly. Kuroki initiated the theory of fuzzy semigroups in his paper [15]. The monograph by Mordeson *et al* [18] deals with the theory of fuzzy semigroups and their use in fuzzy codes, fuzzy finite state machines and fuzzy languages. Due to their possibilities of applications, semigroups and related structure are presently extensively investigated in fuzzy settings. Kehayopulu in [9] characterized regular, left regular and right regular ordered semigroups by means of fuzzy left ideals, right ideals and bi-ideals. Murali [19] defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set played a vital role in generating different types of fuzzy subgroups. Using these ideas, Bhakat and Das ([1], [2], [3]) introduced the concept of (α, β)-fuzzy subgroups by using the "belong to" ($\in$) relation and "quasi-coincident with" (q) relation between a fuzzy point and a fuzzy subgroup and introduced the concept of $(\in, \in\vee q)$-fuzzy subgroup. Kazanci and Yamak [8] studied $(\in, \in\vee q)$-fuzzy bi-ideals of a semigroup. Zhan and Jun introduced the concept of $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy interior ideals of a semigroup (see [30]) and gave some properties of prime $(\in, \in\vee q)$-fuzzy interior ideals. Following the terminology given by Zadeh, fuzzy sets in an ordered semigroup S were first considered by Kehayopulu and Tsingelis in [10]. They defined fuzzy analogous for several notions, which have proven useful in the theory of ordered semigroups. The theory of fuzzy sets on ordered semigroups has been recently developed. For more details, the reader is referred to ([5], [6], [9], [11], [12], [17], [22], [23], [26]).

In 1975, Zadeh [29] introduced the concept of interval valued fuzzy subsets, where the values of the membership functions are intervals of numbers instead of numbers. In [4], Biswas defined interval valued fuzzy subgroups of the same nature of Rosenfeld's fuzzy subgroups. In [7], kar *et al*, defined and studied the interval valued prime fuzzy ideals of semigroups. In [16], Ma and Zhan introduced the concept of interval-valued $(\in, \in\vee q)$-fuzzy h-ideals in hemirings and developed some basic results using these ideals. In [21], Shabir and Khan investigated the interval valued fuzzy ideals generated by an interval valued fuzzy subset in ordered semigroups. Since then, Khan *et al* defined and studied interval valued fuzzy generalized bi-ideals of ordered semigroups (see [13]). In [25], Tang *et al* characterized ordered semigroups by new type of interval valued fuzzy quasi-ideals.

In this paper, we first introduce the notion of (k^*, q)-quasi-coincident in ordered semigroups. Then, using the idea of (k^*, q)-quasi-coincident of a fuzzy point with a fuzzy set, we also introduce the notions of interval valued $(\in, \in\vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideals and interval valued $(\in, \in\vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideals of ordered semigroups, and study their related properties. Finally, we characterize different classes of ordered semigroups by using the properties of these interval valued fuzzy ideals.

2. Preliminaries and some notations

A partially ordered semigroup (briefly ordered semigroup) is a pair $(S, \cdot)$ comprising of a semigroup S and a partial order $\leq$ (on S) that is compatible with the binary operation, i.e. for all $a, b, s \in S$, $a \leq b$ implies that $sa \leq sb$ and $as \leq bs$.

For any nonempty subset A of an ordered semigroup S, we define

$$(A] = \{a \in S \mid a \leq b \text{ for some } b \in A\}.$$

For any two nonempty subsets A, B of S, we have (1)$A \subseteq (A]$; (2) If $A \subseteq B$, then $(A] \subseteq (B]$; (3) $(A](B] \subseteq (AB]$ [24].

Let A be a nonempty subset of an ordered semigroup S. Then A is called a subsemigroup of S if $A^2 \subseteq A$. A is called a bi-ideal of S if (1) for any $a \in S$ and $b \in A$, if $a \leq b$, then $a \in A$, (2) for all $a, b \in A \Rightarrow ab \in A$ and (3) $ASA \subseteq A$ [9]. A is called a quasi-ideal of S if $(AS] \cap (SA] \subseteq A$ [25].

By an interval number $\tilde{a}$, we mean an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all intervals numbers is denoted by $D[0,1]$. The interval $[a, a]$ can be identified by the number $a \in [0,1]$. For the interval numbers $\tilde{a}_i = [a_i^-, a_i^+]$, $\tilde{b}_i =$

$[b_i^-, b_i^+] \in D[0,1], i \in I$, we define

(1) $\min\{\tilde{a}_i, \tilde{b}_i\} = [\min(a_i^-, b_i^-), \min(a_i^+, b_i^+)]$,

(2) $\max\{\tilde{a}_i, \tilde{b}_i\} = [\max(a_i^-, b_i^-), \max(a_i^+, b_i^+)]$,

(3) $\tilde{a}_1 \leq \tilde{a}_2 \Leftrightarrow a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$,

(4) $\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,

(5) If $\tilde{a}_1 \geq \tilde{a}_2$, then

$$\tilde{a}_1 - \tilde{a}_2 = \begin{cases} [a_1^- - a_2^-, a_1^+ - a_2^+], & \text{if } a_1^- - a_2^- \leq a_1^+ - a_2^+, \\ [a_1^+ - a_2^+, a_1^- - a_2^-], & \text{if } a_1^- - a_2^- > a_1^+ - a_2^+. \end{cases}$$

(6) $k\tilde{a} = [ka^-, ka^+]$, whenever $0 \leq k \leq 1$,

(7) $\tilde{k}^* = [k^{*-}, k^{*+}]$, whenever $0 < k^* \leq 1$.

Then, it is clear that $(D[0,1], \leq, \wedge, \vee)$ forms a complete lattice with $\tilde{0} = [0,0]$ as its least element and $\tilde{1} = [1,1]$ as its greatest element. The interval valued fuzzy sets provide a more adequate description of uncertainty than the traditional fuzzy sets.

Let S be an ordered semigroup. A mapping $\tilde{f} : S \to D[0,1]$ is called an interval valued fuzzy subset of S, where $\tilde{f}(x) = [f^-(x), f^+(x)]$ for all $x \in S$, f^- and f^+ are two fuzzy sets of S such that $f^-(x) \leq f^+(x)$ for all $x \in S$. The set of all interval valued fuzzy sets of S is denoted by IVF(S).

For any ordered semigroup $(S, \cdot, \leq)$ and $x \in S$, we define

$$A_x = \{(y,z) \in S \times S \mid x \leq yz\}.$$

Then product $\tilde{f} \circ \tilde{g}$ of any fuzzy subsets $\tilde{f}$ and $\tilde{g}$ of S is defined by

$$(\tilde{f} \circ \tilde{g})(x) = \begin{cases} \bigvee_{(y,z) \in A_x} \min\{\tilde{f}(y), \tilde{g}(z)\}, & A_x \neq \varnothing, \\ [0,0], & A_x = \varnothing, \end{cases}$$

for all $x \in S$ [25].

Definition 2.1 ([25]). Let $\tilde{f}$ be an interval valued fuzzy subset of an ordered semigroup S and $\tilde{u} \in D[0,1]$. Then the crisp set

$$\tilde{f}_{\tilde{u}} = \{x \in S \mid \tilde{f}(x) \geq \tilde{u}\}$$

is called a level subset of $\tilde{f}$.

Definition 2.2 ([25]). Let A be a nonempty subset of S. We denote by $\tilde{f}_A$, the interval valued characteristic function of A, that is the mapping of S into $D[0,1]$ defined by

$$\tilde{f}_A(x) = \begin{cases} [1,1], & \text{if } x \in A, \\ [0,0], & \text{if } x \notin A. \end{cases}$$

Clearly $\tilde{f}_A$ is a fuzzy subset of S.

Definition 2.3 ([25]). Let S be an ordered semigroup, $a \in S$ and $\tilde{u} \in D(0,1]$. An interval valued ordered fuzzy point $a_{\tilde{u}}$ of S is defined by

$$\tilde{a}_{\tilde{u}}(x) = \begin{cases} \tilde{u}, & \text{if } x \in (a], \\ [0,0], & \text{if } x \notin (a]. \end{cases}$$

Then $a_{\tilde{u}}$ is a mapping from S into D[0,1]. For any interval valued fuzzy subset $\tilde{f}$ of

S, we shall also denote $a_{\tilde{u}} \subseteq \tilde{f}$ by $a_{\tilde{u}} \in \tilde{f}$ in the sequel. Then $a_{\tilde{u}} \in \tilde{f}$ if and only if $\tilde{f}(a) \geq \tilde{u}$.

Definition 2.4. An interval valued fuzzy subset $\tilde{f}$ of S is called comparable if $\tilde{f}(x)$ and $\tilde{f}(y)$ are comparable for all $x, y \in S$;

$$\tilde{f}(x) \geq \tilde{f}(y) \text{ or } \tilde{f}(x) < \tilde{f}(y).$$

Throughout the paper, we consider that $\tilde{f}$ is a comparable interval valued fuzzy subset of S unless otherwise specified.

3. Interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideals

In this section, we first define $(\tilde{k^*}, q)$-quasi-coincident and investigate mainly the properties of interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideals of an ordered semigroup.

Definition 3.1. An interval valued ordered fuzzy point $a_{\tilde{u}}$ of an ordered semigroup S, for any $\tilde{k^*} \in D(0,1]$, is said to be $(\tilde{k^*}, q)$-quasi-coincident with an interval valued fuzzy subset $\tilde{f}$ of S, written as $a_{\tilde{u}}(\tilde{k^*}, q)\tilde{f}$, if

$$\tilde{f}(a) + \tilde{u} > \tilde{k^*}.$$

Let $(S, \cdot, \leq)$ be an ordered semigroup and $0 \leq \tilde{k} < \tilde{k^*} \leq 1$. For an ordered interval valued fuzzy point $x_{\tilde{u}}$, we say that

$x_{\tilde{u}}(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$ if $\tilde{f}(x) + \tilde{u} + \tilde{k} > \tilde{k^*}$;

$x_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$ if $x_{\tilde{u}} \in \tilde{f}$ or $x_{\tilde{u}}(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$;

$x_{\tilde{u}}\overline{\alpha}\tilde{f}$ if $x_{\tilde{u}}\alpha\tilde{f}$ does not hold for $\alpha \in \{(\tilde{k^*}, q_{\tilde{k}}), \in \vee(\tilde{k^*}, q_{\tilde{k}})\}$.

Definition 3.2. An interval valued fuzzy subset $\tilde{f}$ of an ordered semigroup S is called an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy subsemigroup of S if $x_{\tilde{u}} \in \tilde{f}$ and $y_{\tilde{v}} \in \tilde{f}$ imply $(xy)_{\min\{\tilde{u},\tilde{v}\}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$ for all $\tilde{u}, \tilde{v} \in D(0,1]$ and $x, y \in S$.

Definition 3.3. An interval valued fuzzy subset $\tilde{f}$ of an ordered semigroup S is called an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideal of S if

(1) $x \leq y$, $y_{\tilde{u}} \in \tilde{f}$ imply $x_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$;

(2) $x_{\tilde{u}} \in \tilde{f}$, $y_{\tilde{v}} \in \tilde{f}$ imply $(xy)_{\min\{\tilde{u},\tilde{v}\}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$;

(3) $x_{\tilde{u}} \in \tilde{f}$, $z_{\tilde{v}} \in \tilde{f}$ imply $(xyz)_{\min\{\tilde{u},\tilde{v}\}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$;

for all $x, y, z \in S$ and $\tilde{u}, \tilde{v} \in D(0,1]$.

Theorem 3.4. *An interval valued fuzzy subset $\tilde{f}$ of an ordered semigroup S is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideal of S if*

(1) $x \leq y \Rightarrow \tilde{f}(x) \geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$;

(2) $\tilde{f}(xy) \geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$;

(3) $\tilde{f}(xyz) \geq \min\{\tilde{f}(x), \tilde{f}(z), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$;

for all $x, y, z \in S$.

Proof. Let $\tilde{f}$ be an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideal of S. Suppose condition (1) is not true, i.e., $\tilde{f}(x) < \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$ for some $x, y \in S$. Choose $\tilde{u} \in D(0, \frac{k^*-k}{2}]$ such that $\tilde{f}(x) < \tilde{u} \leq \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$.

Then $x \in S$ and $y_{\tilde{u}} \in \tilde{f}$, but $x_{\tilde{u}} \overline{\in \vee(\tilde{k^*}, q_{\tilde{k}})} \tilde{f}$, which is a contradiction. Thus $\tilde{f}(x) \geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$. Next suppose that condition (2) is not true, i.e., $\tilde{f}(xy) < \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$ for some $x, y \in S$. Choose $\tilde{u} \in D(0, \frac{k^*-k}{2}]$ such that $\tilde{f}(xy) < \tilde{u} \leq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$. Then $x_{\tilde{u}} \in \tilde{f}$, $y_{\tilde{u}} \in \tilde{f}$, but $(xy)_{\tilde{u}} \overline{\in \vee(\tilde{k^*}, q_{\tilde{k}})}$, which is not possible. Thus we have shown that $\tilde{f}(xy) \geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$.

In a similar way, we may show that $\tilde{f}(xyz) \geq \min\{\tilde{f}(x), \tilde{f}(z), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$.

Conversely, suppose that conditions (1), (2) and (3) hold. Let $x, y \in S$ and $x \leq y$ be such that $y_{\tilde{u}} \in \tilde{f}$. Then $\tilde{f}(x) \geq \tilde{u}$. So

$$\tilde{f}(x) \geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} \geq \min\{\tilde{u}, [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}.$$

Now if $\tilde{u} \leq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then $\tilde{f}(x) \geq \tilde{u}$ implying that $x_{\tilde{u}} \in \tilde{f}$. If $\tilde{u} > [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then $\tilde{f}(x) \geq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$. Thus

$$\tilde{f}(x) + \tilde{u} > [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}] + [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}] = [k^{*-} - k^-, k^{*+} - k^+].$$

This implies that $x_{\tilde{u}}(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$. So $x_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$.
Let $\tilde{f}(xy) \geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$ for all $x, y \in S$. Let $x_{\tilde{u}} \in \tilde{f}$ and $y_{\tilde{v}} \in \tilde{f}$ for all $\tilde{u}, \tilde{v} \in D(0, 1]$. Then $\tilde{f}(x) \geq \tilde{u}$ and $\tilde{f}(y) \geq \tilde{v}$. Thus

$$\begin{aligned} \tilde{f}(xy) &\geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} \\ &\geq \min\{\tilde{u}, \tilde{v}, [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\}. \end{aligned}$$

Now if $\min\{\tilde{u}, \tilde{v}\} \leq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then $\tilde{f}(xy) \geq \min\{\tilde{u}, \tilde{v}\}$, which implies that $(xy)_{\min\{\tilde{u},\tilde{v}\}} \in \tilde{f}$. If $\min\{\tilde{u}, \tilde{v}\} > [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then $\tilde{f}(xy) \geq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, and we have

$$\begin{aligned} \tilde{f}(xy) + \min\{\tilde{u}, \tilde{v}\} &> [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}] + [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}] \\ &= [k^{*-} - k^-, k^{*+} - k^+]. \end{aligned}$$

This implies that $(xy)_{\min\{\tilde{u},\tilde{v}\}}(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$. Hence $(xy)_{\min\{\tilde{u},\tilde{v}\}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$.

Similarly we may show that $(xyz)_{\min\{\tilde{u},\tilde{v}\}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$. Therefore, $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideal of S. □

As a consequence of the so-called *Transfer Principle for Fuzzy Sets* in [14], we have the following theorem.

Theorem 3.5. *Let $\tilde{f}$ be an interval valued fuzzy subset of an ordered semigroup S. Then $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideal of S if and only if for all $\tilde{u} \in D(0, \frac{k^*-k}{2}]$, the crisp set $\tilde{f}_{\tilde{u}}$ $(\neq \varnothing)$ is a bi-ideal of S.*

Definition 3.6. Let $\tilde{f}$ be an interval valued fuzzy subset of an ordered semigroup S. The set

$$[\tilde{f}]_{\tilde{u}} = \{x \in S \mid x_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}\}, \text{where} \tilde{u} \in (0, 1]$$

is called an $(\in \vee(\tilde{k^*}, q_{\tilde{k}}))$-*level subset* of $\tilde{f}$.

Theorem 3.7. *Let $\tilde{f}$ be an interval valued fuzzy subset of an ordered semigroup S. Then $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideal of S if and only if the $(\in \vee(\tilde{k^*}, q_{\tilde{k}}))$-level subset $[\tilde{f}]_{\tilde{u}}$ of $\tilde{f}$ is a bi-ideal of S for all $\tilde{u} \in D(0,1]$.*

Proof. Suppose $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideal of S. To show that $[\tilde{f}]_{\tilde{u}}$ is a bi-ideal of S, take any $y \in [\tilde{f}]_{\tilde{u}}$ and $x \leq y$. As $y \in [\tilde{f}]_{\tilde{u}}$, we have $y_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$, that is $\tilde{f}(y) \geq \tilde{u}$ or $\tilde{f}(y) + \tilde{u} > [k^{*-} - k^-, k^{*+} - k^+]$. Now, by Theorem 3.4, $\tilde{f}(x) \geq \tilde{f}(y) \geq \tilde{u}$ or $\tilde{f}(x) \geq \tilde{f}(y) \geq [k^{*-} - k^-, k^{*+} - k^+]$, that is $x_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$. Then $x \in [\tilde{f}]_{\tilde{u}}$. Next, take any $x, y \in [\tilde{f}]_{\tilde{u}}$ for $\tilde{u} \in D(0,1]$. Thus $x_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$ and $y_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$, that is $\tilde{f}(x) \geq \tilde{u}$ or $\tilde{f}(x) + \tilde{u} > [k^{*-} - k^-, k^{*+} - k^+]$ and $\tilde{f}(y) \geq \tilde{u}$ or $\tilde{f}(y) + \tilde{u} > [k^{*-} - k^-, k^{*+} - k^+]$. Since $\tilde{f}$ is an $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy bi-ideal of S, by Theorem 3.4, $\tilde{f}(xy) \geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$.

Case (i). Let $\tilde{f}(x) \geq \tilde{u}$ and $\tilde{f}(y) \geq \tilde{u}$. If $\tilde{u} > [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then

$$\begin{aligned}\tilde{f}(xy) &\geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} \\ &\geq \min\{\tilde{u}, [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} = [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}],\end{aligned}$$

and thus $(xy)_{\tilde{u}}(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$. If $\tilde{u} \leq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then

$$\begin{aligned}\tilde{f}(xy) &\geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} \\ &\geq \min\{\tilde{u}, [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} = \tilde{u},\end{aligned}$$

and so $(xy)_{\tilde{u}} \in \tilde{f}$. Hence $(xy)_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$.

Case (ii). Let $\tilde{f}(x) \geq \tilde{u}$ and $\tilde{f}(x) + \tilde{u} \geq [k^{*-} - k^-, k^{*+} - k^+]$.
If $\tilde{u} > [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then

$$\begin{aligned}&\tilde{f}(xy) \geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} \\ &\geq \min\{([k^{*-} - k^-, k^{*+} - k^+] - \tilde{u}), [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} \\ &= [k^{*-} - k^-, k^{*+} - k^+] - \tilde{u},\end{aligned}$$

i.e., $f(\tilde{x}y) + \tilde{u} > [k^{*-} - k^-, k^{*+} - k^+]$ and thus $(xy)_{\tilde{u}}(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$. If $\tilde{u} \leq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then

$$\begin{aligned}&\tilde{f}(xy) \geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} \\ &\geq \min\{\tilde{u}, ([k^{*-} - k^-, k^{*+} - k^+] - \tilde{u}), [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} \\ &= \tilde{u}.\end{aligned}$$

So $(xy)_{\tilde{u}} \in \tilde{f}$. Hence $(xy)_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$.

Case (iii). Let $\tilde{f}(x)+\tilde{u} > [k^{*-}-k^{-}, k^{*+}-k^{+}]$ and $\tilde{f}(y) \geq \tilde{u}$. The proof in this case is similar to the proof of case (ii).

Case (iv). Let $\tilde{f}(x)+\tilde{u} > [k^{*-}-k^{-}, k^{*+}-k^{+}]$ and $\tilde{f}(y)+\tilde{u} > [k^{*-}-k^{-}, k^{*+}-k^{+}]$. The proof in this case is similar to the proof of cases (ii) and (iii).

Thus, in any case, we have $(xy)_{\tilde{u}} \in \vee(\tilde{k^{*}}, q_{\tilde{k}})\tilde{f}$, and so $xy \in [\tilde{f}]_{\tilde{u}}$. Furthermore, we may show that $(xyz)_{\tilde{u}} \in \vee(\tilde{k^{*}}, q_{\tilde{k}})\tilde{f}$. On the lines similar to the above proof. Hence $xyz \in [\tilde{f}]_{\tilde{u}}$.

Conversely, let $[\tilde{f}]_{\tilde{u}}$ is a bi-ideal of S for all $\tilde{u} \in D(0,1]$. If possible, let $\tilde{f}(x) < \min\{\tilde{f}(y), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\}$ for some $x, y \in S$ with $x \leq y$ and $y \in \tilde{f}_{\tilde{u}}$. Then there exists $\tilde{u} \in D(0, \frac{k^{*}-k}{2}]$ such that $\tilde{f}(x) < \tilde{u} \leq \min\{\tilde{f}(y), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\}$. Now $y \in [\tilde{f}]_{\tilde{u}}$, which implies that $x \in [\tilde{f}]_{\tilde{u}}$. Thus $\tilde{f}(x) \geq \tilde{u}$ or $\tilde{f}(x)+\tilde{u} > [k^{*-}-k^{-}, k^{*+}-k^{+}]$, which is not possible. So $\tilde{f}(x) \geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\}$.

Next we have to show that $\tilde{f}(xy) \geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\}$. Suppose to the contrary that $\tilde{f}(xy) < \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\}$ for some $x, y \in S$. Then there exists $\tilde{u} \in D(0, \frac{k^{*}-k}{2}]$ such that

$$\tilde{f}(xy) < \tilde{u} \leq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\}.$$

Thus we have $x, y \in [\tilde{f}]_{\tilde{u}}$, which implies that $xy \in [\tilde{f}]_{\tilde{u}}$. So $\tilde{f}(xy) \geq \tilde{u}$ or $\tilde{f}(xy)+\tilde{u} > [k^{*-}-k^{-}, k^{*+}-k^{+}]$. This is a contradiction. Hence

$$\tilde{f}(xy) \geq \min\{\tilde{f}(x), \tilde{f}(y), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\} \text{ for all } x, y \in S.$$

In a similar way, we may show that $\tilde{f}(xyz) \geq \min\{\tilde{f}(x), \tilde{f}(z), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\}$ for all $x, y, z \in S$. Therefore $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^{*}}, q_{\tilde{k}}))$-fuzzy bi-ideal of S. □

4. Interval valued $(\in, \in \vee(\tilde{k^{*}}, q_{\tilde{k}}))$-fuzzy quasi-ideals

In this section, we study mainly the interval valued $(\in, \in \vee(\tilde{k^{*}}, q_{\tilde{k}}))$–fuzzy quasi-ideals of ordered semigroups, and investigate its related properties.

Definition 4.1. An interval valued fuzzy subset $\tilde{f}$ of an ordered semigroup S is called an *interval valued* $(\in, \in \vee(\tilde{k^{*}}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S, if

(1) $x \leq y$, $y \in \tilde{f}$ imply $x \in \vee(\tilde{k^{*}}, q_{\tilde{k}})$,

(2) $x \leq yp$, $x \leq qz$ and $y_{\tilde{u}}, z_{\tilde{v}} \in \tilde{f}$ imply $x_{\min\{\tilde{u},\tilde{v}\}} \in \vee(\tilde{k^{*}}, q_{\tilde{k}})$,

for all $x, y, z, p, q \in S$ and $\tilde{u}, \tilde{v} \in D(0,1]$.

Theorem 4.2. *Let $\tilde{f}$ be an interval valued fuzzy subset of S. Then $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^{*}}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S if and only if*

(1) $x \leq y \Rightarrow \tilde{f}(x) \geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\}$,

(2) $\tilde{f}(x) \geq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\}$,

for all $x, y \in S$.

Proof. Suppose that $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^{*}}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S. Let $x, y \in S$ and $x \leq y$. If possible, let $\tilde{f}(x) < \min\{\tilde{f}(y), [\frac{k^{*-}-k^{-}}{2}, \frac{k^{*+}-k^{+}}{2}]\}$ for some $x, y \in S$. Then there exists $\tilde{u} \in D(0, \frac{k^{*}-k}{2}]$ such that

$$\tilde{f}(x) < \tilde{u} \leq \min\{\tilde{f}(y), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}.$$

This implies that $y_{\tilde{u}} \in \tilde{f}$ but $x_{\tilde{u}} \overline{\notin \vee(\tilde{k^*}, q_{\tilde{k}})}$. Thus

$$\tilde{f}(x) \geq \min\{\tilde{f}(y), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}.$$

Furthermore,

$$\tilde{f}(x) \geq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\} \text{ for all } x \in S.$$

Indeed, if

$$\tilde{f}(x) < \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\} \text{ for some } x \in S.$$

Then there exists $\tilde{u} \in D(0, \frac{k^*-k}{2}]$ such that

$$\tilde{f}(x) < \tilde{u} \leq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}.$$

If $\min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x)\} \leq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then

$$\tilde{f}(x) < \tilde{u} \leq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x)\}.$$

Since $(\tilde{f} \circ \tilde{S})(x) \geq \tilde{u} > [0,0]$, $(\tilde{S} \circ \tilde{f})(x) \geq \tilde{u} > [0,0]$, there exist $y, z, p, q \in S$ such that $x \leq yp$, $x \leq qz$,

$$\tilde{u} \leq \bigvee_{x \leq yp} \min\{\tilde{f}(y), \tilde{S}(p)\} = \tilde{f}(y)$$

and

$$\tilde{u} \leq \bigvee_{x \leq qz} \min\{\tilde{S}(q), \tilde{f}(z)\} = \tilde{f}(z).$$

Then $y_{\tilde{u}} \in \tilde{f}$ and $z_{\tilde{u}} \in \tilde{f}$ but $x_{\tilde{u}} \notin \tilde{f}$. This is a contradiction. Thus

$$\tilde{f}(x) \geq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x)\}.$$

Now, if $\min\{(\tilde{f} \circ \tilde{S})(x) and (\tilde{S} \circ \tilde{f})(x)\} > [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then

$$\tilde{f}(x) + \tilde{u} < [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}] + [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}] = [k^{*-} - k^-, k^{*+} - k^+].$$

This implies that $x_{\tilde{u}} \overline{(\tilde{k^*}, q_{\tilde{k}})}$, which is a contradiction. So

$$\tilde{f}(x) \geq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}.$$

Conversely, suppose that the conditions (1) and (2) hold. Let $x, y \in S$ and $\tilde{u} \in D(0,1]$ be such that $x \leq y$. If $y_{\tilde{u}} \in \tilde{f}$, then $\tilde{f}(y) \geq \tilde{u}$, and we have $\tilde{f}(x) \geq \min\{\tilde{u}, [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$. Now, if $\tilde{u} \leq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then $\tilde{f}(x) \geq \tilde{u}$. It implies that $x_{\tilde{u}} \in \tilde{f}$. If $\tilde{u} > [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then $\tilde{f}(x) \geq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$. So

$$\tilde{f}(x) + \tilde{u} > [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}] + [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}] = [k^{*-} - k^-, k^{*+} - k^+].$$

This implies that $x_{\tilde{u}}(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$. Hence $x_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$. Next suppose that $x, y, z, p, q \in S$ and $\tilde{u}, \tilde{v} \in D(0, 1]$ be such that $x \leq yp$ and $x \leq qz$ and $y_{\tilde{u}}, z_{\tilde{v}} \in \tilde{f}$. Then

$$\begin{aligned}\tilde{f}(x) &\geq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}\\ &= \min\left\{\bigvee_{(y,p)\in A_x} \min\{\tilde{f}(y), \tilde{S}(p)\}, \bigvee_{(q,z)\in A_x} \min\{\tilde{S}(q), \tilde{f}(z)\}, [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\right\}\\ &\geq \min\{\tilde{f}(y), \tilde{f(z)}, [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\} = \min\{\tilde{u}, \tilde{v}, [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}.\end{aligned}$$

Now if $\min\{\tilde{u}, \tilde{v}\} \leq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then $\tilde{f}(x) \geq \min\{\tilde{u}, \tilde{v}\}$. Therefore, $x_{\min\{\tilde{u},\tilde{v}\}} \in \tilde{f}$. If $\min\{\tilde{u}, \tilde{v}\} > [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then $\tilde{f}(x) \geq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$. Thus $\tilde{f}(x) + \min\{\tilde{u}, \tilde{v}\} > [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}] + [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}] = [k^{*-} - k^-, k^{*+} - k^+]$. Therefore, $x_{\min\{\tilde{u},\tilde{v}\}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$. Hence $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S.

In the above theorem, we characterize the interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal in terms of the multiplications $\tilde{f} \circ \tilde{S}$ and $\tilde{S} \circ \tilde{f}$. A natural question is if an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal $\tilde{f}$ can be characterize only in terms of interval valued fuzzy subset $\tilde{f}$ itself. The following theorems give the answer. □

Theorem 4.3. *Let $\tilde{f}$ be an interval valued fuzzy subset of S. Then $\tilde{f}$ is an $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S if and only if*

(1) $x \leq y \Rightarrow \tilde{f}(x) \geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$,

(2) $x \leq yp$ *and* $x \leq qz \Rightarrow \tilde{f}(x) \geq \min\{(\tilde{f}(y), \tilde{f}(z), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$,

for all $x, y, z, p, q \in S$.

Proof. Suppose $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S. The proof of first part of the theorem is similar as the proof of first part of Theorem 4.2. So we omit it. Next we have to show that condition (2) hold. By Theorem 4.2(2), we have $\tilde{f}(x) \geq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$. Let $x, y, z, p, q \in S$ and $\tilde{u}, \tilde{v} \in D(0, 1]$ be such that $x \leq yp$ and $x \leq qz$ and $y_{\tilde{u}}, z_{\tilde{v}} \in \tilde{f}$. Then

$$\begin{aligned}\tilde{f}(x) &\geq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}\\ &= \min\left\{\bigvee_{(y,p)\in A_x} \min\{\tilde{f}(y), \tilde{S}(p)\}, \bigvee_{(q,z)\in A_x} \min\{\tilde{S}(q), \tilde{f}(z)\}, [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\right\}\\ &\geq \min\{\tilde{f}(y), \tilde{f(z)}, [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}.\end{aligned}$$

Conversely, suppose that the conditions (1) and (2) hold. Let $x \in S$. Then

$$\tilde{f}(x) \geq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}.$$

Indeed, if $A_x = \varnothing$, then $\min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x)\} = [0, 0]$. Thus we have

$$\min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\} = [0, 0] \leq \tilde{f}(x).$$

Let $A_x \neq \varnothing$. Since $\tilde{f}$ is comparable, we consider the following cases.

Case (i). If $\tilde{f}(x) \geq \min\{(\tilde{f} \circ \tilde{S})(x), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$, then

$$\tilde{f}(x) \geq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}.$$

Case (ii). Let $\tilde{f}(x) < \min\{(\tilde{f} \circ \tilde{S})(x), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$. Then there exists $(y, p) \in A_x$ such that $(\tilde{f} \circ \tilde{S})(x) = \bigvee_{(y,p)\in A_x} \min\{\tilde{f}(y), \tilde{S}(p)\} = \tilde{f}(y)$. So, we have

$$\tilde{f}(x) < \min\{\tilde{f}(y), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}. \tag{A1}$$

So we can show that $\tilde{f}(x) \geq \min\{\tilde{S}(q), \tilde{f}(z), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$ for any $(q,z) \in A_x$. Since there exist $x, y, z, p, q \in S$ such that $x \leq yp$ and $x \leq qz$, by condition (2), we have $\tilde{f}(x) \geq \min\{\tilde{f}(y), \tilde{f}(z), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$. Since $\tilde{f}$ is comparable, we have $\min\{\tilde{f}(y), \tilde{f}(z)\} = \tilde{f}(y)$ or $f(\tilde{z})$. If $\min\{\tilde{f}(y), \tilde{f}(z)\} = \tilde{f}(y)$, then $\tilde{f}(x) \geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$, which is not possible by equation (A1). Hence $\tilde{f}(x) \geq \min\{\tilde{f}(z), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} = \min\{\tilde{S}(q), \tilde{f}(z), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$ for any $(q,z) \in A_x$, and we have

$$\begin{aligned}
\tilde{f}(x) &\geq \min\{\tilde{S}(q), \tilde{f}(z), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} \\
&= \min\{(\tilde{S} \circ \tilde{f})(x), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}] \\
&\geq \min\{(\tilde{f} \circ \tilde{S})(x), (\tilde{S} \circ \tilde{f})(x), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}.
\end{aligned}$$

Therefore, $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S by Theorem 4.2. □

Similar to the proof of Theorem 4.3, we have the following theorem.

Theorem 4.4. *Let $\tilde{f}$ be an interval valued fuzzy subset of S. Then $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S if and only if*

(1) $x \leq y \Rightarrow \tilde{f}(x) \geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$,

(2) $x \leq yp$ *and* $x \leq qz \Rightarrow \tilde{f}(x) \geq \max\{\min(\tilde{f}(y), \tilde{f}(z)), \min(\tilde{f}(p), \tilde{f}(q)), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$, *for all* $x, y, z, p, q \in S$.

Theorem 4.5. *Let S be an ordered semigroup. If $\tilde{k}_1, \tilde{k}_2 \in D(0,1)$ and $\tilde{k}_1 > \tilde{k}_2$, then each interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}_1}))$-fuzzy quasi-ideal of S is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}_2}))$-fuzzy quasi-ideal of S*

Proof. The proof is straightforward □

Theorem 4.6. *Let $\tilde{f}$ be an interval valued fuzzy subset of S. Then $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S if and only if the crisp set $\tilde{f}_{\tilde{u}}$ $(\neq \varnothing)$ of $\tilde{f}$ is a quasi-ideal of S for all $\tilde{u} \in D(0, \frac{k^*-k}{2}]$.*

Proof. It is clear by the Transfer Principle for Fuzzy Sets in [14]. □

Lemma 4.7. *Let A be a nonempty subset of S. Then A is a quasi-ideal of S if and only if the interval valued characteristic function $\tilde{f}_A$ of A is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S.*

Proof. Straightforward. □

Definition 4.8. An interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal $\tilde{f}$ of an ordered semigroup S is called prime if for all $x, y \in S$ and $\tilde{u} \in D(0,1]$,

$$(xy)_{\tilde{u}} \in \tilde{f} \Rightarrow x_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}})\tilde{f} \text{ or } \ y_{\tilde{u}} \in \vee(\tilde{k^*}, q_{\tilde{k}}).$$

Theorem 4.9. *Let $\tilde{f}$ be an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S. Then $\tilde{f}$ is prime if and only if* $\min\{\tilde{f}(x), \tilde{f}(y)\} \geq \min\{\tilde{f}(xy), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$ *for all $x, y \in S$.*

Proof. Let $\tilde{f}$ be a prime interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S and $x, y \in S$. Let $(xy)_{\tilde{u}} \in \tilde{f}$. If possible, let

$$\min\{\tilde{f}(x), \tilde{f}(y)\} < \min\{\tilde{f}(xy), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}$$

for some $x, y \in S$. Then there exists $\tilde{u} \in D(0, \frac{k^*-k}{2}]$ such that

$$\min\{\tilde{f}(x), \tilde{f}(y)\} < \tilde{u} \leq \min\{\tilde{f}(xy), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}.$$

Then $(xy)_{\tilde{u}} \in \tilde{f}$ but neither $x_{\tilde{u}} \overline{\in \vee(\tilde{k^*}, q_{\tilde{k}})} \tilde{f}$ nor $y_{\tilde{u}} \overline{\in \vee(\tilde{k^*}, q_{\tilde{k}})} \tilde{f}$, which is a contradiction. Thus, $\min\{\tilde{f}(x), \tilde{f}(y)\} \geq \min\{\tilde{f}(xy), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$.

Conversely, suppose that $\min\{\tilde{f}(x), \tilde{f}(y)\} \geq \min\{\tilde{f}(xy), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$ for all $x, y \in S$. Let $(xy)_{\tilde{u}} \in \tilde{f}$ and $\tilde{u} \in D(0,1]$. Then we have

$$\min\{\tilde{f}(x), \tilde{f}(y)\} \geq \min\{\tilde{u}, [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\}.$$

Now if $\tilde{u} \leq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then $\min\{\tilde{f}(x), \tilde{f}(y)\} \geq \tilde{u}$ implying that $x_{\tilde{u}} \in \tilde{f}$ or $y_{\tilde{u}} \in \tilde{f}$. If $\tilde{u} > [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, then $\min\{\tilde{f}(x), \tilde{f}(y)\} \geq [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]$, which implies that $x_{\tilde{u}}(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$ or $y_{\tilde{u}}(\tilde{k^*}, q_{\tilde{k}})\tilde{f}$. Therefore, $\tilde{f}$ is prime. $\square$

Theorem 4.10. *Let $\tilde{f}$ interval valued fuzzy subset of S. Then $\tilde{f}$ is a prime interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S if and only if the crisp set $\tilde{f}_{\tilde{u}}$ $(\neq \varnothing)$ of $\tilde{f}$ is a prime quasi-ideal of S for all $\tilde{u} \in D(0, \frac{k^*-k}{2}]$*

Proof. The proof is easy by Theorems 4.6 and 4.9, and thus we omit the details. $\square$

5. Characterizations of some types of ordered semigroups

In this section, we study the properties of $(\tilde{k^*}, \tilde{k})$-lower parts of $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideals of an ordered semigroup S. Also, we characterize the regular, intra-regular and bi-regular ordered semigroup in terms of interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideals of S.

Definition 5.1. Let $\tilde{f}$ be an interval valued fuzzy subset of S. Then

$$\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(x) = \min\left\{\tilde{f}(x), \left[\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}\right]\right\}$$

for all $x \in S$ and $0 \leq \tilde{k} < \tilde{k^*} \leq 1$, is called a $(\tilde{k^*}, \tilde{k})$-*lower part* of $\tilde{f}$.

Clearly, $\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}$ is a interval valued fuzzy subset of S.

For any two interval valued fuzzy subsets $\tilde{f}$ and $\tilde{g}$ of S, we define $\tilde{f}(\cap)_{\tilde{k}}^{\tilde{k^*}}\tilde{g}$ and $\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{g}$ as follows:

$$(\tilde{f}(\cap)_{\tilde{k}}^{\tilde{k^*}}\tilde{g})(x) = \min\left\{(\tilde{f} \cap \tilde{g})(x), \left[\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}\right]\right\},$$

$$(\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{g})(x) = \min\left\{(\tilde{f} \circ \tilde{g})(x), \left[\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}\right]\right\}$$

for all $x \in S$.

The following two lemmas can be easily obtained.

Lemma 5.2. *Let $\tilde{f}$ and $\tilde{g}$ be two interval valued fuzzy subsets of an ordered semigroup S. Then*

(1) $(\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}})_{\tilde{k}}^{\tilde{k^*}} = \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}$ *and* $\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}} \subseteq \tilde{f}$.

(2) *If* $\tilde{f} \subseteq \tilde{g}$ *and* $\tilde{h} \in IVF(S)$, *then* $\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{h} \subseteq \tilde{g}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{h}$, $\tilde{h}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f} \subseteq \tilde{h}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{g}$.

(3) $\tilde{f}(\cap)_{\tilde{k}}^{\tilde{k^*}}\tilde{g} = \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}} \cap \underline{\tilde{g}_{\tilde{k}}^{\tilde{k^*}}}$.

(4) $\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{g} = \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}} \circ \underline{\tilde{g}_{\tilde{k}}^{\tilde{k^*}}}$.

(5) $\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{S} = \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}} \circ \tilde{S}$ *and* $\tilde{S}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f} = \tilde{S} \circ \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}$.

Lemma 5.3. *Let A and B be any two nonempty subsets of an ordered semigroup S. Then*

(1) $\tilde{f}_A(\cap)_{\tilde{k}}^{\tilde{k^*}}\tilde{f}_B = (\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}})_{A\cap B}$.

(2) $\tilde{f}_A(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f}_B = (\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}})_{(AB]}$.

Theorem 5.4. *Let $\tilde{f}$ be an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S. Then the $(\tilde{k^*}, \tilde{k})$-lower part $\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}$ of $\tilde{f}$ is an interval valued fuzzy quasi-ideal of S.*

Proof. Let $\tilde{f}$ be an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S and $x, y \in S$ with $x \leq y$. By Theorem 4.2(1), we have $\tilde{f}(x) \geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$. Thus

$$\begin{aligned}
\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(x) &= \min\{\tilde{f}(x), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} \\
&\geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}], [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} \\
&= \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} = \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(y).
\end{aligned}$$

Let $x, y \in S$. By Theorem 4.2(2), we have

$$\tilde{f}(x) \geq \min\{(\tilde{f}\circ\tilde{S})(x), (\tilde{S}\circ\tilde{f})(x), [\tfrac{k^{*-}-k^-}{2}, \tfrac{k^{*+}-k^+}{2}]\},$$

and thus

$$\begin{aligned}
\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(x) &= \min\{\tilde{f}(x), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} \\
&\geq \min\{(\tilde{f}\circ\tilde{S})(x), (\tilde{S}\circ\tilde{f})(x), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}\} \\
&= \min\{(\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{S})(x), (\tilde{S}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f})(x)\} \\
&= \min\{(\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}\circ\tilde{S})(x), \tilde{S}\circ\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(x)\}.
\end{aligned}$$

Hence $\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}$ is an interval valued fuzzy quasi-ideal of S. □

Theorem 5.5. *Let $\tilde{f}$ be an interval valued fuzzy subset of S. Then $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S if and only if*

(1) $x \leq y \Rightarrow \tilde{f}(x) \geq \min\{\tilde{f}(y), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\}$.

(2) $\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{S} \cap \tilde{S}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f} \subseteq \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}$, *for all $x, y \in S$.*

Proof. The proof is obvious by Lemma 5.2 and Theorem 4.2. □

Definition 5.6 ([26]). An ordered semigroup S is called regular ordered semigroup if for each $a \in S$, there exists an element $x \in S$ such that $a \leq axa$, or equivalently $A \subseteq (ASA]$, $\forall A \subseteq S$.

Definition 5.7 ([26]). An ordered semigroup S is called intra-regular ordered semigroup if for each $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2y$, or equivalently $a \in (Sa^2S]$, $\forall a \in S$.

Definition 5.8 ([25]). An ordered semigroup S is called left (resp. right) regular if for each $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2x$), that is, $a \in (Sa^2]$ (resp. $a \in (a^2S]$). An ordered semigroup S is called a bi-regular if it is both left and right regular.

Clearly, an ordered semigroup S is bi-regular if and only if $a \in (Sa^2] \cap (a^2S]$ for each $a \in S$.

Lemma 5.9 ([27]). *An ordered semigroup S is regular and intra-regular ordered semigroup if and only if $A = (A^2]$ for each quasi-ideal A of S.*

Theorem 5.10. *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is regular and intra regular.*
(2) *$\tilde{f}(\cap)_{\tilde{k}}^{\tilde{k^*}}\tilde{g} \subseteq (\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{g}) \cap (\tilde{g}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f})$ for each interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal $\tilde{f}$ and $\tilde{g}$ of S.*
(3) *$\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f} = \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}$ for each interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal $\tilde{f}$ of S.*

Proof. (1) $\Rightarrow$ (2) Let $\tilde{f}$ and $\tilde{g}$ be two interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideals of S and $a \in S$. Since S is regular and intra-regular, there exists $x \in S$ such that $a \leq axa$ and there exist $y, z \in S$ such that $a \leq ya^2z$. Thus $a \leq axa \leq axaxa \leq ax(ya^2z)xa = (axya)(azxa)$, that is $(axya, azxa) \in A_a$. Since $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S and $axya \leq a(xya), axya \leq (axy)a$, by Theorem 4.3, we have

$$\begin{aligned}
\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(axya) &= \min\{\tilde{f}(axya), [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} \\
&\geq \min\{\tilde{f}(a), \tilde{f}(a), [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} \\
&= \min\{\tilde{f}(a), [\frac{k^{*-} - k^-}{2}, \frac{k^{*+} - k^+}{2}]\} = \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a).
\end{aligned}$$

In a similar way, we can get $\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(azxa) \geq \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a)$. Thus we have

$$\begin{aligned}
(\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{g})(a) = (\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}} \circ \underline{\tilde{g}_{\tilde{k}}^{\tilde{k^*}}})(a) = \bigvee_{(b,c) \in A_a} \min\{\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(b), \underline{\tilde{g}_{\tilde{k}}^{\tilde{k^*}}}(c)\} \\
\geq \min\{\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(axya), \underline{\tilde{g}_{\tilde{k}}^{\tilde{k^*}}}(azxa)\} \geq \min\{\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a), \underline{\tilde{g}_{\tilde{k}}^{\tilde{k^*}}}(a)\} = (\tilde{f}(\cap)_{\tilde{k}}^{\tilde{k^*}}\tilde{g})(a).
\end{aligned}$$

In a similar way, we can show that $\tilde{f}(\cap)_{\tilde{k}}^{\tilde{k^*}}\tilde{g} \subseteq \tilde{g}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f}$. Hence $\tilde{f}(\cap)_{\tilde{k}}^{\tilde{k^*}}\tilde{g} \subseteq (\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{g}) \cap (\tilde{g}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f})$.

(2) ⇒ (3) Take $\tilde{f} = \tilde{g}$, we get $\tilde{f}(\circ)^{\tilde{k^*}}_{\tilde{k}}\tilde{f} \supseteq \tilde{f}(\cap)^{\tilde{k^*}}_{\tilde{k}}\tilde{f} = \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}} \cap \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}} = \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}$. On the other hand, by Lemma 5.2(2) and Theorem 5.5, we have $\tilde{f}(\circ)^{\tilde{k^*}}_{\tilde{k}}\tilde{f} \subseteq (\tilde{f}(\circ)^{\tilde{k^*}}_{\tilde{k}}\tilde{S}) \cap (\tilde{S}(\circ)^{\tilde{k^*}}_{\tilde{k}}\tilde{f}) \subseteq \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}$. Thus $\tilde{f}(\circ)^{\tilde{k^*}}_{\tilde{k}}\tilde{f} = \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}$.

(3) ⇒ (1) Let A be a quasi-ideal of S. By Lemma 4.7, the interval valued characteristic function $\tilde{f}_A$ of A is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S. Then, by Lemma 5.3, we have $(\underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}})_{(A^2]} = \tilde{f}_A(\circ)^{\tilde{k^*}}_{\tilde{k}}\tilde{f}_A = (\underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}})_A$, which implies that $(A^2] = A$. By Lemma 5.9, S is regular and intra-regular ordered semigroup. □

Definition 5.11. Let S be an ordered semigroup and $\tilde{f}$ an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S. Then $\tilde{f}$ is called *completely semiprime* if $\underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a) \geq \tilde{f}(a^2)$ for any $a \in S$.

Theorem 5.12. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is bi-regular.*

(2) *$\underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a) = \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a^2)$ for each interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal $\tilde{f}$ of S.*

(3) *$\underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a^n) = \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a^{n+1})$ for each interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal $\tilde{f}$ of S, $n \in Z^+$ (set of all positive integers).*

(4) *Each interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal $\tilde{f}$ of S is completely semiprime.*

Proof. (1) ⇒ (2) Let S be a bi-regular ordered semigroup and $\tilde{f}$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S. Since S is bi-regular, $a \in S$ there exist $x, y \in S$ such that $a \leq a^2x$ and $a \leq ya^2$. Then, by Theorem 4.3, we have

$$\begin{aligned}
\underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a) &= \min\{\tilde{f}(a), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} \\
&\geq \min\{\tilde{f}(a^2), \tilde{f}(a^2), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} \\
&= \min\{\tilde{f}(a^2), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} = \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a^2) \\
&\geq \min\{(\tilde{f} \circ \tilde{S})(a^2), (\tilde{S} \circ \tilde{f})(a^2), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} \\
&= \min\{\tilde{f}(a), \tilde{f}(a), [\frac{k^{*-}-k^-}{2}, \frac{k^{*+}-k^+}{2}]\} = \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a).
\end{aligned}$$

Therefore $\underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a) = \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a^2)$.

(2) ⇒ (3) Let $a \in S$ and $n \in Z^+$. Then, by Lemma 5.2, we have

$$\begin{aligned}
(\tilde{f}(\circ)^{\tilde{k^*}}_{\tilde{k}}\tilde{S})(a^{4n}) &= (\underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}} \circ \tilde{S})(a^{4n}) = \bigvee_{(b,c)\in A_{a^{4n}}} \min\{\underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(b), \tilde{S}(y)\} \\
&\geq \min\{\underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a^{n+1}), \tilde{S}(a^{3n-1})\} = \underline{\tilde{f}^{\tilde{k^*}}_{\tilde{k}}}(a^{n+1}).
\end{aligned}$$

In a similar way, we can show that $(\tilde{S}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f})(a^{4n}) \geq \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^{n+1})$. Thus, by hypothesis and Theorem 5.5, we have

$$\begin{aligned}\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^n) &= \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^{2n}) = \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^{4n}) \\ &\geq \min\{(\tilde{f}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{S})(a^{4n}), (\tilde{S}(\circ)_{\tilde{k}}^{\tilde{k^*}}\tilde{f})(a^{4n})\} \\ &\geq \min\{\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^{n+1}), \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^{n+1})\} = \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^{n+1}).\end{aligned}$$

Similarly, we can show that $\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^{n+1}) \geq \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^n)$. Hence $\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^n) = \underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}}(a^{n+1})$.

(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) Let $a \in S$. We consider the quasi-ideal $A(a^2)$ of S generated by a^2. By Lemma 4.7, the interval valued characteristic function $\tilde{f}_{A(a^2)}$ of $A(a^2)$ is an interval valued $(\in, \in \vee(\tilde{k^*}, q_{\tilde{k}}))$-fuzzy quasi-ideal of S. So $(\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}})_{A(a^2)}(a) \geq \tilde{f}_{A(a^2)}(a^2) = [1, 1]$. But $(\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}})_{A(a^2)}(a) \leq [1, 1]$ for all $a \in S$. Thus $(\underline{\tilde{f}_{\tilde{k}}^{\tilde{k^*}}})_{A(a^2)}(a) = [1, 1]$, and we obtain that $a \in A(a^2) = (a^2 \cup ((a^2S] \cap (Sa^2])]$. Then $a \leq b$ for some $b \in (a^2 \cup ((a^2S] \cap (Sa^2])]$. If $b = a^2$, then $a \leq a^2 = aa \leq a^2aa^2 \in a^2Sa^2 \subseteq (a^2S] \cap (Sa^2]$, that is, $a \in (a^2S] \cap (Sa^2]$. If $b \in (a^2S] \cap (Sa^2]$, then $a \in (a^2S] \cap (Sa^2]$. Hence, S is bi-regular. $\square$

Acknowledgments

This work was supported by the University Natural Science Project of Anhui Province (No. KJ2015A161, 2014KJ004, 2015KJ003, 2015KJ016).

References

[1] S. K. Bhakat, $(\in \vee q)$-level subset, Fuzzy Sets and Systems 103 (1999) 529–533.
[2] S. K. Bhakat and P. Das, On the definition of a fuzzy subgroup, Fuzzy Sets and Systems 51 (2) (1992) 235–241.
[3] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy subgroup, Fuzzy Sets and Systems 80 (3) (1996) 359–368.
[4] R. Biswas, Rosenfeld's fuzzy subgroups with interval valued membership functions, Fuzzy Sets and Systems 63 (1994) 87–90.
[5] B. Davvaz and A. Khan, Characterization of regular ordered semigroups in terms of (α, β)-fuzzy generalized bi-ideals, Inform. Sci. 181 (2011) 1759–1770.
[6] Y. B. Jun, A. Khan and M. Shabir, Ordered semigroups characterized by their $(\in, \in \vee q)$-fuzzy bi-ideals, Bull. Malays. Math. Sci. Soc. 32 (3) (2009) 391–408.
[7] S. Kar, K. P. Shum and P. Sarkar, Interval-valued prime fuzzy ideals of semigroups, Lobachevskii Journal of Mathematics 34 (1) (2013) 11–19.
[8] O. Kazanci and S. Yamak, Generalized fuzzy bi-ideals of semigroup, Soft Computing 12 (2008) 1119–1124.
[9] N. Kehayopulu and M. Tsingelis, Fuzzy bi-ideals in ordered semigroups, Inform. Sci. 171 (2005) 13–28.
[10] N. Kehayopulu and M. Tsingelis, The embedding of an ordered groupoid into a poe-groupoid in terms of fuzzy sets, Inform. Sci. 152 (2003) 231–236.
[11] N. Kehayopulu and M. Tsingelis, Fuzzy right, left, quasi-ideals, bi-ideals in ordered semigroups, Lobachevskii J. Math. 30 (2009) 17–22.
[12] A. Khan, Y. B. Jun, N. H. Sarmin and F. M. Khan, Ordered semigroups characterized by $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals, Neural Computing and Applications 21 (2012) 121–132.
[13] A. Khan, Y. B. Jun, N. H. Sarmin and H. U. Khan, Interval valued fuzzy generalized bi-ideals of ordered semigroups redefined, World Applied Sciences Journal 27 (12) (2013) 1737–1751.
[14] M. Kondo and W. A. Dudek, On the transfer principle in fuzzy theory, Mathware & Soft Computing, 12 (2005) 41–55.

[15] N. Kuroki, On fuzzy semigroups, Inform. Sci. 53 (1991) 203–236.
[16] X. Ma and J. Zhan, On fuzzy h-ideals of hemirings, J. Syst. Sci. Complex. 20 (2007) 470–478.
[17] G. Mohanraj, D. Krishnaswamy and R. Hema, On fuzzy m-systems and n-systems of ordered semigroup, Ann. Fuzzy Math. Inform. 7 (1) (2014) 173–179.
[18] J. N. Mordeson, D. S. Malik and N. Kuroki, Fuzzy Semigroups, Studies in Fuzziness and Soft Computing, vol. 131, Springer-Verlag, Berlin 2003.
[19] V. Murali, Fuzzy points of equivalent subsets, Inform. Sci. 158 (2004) 277-288.
[20] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[21] M. Shabir and I. A. Khan, Interval valued fuzzy ideals generated by an interval valued fuzzy subset in ordered semigroups, Mathware and Soft Computing 15 (3) (2008) 263–272.
[22] M. Shabir and A. Khan, Fuzzy quasi-ideal in ordered semigroups, Bull. Malays. Math. Sci. Soc. 34 (1) (2011) 87–102.
[23] J. Tang and Y. Luo, A new type of generalized fuzzy quasi-ideals of ordered semigroups, Ann. Fuzzy Math. Inform. 8 (6) (2014) 941–963.
[24] J. Tang and X. Y. Xie, Characterizations of regular ordered semigroups by generalized fuzzy ideals, J. Intell. Fuzzy Systems 26 (1) (2014) 239–252.
[25] J. Tang, X. Y. Xie and Y. Luo, Characterizations of ordered semigroups by new type of Interval valued fuzzy quasi ideals, Journal of Applied Mathematics 2014 (2014) Article ID 867459, 14 pages.
[26] X. Y. Xie and J. Tang, Regular ordered semigroups and intra-regular ordered semigroups in terms of fuzzy subsets, Iran. J. Fuzzy Syst. 7 (2) (2010) 121–140.
[27] X. Y. Xie, An introduction to ordered semigroup theory, Kexue Press, Beijing 2001.
[28] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.
[29] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Information and Control 18 (1975) 199–249.
[30] J. Zhan and Y. B. Jun, Generalized fuzzy interior ideals of semigroups. Neural Computing & Applications 19 (2010) 515–519.

Jian Tang (tangjian0901@126.com)
School of Mathematics and Statistics, Fuyang Normal University, Fuyang, Anhui, 236037, P. R. China

Noor Mohammad Khan (nm_khan123@yahoo.co.in)
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

Mohammad Aasim Khan (khan.aasim09@gmail.com)
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India