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Characterization of regular semigroup through interval valued fuzzy ideals

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Abstract. In this paper, we characterize regular semigroup in terms of its left, right and quasi-ideals as well as in terms of their interval valued fuzzy(in sort, i-v fuzzy) left and right ideals. For a regular semigroup the relationship between its i-v fuzzy subset, i-v fuzzy ideal, and i-v fuzzy interior ideal is established. A regular semigroup is also characterized using its bi-ideal, i-v fuzzy bi-ideal and i-v fuzzy quasi-ideals. Idempotent condition of i-v fuzzy subset is utilized in characterizing a regular semigroup in terms of its i-v fuzzy bi-ideal, i-v fuzzy quasi-ideal, i-v fuzzy interior ideal and generalized i-v fuzzy bi-ideal. Equivalent conditions on regular semigroup are also established using the idempotent property of these ideals.

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1. Introduction

Zadeh[12] introduced the fundamental concepts of fuzzy set in 1965. In 1975[13], the same author introduced a new type of fuzzy subset viz. interval valued fuzzy subset (in short, i-v fuzzy subset) where the values of the membership function are closed intervals, instead of a single number. Fuzzy semigroup has been first considered by Kurki[8]. In 1971, Rosenfeld[9] defined fuzzy subgroup and gave some of its properties. In [1], Biswas defined interval valued fuzzy subgroup and investigated some elementary properties. Subsequently, Jun and Kim [5] and Davvaz [2] applied a few concepts of i-v fuzzy subsets in near-rings. Kar et al.[6], introduced the concept of i-v fuzzy quasi-ideals and bi-ideals of semirings. Iseki[4], discussed the characterizations of regular semigroups. In [10] initiated the concept of quasi-ideals in rings and semigroups. In [7], the idea of i-v fuzzy completely, regular subsemigroups of semigroups. Dutta et al.[3], introduced the concept of i-v fuzzy semisimple ideals of a semiring. Thillaigovindan and Chinnadurai[11] introduced the notions of i-v fuzzy ideal (bi-ideal, quasi-ideal, interior-ideal, generalized bi-ideal) of semigroup. In this paper, we characterize regular semigroup in terms of i-v fuzzy ideal, i-v fuzzy interior ideal, i-v fuzzy quasi-ideal, i-v fuzzy bi-ideal and generalized i-v fuzzy bi-ideal. Finally, we characterize regular semigroup through its i-v fuzzy subsets and obtain equivalent conditions on regular semigroup through these subsets.

2. Preliminaries

In this section, we recall some definitions and results that are necessary for the development of the concepts studied in this paper.

Definition 2.1. Let S be a semigroup. Let A and B be subsets of S, the product of A and B is defined as $AB = \{\, ab \in S \mid a \in A \text{ and } b \in B\,\}$. A nonempty subset A of S is called a subsemigroup of S if $AA \subseteq A$. A nonempty subset A of S is called a left (resp. right)ideal of S if $SA \subseteq A$ (resp.$AS \subseteq A$). A is called a two-sided ideal (simply, ideal) of S if it is both a left and a right ideal of S. A nonempty subset A of S is called an interior ideal of S if $SAS \subseteq A$, and a quasi-ideal of S if $AS \cap SA \subseteq A$. A subsemigroup A of S is called a bi-ideal of S if $ASA \subseteq A$. A semigroup S is called regular if for each element $a \in S$ there exists $x \in S$ such that $a = axa$. A function μ from a nonempty set A into the unit interval $[\,0,\ 1\,]$ is called a uzzy subset of A.

The following definitions and results quoted from [8] are essential for the development of the subject matter of this paper.

Definition 2.2 ([11])**.** An interval number $\overline{a}$ on $[0, 1]$ is a closed subinterval of $[\,0,\ 1\,]$, that is, $\overline{a} = [a^-,\ a^+]$ such that $0 \le a^- \le a^+ \le 1$ where a^- and a^+ are the lower and upper end points of $\overline{a}$ respectively.

In this notation $\overline{0} = [0,\ 0]$ and $\overline{1} = [1,\ 1]$. For any two interval numbers $\overline{a} = [a^-,\ a^+]$ and $\overline{b} = [b^-,\ b^+]$ on $[\,0,\ 1\,]$, define (1) $\overline{a} \le \overline{b}$ if and only if $a^- \le b^-$ and $a^+ \le b^+$, (2) $\overline{a} = \overline{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$.

Definition 2.3 ([11])**.** Let X be any set. A mapping $\overline{A} : X \to D[0,\ 1]$ is called an interval-valued fuzzy subset (briefly, i-v fuzzy subset) of X, where $D[0,\ 1]$ denotes the family of all closed subintervals of $[0, 1]$ and $\overline{A}(x) = [A^-(x),\ A^+(x)]$ for all $x \in X$, where A^- and A^+ are fuzzy subsets of X such that $A^-(x) \le A^+(x)$ for all $x \in X$.

Thus $\overline{A}(x)$ is an interval (a closed subset of $[0,\ 1]$) and not a number from the interval $[0,\ 1]$ as in the case of fuzzy subset.

Definition 2.4 ([11])**.** A mapping $min^i : D[0,\ 1] \times D[0,\ 1] \to D[0,\ 1]$ defined by $min^i(\overline{a}, \overline{b}) = [min\{a^-,\ b^-\}, min\{a^+,\ b^+\}]$ for all $\overline{a}, \overline{b} \in D[0,\ 1]$ is called an interval min-norm. A mapping $max^i : D[0,\ 1] \times D[0,\ 1] \to D[0,\ 1]$ defined by $max^i(\overline{a}, \overline{b}) = [max\{a^-,\ b^-\}, max\{a^+,\ b^+\}]$ for all $\overline{a},\ \overline{b} \in D[0,\ 1]$ is called an interval max-norm.

Definition 2.5 ([11])**.** Let $\overline{A}$, $\overline{B}$, $\overline{A}_j$ $(j \in \Omega)$ be interval valued fuzzy subsets of X. The following are defined by

(1) $\overline{A} \le \overline{B}$ if and only if $\overline{A}(x) \le \overline{B}(x)$,
(2) $\overline{A} = \overline{B}$ if and only if $\overline{A}(x) = \overline{B}(x)$,
(3) $(\overline{A} \cup \overline{B})(x) = max^i\{\overline{A}(x), \overline{B}(x)\}$,

(4) $(\overline{A}\cap\overline{B})(x)=min^i\{\overline{A}(x),\overline{B}(x)\}$,
(5) $(\bigcap_{j\in\Omega}\overline{A}_j)(x)=\inf^i\{\overline{A}_j(x)\,|\,j\in\Omega\}$,
(6) $(\bigcup_{j\in\Omega}\overline{A}_j)(x)=\sup^i\{\overline{A}_j(x)\,|\,j\in\Omega\}$,

where $\inf^i\{\overline{A}_j(x)\,|\,j\in\Omega\}=[\inf_{j\in\Omega}\{A_j^-(x)\},\ \inf_{j\in\Omega}\{A_j^+(x)\}]$ is the interval valued infimum norm and $\sup^i\{\overline{A}_j(x)\,|\,j\in\Omega\}=[\sup_{j\in\Omega}\{A_j^-(x)\},\ \sup_{j\in\Omega}\{A_j^+(x)\}]$ is the interval valued supremum norm.

Definition 2.6 ([11])**.** Let '.'be a binary composition in a set S. The *product* $\overline{A}\circ\overline{B}$ of any two i-v fuzzy subsets $\overline{A}$, $\overline{B}$ of S is defined by

$$(\overline{A}\circ\overline{B})(x)=\begin{cases}\sup\limits_{x=a.b}{}^i\{min^i\{\overline{A}(a),\ \overline{B}(b)\}\}, & \text{if } x \text{ can be expressed as } x=a.b\\ \overline{0} & \text{otherwise.}\end{cases}$$

Since semigroup S is associative, the operation $\circ$ is associative. We denote xy instead of $x.y$ and $\overline{A}\,\overline{B}$ for $\overline{A}\circ\overline{B}$.

Definition 2.7 ([11])**.** Let I be a subset of a semigroup S. Define a function $\overline{\chi}_I:\ S\to D[0,\,1]$ by

$$\overline{\chi}_I(x)=\begin{cases}\overline{1} & \text{if } x\in I,\\ \overline{0} & \text{otherwise,}\end{cases}$$

for all $x\in S$. Clearly $\overline{\chi}_I$ is an i-v fuzzy subset of S. Throughout this paper $\overline{\chi}_S$ is denoted by $\overline{S}$ and S will denote a semigroup unless otherwise mentioned.

Definition 2.8 ([11])**.** An i-v fuzzy subset $\overline{\lambda}$ of S is called an i-v fuzzy subsemigroup of S if $\overline{\lambda}(ab)\geq min^i\{\overline{\lambda}(a),\ \overline{\lambda}(b)\}$ for all $a,\ b\in S$.

Definition 2.9 ([11])**.** An i-v fuzzy subset $\overline{\lambda}$ of S is called an i-v fuzzy left (resp. right) ideal of S if $\overline{\lambda}(ab)\geq\overline{\lambda}(b)$(resp. $\overline{\lambda}(ab)\geq\overline{\lambda}(a)$) for all $a,\ b\in S$. An i-v fuzzy subset $\overline{\lambda}$ of S is called an i-v fuzzy two-sided ideal (simply, i-v fuzzy ideal) of S if it is both an i-v fuzzy left ideal and an i-v fuzzy right ideal of S.

Example 2.10. Let $S=\{\,a,\,b,\,c,\,d\,\}$ be a semigroup with the multiplication table given below:

$\bullet$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $\overline{\lambda}$ be an i-v fuzzy subset of S such that $\overline{\lambda}(a)=[0.7,\,0.8]$, $\overline{\lambda}(c)=[0.4,\,0.5]\overline{\lambda}(b)=\overline{\lambda}(d)$. Then $\overline{\lambda}$ is an i-v fuzzy ideal of S.

Definition 2.11 ([11])**.** An i-v fuzzy subsemigroup $\overline{\lambda}$ of S is called an i-v fuzzy bi-ideal of S if $\overline{\lambda}(xyz)\geq min^i\{\overline{\lambda}(x),\ \overline{\lambda}(z)\}$ for all $x,\ y,\ z\in S$.

Definition 2.12 ([11])**.** An i-v fuzzy subset $\overline{\lambda}$ of S is called an i-v fuzzy interior ideal of S if $\overline{\lambda}(xay)\geq\overline{\lambda}(a)$ for all $x,\ a,\ y\in S$.

Definition 2.13 ([11]). An i-v fuzzy subsemigroup $\overline{\lambda}$ of S is called an i-v fuzzy quasi-ideal of S if $(\overline{\lambda}\ \overline{S}) \cap (\overline{S}\ \overline{\lambda}) \leq \overline{\lambda}$.

Lemma 2.14 ([11]). *Let A and B be nonempty subsets of S. Then the following properties hold:*
(1) $\overline{\chi}_A \cap \overline{\chi}_B = \overline{\chi}_{A\cap B}$.
(2) $\overline{\chi}_A \overline{\chi}_B = \overline{\chi}_{AB}$.

Lemma 2.15 ([11]). *Every i-v fuzzy ideal of S is an i-v fuzzy interior ideal of S.*

Lemma 2.16 ([11]). *Every i-v fuzzy quasi-ideal of S is an i-v fuzzy bi-ideal of S.*

Proposition 2.17 ([11]). *Let A be a nonempty subset of S. A is a bi-ideal of S if and only if $\overline{\chi}_A$ is an i-v fuzzy bi-ideal of S.*

Lemma 2.18 ([11]). *Let $\overline{\lambda}$, $\overline{\mu}$ and $\overline{\nu}$ be i-v fuzzy subsets of S. If $\overline{\lambda} \leq \overline{\mu}$, then $\overline{\lambda}\ \overline{\nu} \leq \overline{\mu}\ \overline{\nu}$ and $\overline{\nu}\ \overline{\lambda} \leq \overline{\nu}\ \overline{\mu}$.*

Proposition 2.19 ([11]). *Let $\overline{\lambda}$ be an i-v fuzzy subset of S. $\overline{\lambda}$ is an i-v fuzzy left ideal (resp. subsemigroup, right ideal) of S, if and only if $\overline{S}\ \overline{\lambda} \leq \overline{\lambda}$ (resp. $\overline{\lambda}\ \overline{\lambda} \leq \overline{\lambda}$, $\overline{\lambda}\ \overline{S} \leq \overline{\lambda}$).*

3. Interval valued fuzzy ideals in regular semigroup

In this section, we characterize regular semigroup in terms of i-v fuzzy ideal, i-v fuzzy interior ideal, i-v fuzzy quasi-ideal, i-v fuzzy bi-ideal and generalized i-v fuzzy bi-ideal.

Definition 3.1. An i-v fuzzy subset $\overline{\lambda}$ of S is called a generalized i-v fuzzy bi-ideal of S, if $\overline{\lambda}(xyz) \geq min^i\{\overline{\lambda}(x), \overline{\lambda}(z)\}$ for all $x, y, z \in S$.

The following theorem gives a characterization of a regular semigroup in terms of its right, left and quasi-ideals.

Theorem 3.2. *For a semigroup S, the following conditions are equivalent :*
(1) *S is regular.*
(2) *$R \cap L = RL$ for every right ideal R of S and every left ideal L of S.*
(3) *$Q = QSQ$ for every quasi-ideal Q of S.*

Proof. The equivalence of (1) and (2) is due to Iseki([4], Theorem 1) and the equivalence of (1) and (3) follows from Steinfeld([10], p.10). □

The next theorem characterizes a regular semigroup through its i-v fuzzy right and left ideals.

Theorem 3.3. *The following conditions are equivalent:*
(1) *S is regular.* (2) *$\overline{\lambda}\ \overline{\mu} = \overline{\lambda} \cap \overline{\mu}$ for every i-v fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy left ideal $\overline{\mu}$ of S.*

Proof. (1) $\Rightarrow$ (2). Let $\overline{\lambda}$ and $\overline{\mu}$ be any i-v fuzzy right ideal and i-v fuzzy left ideal of a regular semigroup S respectively. Let $x \in S$. Then

$$\begin{aligned}(\overline{\lambda}\,\overline{\mu})(x) &= \sup_{x=ab}{}^{i}\{min^{i}\{\overline{\lambda}(a),\,\overline{\mu}(b)\}\}\\ &\leq \sup_{x=ab}\{min^{i}\{\overline{\lambda}(x),\overline{\mu}(x)\}\}\\ &= \sup_{x=ab}\{(\overline{\lambda}\cap\overline{\mu})(x)\}\end{aligned}$$

and so $\overline{\lambda}\,\overline{\mu} \leq \overline{\lambda}\cap\overline{\mu}$. Again let $a \in S$. Then since S is regular, there exists $x \in S$ such that $a = axa$. Now,

$$\begin{aligned}(\overline{\lambda}\,\overline{\mu})(a) &= \sup_{a=pq}{}^{i}\{min^{i}\{\overline{\lambda}(p),\,\overline{\mu}(q)\}\}\\ &\geq min^{i}\{\overline{\lambda}(a),\,\overline{\mu}(xa)\}\\ &\geq min^{i}\{\overline{\lambda}(a),\,\overline{\mu}(a)\}\\ &= (\overline{\lambda}\cap\overline{\mu})(a)\end{aligned}$$

and hence $\overline{\lambda}\,\overline{\mu} \geq \overline{\lambda}\cap\overline{\mu}$. Thus $\overline{\lambda}\cap\overline{\mu} = \overline{\lambda}\,\overline{\mu}$.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let R and L be any right ideal and left ideal of S respectively. Since $\overline{\chi}_R$ and $\overline{\chi}_L$ are respectively i-v fuzzy right ideal and i-v fuzzy left ideal, $\overline{\chi}_R \cap \overline{\chi}_L = \overline{\chi}_R\,\overline{\chi}_L$. By Lemma 2.13, $\overline{\chi}_{R\cap L} = \overline{\chi}_{RL}$. Thus $RL = R \cap L$ and hence by Theorem 3.2, S is regular. □

Now, we characterize regular semigroup using i-v fuzzy ideal and i-v fuzzy interior ideal.

Theorem 3.4. *Let $\overline{\lambda}$ be an i-v fuzzy subset of a regular semigroup S. Then the following conditions are equivalent :*
(1) *$\overline{\lambda}$ is an i-v fuzzy ideal of S.*
(2) *$\overline{\lambda}$ is an i-v fuzzy interior ideal of S.*

Proof. (1) $\Rightarrow$ (2) follows by Lemma 2.15.
(2) $\Rightarrow$ (1). Assume that (2) holds. Let $a, b \in S$. Then, since S is regular there exist elements $x, y \in S$ such that $a = axa$ and $b = byb$. Thus we have, $\overline{\lambda}(ab) = \overline{\lambda}((axa)b) = \overline{\lambda}((ax)ab) \geq \overline{\lambda}(a)$ and $\overline{\lambda}(ab) = \overline{\lambda}(a(byb)) = \overline{\lambda}(ab(yb)) \geq \overline{\lambda}(b)$. This implies that $\overline{\lambda}$ is an i-v fuzzy ideal of S. □

The following theorem characterizes regular semigroup in terms of its i-v fuzzy bi-ideal and i-v fuzzy quasi-ideal.

Theorem 3.5. *For a semigroup S, the following conditions are equivalent :*
(1) *S is regular.*
(2) *$\overline{\lambda} = \overline{\lambda}\,\overline{S}\,\overline{\lambda}$ for every i-v fuzzy bi-ideal $\overline{\lambda}$ of S.*
(3) *$\overline{\lambda} = \overline{\lambda}\,\overline{S}\,\overline{\lambda}$ for every i-v fuzzy quasi-ideal $\overline{\lambda}$ of S.*

Proof. (1) $\Rightarrow$ (2). Let $\overline{\lambda}$ be any i-v fuzzy bi-ideal of S and $a \in S$. Since S is regular, there exists $x \in S$ such that $a = axa$. Then

$$\begin{aligned}(\overline{\lambda}\,\overline{S}\,\overline{\lambda})(a) &= \sup_{a=yz}{}^{i}\{min^{i}\{\overline{\lambda}(y),\,(\overline{S}\,\overline{\lambda})(z)\}\}\\ &\geq min^{i}\{\overline{\lambda}(a),\ \sup_{xa=pq}{}^{i}\{min^{i}\{\overline{S}(p),\,\overline{\lambda}(q)\}\}\}\\ &\geq min^{i}\{\overline{\lambda}(a),\overline{\lambda}(a)\}\\ &= \overline{\lambda}(a)\end{aligned}$$

and hence $\overline{\lambda}\ \overline{S}\ \overline{\lambda} \geq \overline{\lambda}$. Since $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S, $\overline{\lambda}\ \overline{S}\ \overline{\lambda} \leq \overline{\lambda}$. Thus $\overline{\lambda}\,\overline{S}\,\overline{\lambda} = \overline{\lambda}$ and (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). By Lemma 2.15, any i-v fuzzy quasi-ideal of S is a fuzzy bi-ideal of S and hence (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Let Q be any quasi-ideal in S and let $x \in Q$. Then $\overline{\chi}_Q$ is an i-v fuzzy quasi-ideal of S and

$$\begin{aligned}\overline{\chi}_{QSQ}(x) &= (\overline{\chi}_Q\ \overline{\chi}_S\ \overline{\chi}_Q)(x)\\ &= \overline{\chi}_Q(x) = \overline{1}.\end{aligned}$$

This implies that $x \in QSQ$. Thus $Q \subseteq QSQ$. On the other hand, since Q is a quasi-ideal of S,

$$QS \cap SQ \subseteq Q,\ QSQ \subseteq QSS \subseteq QS$$

and

$$QSQ \subseteq SSQ \subseteq SQ.$$

Thus $QSQ \subseteq QS \cap SQ \subseteq Q$. So $QSQ = Q$. Hence, by Theorem 3.2, S is regular. □

The next result is a characterization of regular semigroup in terms of its bi-ideal and i-v fuzzy bi-ideal.

Theorem 3.6. *For a regular semigroup S, the following conditions are equivalent :*
(1) *Every bi-ideal of S is a right (left, two-sided) ideal of S.*
(2) *Every i-v fuzzy bi-ideal of S is an i-v fuzzy right (left, two-sided) ideal of S.*

Proof. (1) $\Rightarrow$ (2). Let $\overline{\lambda}$ be any i-v fuzzy bi-ideal of S and $a, b \in S$. Since the set aSa is a bi-ideal of S, by the assumption it is a right ideal of S. Since S is regular, $ab \in (aSa)S \subseteq aSa$. Thus, there exists an element $x \in S$ such that $ab = axa$. Since $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S, $\overline{\lambda}(ab) = \overline{\lambda}(axa) \geq min^i\{\overline{\lambda}(a), \overline{\lambda}(a)\} = \overline{\lambda}(a)$. Thus $\overline{\lambda}$ is an i-v fuzzy right ideal of S and hence (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Let A be any bi-ideal of S. By Proposition 2.16, $\overline{\chi}_A$ is an i-v fuzzy bi-ideal of S. Hence by assumption $\overline{\chi}_A$ is an i-v fuzzy right ideal of S. Thus by Lemma 2.17, A is a right ideal of S. Hence (2) $\Rightarrow$ (1). □

Now, we characterize regular semigroup using its i-v fuzzy ideal, i-v fuzzy bi-ideal and i-v fuzzy quasi-ideal.

Theorem 3.7. *For any semigroup S, the following conditions are equivalent :*
(1) *S is regular.*
(2) *$\overline{\lambda} \cap \overline{\mu} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy ideal $\overline{\lambda}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*
(3) *$\overline{\lambda} \cap \overline{\mu} = \overline{\mu}\ \overline{\lambda}\ \overline{\mu}$ for every i-v fuzzy ideal $\overline{\lambda}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

Proof. (1) $\Rightarrow$ (2). Let $\overline{\lambda}$ and $\overline{\mu}$ be an i-v fuzzy ideal and an i-v fuzzy bi-ideal of S respectively. Then $\overline{\mu}\ \overline{\lambda}\ \overline{\mu} \leq \overline{\mu}\ \overline{S}\ \overline{\mu} \leq \overline{\mu}$. Again $\overline{\mu}\ \overline{\lambda}\ \overline{\mu} \leq \overline{S}\ \overline{\lambda}\ \overline{S} \leq \overline{\lambda}$. Thus $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} \leq \overline{\lambda} \cap \overline{\mu}$.

On the other hand, let $a \in S$. Since S is regular, there exists $x \in S$ such that $a = axa = axaxa$. Since $\overline{\lambda}$ is an i-v fuzzy ideal of S, $\overline{\lambda}(xax) \geq \overline{\lambda}(a)$. Then we have,

$$
\begin{aligned}
(\overline{\mu}\,\overline{\lambda}\,\overline{\mu})(a) &= \sup_{a=yz}{}^{i}\{min^{i}\{\overline{\mu}(y),\,(\overline{\lambda}\,\overline{\mu})(z)\}\} \\
&\geq min^{i}\{\overline{\mu}(a),\,(\overline{\lambda}\,\overline{\mu})(xaxa)\} \\
&= min^{i}\{\overline{\mu}(a),\,\sup_{xaxa=pq}{}^{i}\{min^{i}\{\overline{\lambda}(p),\,\overline{\mu}(q)\}\}\} \\
&\geq min^{i}\{\overline{\mu}(a), min^{i}\{\overline{\lambda}(xax), \overline{\mu}(a)\}\} \\
&\geq min^{i}\{\overline{\mu}(a), min^{i}\{\overline{\lambda}(a),\,\overline{\mu}(a)\}\} \\
&= min^{i}\{\overline{\mu}(a),\,\overline{\lambda}(a)\} \\
&= (\overline{\mu}\cap\overline{\lambda})(a)
\end{aligned}
$$

and so $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} \geq \overline{\mu}\cap\overline{\lambda}$ and hence $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} = \overline{\mu}\cap\overline{\lambda}$. Thus (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Let (3) hold. Let $\overline{\mu}$ be any i-v fuzzy quasi-ideal of S. $\overline{S}$ itself being an i-v fuzzy bi-ideal of S, $\overline{\mu} = \overline{\mu}\cap\overline{S} = \overline{\mu}\,\overline{S}\,\overline{\mu}$. Thus, by Theorem 3.5, S is regular and so (3) $\Rightarrow$ (1). □

Definition 3.8. An i-v fuzzy subset $\overline{\lambda}$ of a semigroup S is said to have *idempotent* property if $(\overline{\lambda}\,\overline{\lambda})(x) = \overline{\lambda}(x)$ for all $x \in S$.

Theorem 3.9. *Every i-v fuzzy ideal of a regular semigroup is idempotent.*

Proof. Let $\overline{\mu}$ be an i-v fuzzy ideal of a regular semigroup S. Then, by Lemma 2.14, $\overline{\mu}\,\overline{S}\,\overline{\mu} \leq \overline{\mu}$. Hence $\overline{\mu}$ is an i-v fuzzy bi-ideal of S. Since S is regular, by Theorem 3.5, we have $\overline{\mu} = \overline{\mu}\,\overline{S}\,\overline{\mu} \leq \overline{\mu}\,\overline{\mu} \leq \overline{\mu}\,\overline{S} \leq \overline{\mu}$ and so $\overline{\mu} = \overline{\mu}\,\overline{\mu}$. Thus $\overline{\mu}$ is idempotent. □

Next, we develop equivalent conditions on regular semigroup using its fuzzy ideal, i-v fuzzy ideal, i-v fuzzy bi-ideal, i-v fuzzy interior-ideal and i-v fuzzy quasi-ideal.

Theorem 3.10. *For a semigroup S, the following conditions are equivalent :*

(1) *S is regular.*

(2) *$\overline{\mu}\cap\overline{\lambda} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy quasi-ideal $\overline{\mu}$ and every fuzzy ideal $\overline{\lambda}$ of S.*

(3) *$\overline{\mu}\cap\overline{\lambda} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy quasi-ideal $\overline{\mu}$ and every i-v fuzzy interior ideal $\overline{\lambda}$ of S.*

(4) *$\overline{\mu}\cap\overline{\lambda} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy bi-ideal $\overline{\mu}$ and every i-v fuzzy ideal $\overline{\lambda}$ of S.*

(5) *$\overline{\mu}\cap\overline{\lambda} = \overline{\mu}\,\overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy bi-ideal $\overline{\mu}$ and every i-v fuzzy interior ideal $\overline{\lambda}$ of S.*

Proof. (1) $\Rightarrow$ (5). Assume that (1) holds. Let $\overline{\mu}$ and $\overline{\lambda}$ be any i-v fuzzy bi-ideal and any i-v fuzzy interior ideal of S respectively. Then $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} \leq \overline{\mu}\cap\overline{\lambda}$. Let $a \in S$. Since S is regular, there exists $x \in S$ such that $a = axa$ $(= axaxa)$. Then

$$
\begin{aligned}
(\overline{\mu}\,\overline{\lambda}\,\overline{\mu})(a) &= \sup_{a=yz}{}^{i}\{min^{i}\{\overline{\mu}(y),\,\{\sup_{a=yz}{}^{i}\{min^{i}\{\overline{\mu}(y),\,(\overline{\lambda}\,\overline{\mu})(z)\}\}\} \\
&\geq min^{i}\{\overline{\mu}(a),\,(\overline{\lambda}\,\overline{\mu})(xaxa)\}\} \\
&= min^{i}\{\overline{\mu}(a),\,\sup_{xaxa=p\,q}{}^{i}\{min^{i}\{\overline{\lambda}(p),\,\overline{\mu}(q)\}\}\} \\
&\geq min^{i}\{\overline{\mu}(a), min^{i}\{\overline{\lambda}(xax), \overline{\mu}(a)\}\} \\
&\geq min^{i}\{\overline{\mu}(a), min^{i}\{\overline{\lambda}(a),\,\overline{\mu}(a)\}\}, \\
&\quad \text{[since } \overline{\lambda} \text{ is an i-v fuzzy interior ideal, } \overline{\lambda}(xax) \geq \overline{\lambda}(a).] \\
&= min^{i}\{\overline{\mu}(a), \overline{\lambda}(a)\}
\end{aligned}
$$

$= (\overline{\mu} \cap \overline{\lambda})(a).$

Thus $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} \geq \overline{\mu} \cap \overline{\lambda}$. Hence $\overline{\mu}\,\overline{\lambda}\,\overline{\mu} = \overline{\mu} \cap \overline{\lambda}$. So (1) $\Rightarrow$ (5).

By Theorem 3.7, (1), (2), (3) and (4) are equivalent. It is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2). □

Now, we obtain equivelent conditions on regular semigroup in terms of its i-v fuzzy left, right, quasi and bi-ideals.

Theorem 3.11. *For a semigroup S, the following conditions are equivalent :*

(1) *S is regular.*

(2) *$\overline{\lambda} \cap \overline{\mu} \leq \overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(3) *$\overline{\lambda} \cap \overline{\mu} \leq \overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

(4) *$\overline{\mu} \cap \overline{\nu} \leq \overline{\mu}\,\overline{\nu}$ for every i-v fuzzy left ideal $\overline{\nu}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(5) *$\overline{\mu} \cap \overline{\nu} \leq \overline{\mu}\,\overline{\nu}$ for every i-v fuzzy left ideal $\overline{\nu}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

(6) *$\overline{\lambda} \cap \overline{\mu} \cap \overline{\nu} \leq \overline{\lambda}\,\overline{\mu}\,\overline{\nu}$ for every i-v fuzzy right ideal $\overline{\lambda}$, every i-v fuzzy left ideal $\overline{\nu}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(7) *$\overline{\lambda} \cap \overline{\mu} \cap \overline{\nu} \leq \overline{\lambda}\,\overline{\mu}\,\overline{\nu}$ for every i-v fuzzy right ideal $\overline{\lambda}$, every i-v fuzzy left ideal $\overline{\nu}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

Proof. (1) $\Rightarrow$ (2). Let $\overline{\lambda}$ and $\overline{\mu}$ be an i-v fuzzy right ideal and an i-v fuzzy bi-ideal of S respectively. Suppose $a \in S$. Since S is regular, there exists $x \in S$ such that $a = axa = axaxa$. Then

$$\begin{aligned}(\overline{\lambda}\,\overline{\mu})(a) &= \sup_{a=xy}{}^{i}\{min^{i}\{\overline{\lambda}(x), \overline{\mu}(y)\}\} \\ &\geq min^{i}\{\overline{\lambda}(axax), \overline{\mu}(a)\} \\ &= min^{i}\{\overline{\lambda}(a(xax)), \overline{\mu}(a)\} \\ &\geq min^{i}\{\overline{\lambda}(a), \overline{\mu}(a)\} \\ &= (\overline{\lambda} \cap \overline{\mu})(a).\end{aligned}$$

Thus (1) $\Rightarrow$ (2).

It can be shown in a similar way that (1) $\Rightarrow$ (4).

Clearly (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5).

Now assume that (3) holds. Let $\overline{\lambda}$ be an i-v fuzzy right ideal and $\overline{\mu}$ be an i-v fuzzy left ideal of S. Since every i-v fuzzy left ideal is an i-v fuzzy quasi-ideal of S, by (3), we have $\overline{\lambda} \cap \overline{\mu} \leq \overline{\lambda}\,\overline{\mu}$. Again since $\overline{\lambda}$ is an i-v fuzzy right ideal and $\overline{\mu}$ is an i-v fuzzy left ideal of S, we have $\overline{\lambda}\,\overline{\mu} \leq \overline{\lambda} \cap \overline{\mu}$. Thus $\overline{\lambda}\,\overline{\mu} = \overline{\lambda} \cap \overline{\mu}$. By Theorem 3.3, S is regular hence, we obtain (3) $\Rightarrow$ (1).

Similarly (4) $\Rightarrow$ (1).

Again assume that (1) holds. Let $\overline{\lambda}, \overline{\nu}$ and $\overline{\mu}$ be an i-v fuzzy right ideal, an i-v fuzzy left ideal and an i-v fuzzy bi-ideal of S respectively and let $a \in S$. Since S is regular, there exists $x \in S$ such that $a = axa = axaxa$. Then,

$$\begin{aligned}(\overline{\lambda}\,\overline{\mu}\,\overline{\nu})(a) &= \sup_{a=yz}{}^{i}\{min^{i}\{\overline{\lambda}(y), (\overline{\mu}\,\overline{\nu})(z)\}\} \\ &\geq min^{i}\{\overline{\lambda}(ax), (\overline{\mu}\,\overline{\nu})(axa)\}\end{aligned}$$

$$\begin{aligned} &\geq min^i\{\overline{\lambda}(a), min^i\{\overline{\mu}(a), \overline{\nu}(a)\}\} \\ &= min^i\{\overline{\lambda}(a), \overline{\mu}(a), \overline{\nu}(a)\} \\ &= (\overline{\lambda} \cap \overline{\mu} \cap \overline{\nu})(a) \end{aligned}$$

and hence $\overline{\lambda} \cap \overline{\mu} \cap \overline{\nu} \leq \overline{\lambda}\,\overline{\mu}\,\overline{\nu}$ and so (1) $\Rightarrow$ (6).
It is clear that (6) $\Rightarrow$ (7).

Finally, assume that (7) holds. Let $\overline{\lambda}$ and $\overline{\nu}$ be an i-v fuzzy right ideal and an i-v fuzzy left ideal of S, respectively. Since $\overline{S}$ is itself an i-v fuzzy quasi-ideal of S, we have $(\overline{\lambda} \cap \overline{\nu}) = \overline{\lambda} \cap \overline{S} \cap \overline{\nu} \leq \overline{\lambda}\,\overline{S}\,\overline{\nu} \leq \overline{\lambda}\,\overline{\nu}$, by Proposition 2.18. Clearly $\overline{\lambda}\,\overline{\nu} \leq \overline{\lambda} \cap \overline{\nu}$. hence $\overline{\lambda}\,\overline{\nu} = \overline{\lambda} \cap \overline{\nu}$. It follows from Theorem 3.3, that S is regular thus (7) $\Rightarrow$ (1). □

Now, we characterize regular semigroup in terms of its i-v fuzzy ideal, i-v fuzzy interior ideal, i-v fuzzy quasi-ideal, i-v fuzzy bi-ideal and generalized i-v fuzzy bi-ideal. We also find the equivalent conditions on regular semigroup in terms of these ideals.

First we show that in a regular semigroup every generalized i-v fuzzy bi-ideal is an i-v fuzzy bi-ideal.

Theorem 3.12. *Every generalized i-v fuzzy bi-ideal of a regular semigroup S is an i-v fuzzy bi-ideal of S.*

Proof. Let $\overline{\lambda}$ be any generalized i-v fuzzy bi-ideal of S, and $a, b \in S$. Then, since S is regular, there exists $x \in S$ such that $b = bxb$. Then we have, $\overline{\lambda}(ab) = \overline{\lambda}(a(bxb)) = \overline{\lambda}(a(bx)b) \geq min^i\{\overline{\lambda}(a), \overline{(}b)\}$. This means that $\overline{\lambda}$ is an i-v fuzzy subsemigroup of S. Hence $\overline{\lambda}$ is an i-v fuzzy bi-ideal of S. □

Theorem 3.13. *For a semigroup S, the following conditions are equivalent :*
(1) *S is regular.*
(2) *$\overline{\lambda}=\overline{\lambda}\,\overline{S}\,\overline{\lambda}$ for every generalized i-v fuzzy bi-ideal $\overline{\lambda}$ of S.*
(3) *$\overline{\lambda}=\overline{\lambda}\,\overline{S}\,\overline{\lambda}$ for every i-v fuzzy bi-ideal $\overline{\lambda}$ of S.*
(4) *$\overline{\lambda}=\overline{\lambda}\,\overline{S}\,\overline{\lambda}$ for every i-v fuzzy quasi-ideal $\overline{\lambda}$ of S.*

Proof. (1) $\Rightarrow$ (2). Assume that (1) holds. Let $\overline{\lambda}$ be any generalized i-v fuzzy bi-ideal of S and $a \in S$. Then $a = axa$. Hence we have,

$$\begin{aligned} \left(\overline{\lambda}\,\overline{S}\,\overline{\lambda}\right)(a) &= \sup_{a=yz}{}^{i} \left\{min^i\left\{\left(\overline{\lambda}\,\overline{S}\right)(y), \overline{\lambda}(z)\right\}\right\} \\ &\geq min^i\left\{\left(\overline{\lambda}\,\overline{S}\right)(ax), \overline{\lambda}(a)\right\} \\ &= min^i\left\{\sup_{ax=pq}{}^{i}\left\{min^i\left\{\overline{\lambda}(p), \overline{S}(q)\right\}\right\}, \overline{\lambda}(a)\right\} \\ &\geq min^i\left\{min^i\left\{\overline{\lambda}(a), \overline{S}(x)\right\}, \overline{\lambda}(a)\right\} \\ &= min^i\left\{\overline{\lambda}(a), \overline{\lambda}(a)\right\} \\ &= \overline{\lambda}(a) \end{aligned}$$

and hence $\overline{\lambda}\,\overline{S}\,\overline{\lambda} \geq \overline{\lambda}$. Since $\overline{\lambda}$ is an generalized i-v fuzzy bi-ideal of S, $\overline{\lambda}\,\overline{S}\,\overline{\lambda} \leq \overline{\lambda}$. Thus $\overline{\lambda} = \overline{\lambda}\,\overline{S}\,\overline{\lambda}$ and so (1) $\Rightarrow$ (2).

Since every i-v fuzzy bi-ideal is generalized i-v fuzzy bi-ideal, we have (2) $\Rightarrow$ (3)

and since every i-v fuzzy quasi-ideal is i-v fuzzy bi-ideal, we have (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1). Let A be any quasi-ideal of S. Then we have

$$ASA \subseteq A(SS) \cap (SS)A \subseteq AS \cap SA \subseteq A.$$

Let $a \in A$. Since $\overline{\chi}_A$ is an i-v fuzzy quasi-ideal of S, we have $\overline{\chi}_A\ \overline{S}\ \overline{\chi}_A = \overline{\chi}_A$. Now,

$$\begin{aligned}\overline{\chi}_A(a) &= \left(\overline{\chi}_A\ \overline{S}\ \overline{\chi}_A\right)(a)\\ \overline{1} &= \sup_{a=yz}{}^{i}\left\{min^i\left\{\left(\overline{\chi}_A\ \overline{S}\ \right)(y),\ \overline{\chi}_A(z)\right\}\right\}.\end{aligned}$$

This implies that there exist $b, c \in S$ such that $\left(\overline{\chi}_A\ \overline{S}\right)(b) = \overline{1}$ and $\overline{\chi}_A(c) = \overline{1}$ with $a = bc$. Now,

$$\begin{aligned}\overline{1} &= \left(\overline{\chi}_A\ \overline{S}\right)(b)\\ &= \sup_{b=pq}{}^{i}\left\{min^i\left\{\overline{\chi}_A(p), \overline{S}(q)\right\}\right\}.\end{aligned}$$

This shows that there exist elements d and e such that $\overline{\chi}_A(d) = \overline{1}$ and $\overline{S}(e) = \overline{1}$ with $b = de$. Thus $d, e \in A$ and $e \in S$ and so $a = bc = (de)c \in ASA$. Therefore $A \subseteq ASA$. Hence $A = ASA$. By Theorem 3.7 it follows that S is regular and hence (4) $\Rightarrow$ (1). □

Theorem 3.14. *For a semigroup S, the following conditions are equivalent :*

(1) *S is regular.*

(2) *$\overline{\lambda} \cap \overline{\mu} = \overline{\mu}\overline{\lambda}\overline{\mu}$ for every i-v fuzzy ideal $\overline{\lambda}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

(3) *$\overline{\lambda} \cap \overline{\mu} = \overline{\mu}\overline{\lambda}\overline{\mu}$ for every i-v fuzzy interior ideal $\overline{\lambda}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

(4) *$\overline{\lambda} \cap \overline{\mu} = \overline{\mu}\overline{\lambda}\overline{\mu}$ for every i-v fuzzy ideal $\overline{\lambda}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(5) *$\overline{\lambda} \cap \overline{\mu} = \overline{\mu}\overline{\lambda}\overline{\mu}$ for every i-v fuzzy interior ideal $\overline{\lambda}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(6) *$\overline{\lambda} \cap \overline{\mu} = \overline{\mu}\overline{\lambda}\overline{\mu}$ for every fuzzy ideal $\overline{\lambda}$ and every generalized i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(7) *$\overline{\lambda} \cap \overline{\mu} = \overline{\mu}\overline{\lambda}\overline{\mu}$ for every i-v fuzzy interior ideal $\overline{\lambda}$ and every generalized i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

Proof. (1) $\Rightarrow$ (7). Let $\overline{\lambda}$ and $\overline{\mu}$ be any i-v fuzzy interior ideal and generalized i-v fuzzy bi-ideal of S, respectively. Then

$$\begin{aligned}&\overline{\mu}\overline{\lambda}\overline{\mu} \leq \overline{\mu}\overline{S}\overline{\mu} \leq \overline{\mu}\\ &\overline{\mu}\overline{\lambda}\overline{\mu} \leq \overline{S}\ \overline{\lambda}\ \overline{S} \leq \overline{\lambda}\\ &\overline{\mu}\overline{\lambda}\overline{\mu} \leq \overline{\mu} \cap \overline{\lambda}.\end{aligned}$$

Let $a \in S$. Then, since S is regular, there exists $x \in S$ such that $a = axa(= axaxa)$. Now,

$$\begin{aligned}
\left(\overline{\mu}\overline{\lambda}\overline{\mu}\right)(a) &= \underset{a=yz}{sup^i}\left\{min^i\left\{\overline{\mu}(y), \left(\overline{\lambda}\overline{\mu}\right)(z)\right\}\right\} \\
&\geq min^i\left\{\overline{\mu}(a), \left(\overline{\lambda}\overline{\mu}\right)(xaxa)\right\}, \text{since } a = axaxa \\
&= min^i\left\{\overline{\mu}(a), \underset{xaxa=p\,q}{sup^i}\left\{min^i\left\{\overline{\lambda}(p), \overline{\mu}(q)\right\}\right\}\right\} \\
&\geq min^i\left\{\overline{\mu}(a), min^i\left\{\overline{\lambda}(xax), \overline{\mu}(a)\right\}\right\} \\
&= min^i\left\{\overline{\mu}(a), \overline{\lambda}(a)\right\} \\
&= \left(\overline{\mu} \cap \overline{\lambda}\right)(a).
\end{aligned}$$

Thus $\overline{\mu}\overline{\lambda}\overline{\mu} \geq \overline{\mu} \cap \overline{\lambda}$. Hence $\overline{\mu}\overline{\lambda}\overline{\mu} = \overline{\mu} \cap \overline{\lambda}$ and so (1) $\Rightarrow$ (7).
It is clear that (7) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Let $\overline{\mu}$ be any i-v fuzzy quasi-ideal of S. Then, since $\overline{S}$ itself is an i-v fuzzy ideal of S, we have

$$\overline{\mu} = \overline{\mu} \cap \overline{S} = \overline{\mu}\overline{S}\overline{\mu}.$$

Thus it follows from Theorem 3.5 that S is regular and hence (2) $\Rightarrow$ (1). □

Theorem 3.15. *For a semigroup S, the following conditions are equivalent :*

(1) *S is regular.*

(2) *$\overline{\lambda} \cap \overline{\mu} \leq \overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(3) *$\overline{\lambda} \cap \overline{\mu} \leq \overline{\lambda}\,\overline{\mu}$ for every i-v fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

(4) *$\overline{\mu} \cap \overline{\nu} \leq \overline{\mu}\,\overline{\nu}$ for every i-v fuzzy left ideal $\overline{\nu}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(5) *$\overline{\mu} \cap \overline{\nu} \leq \overline{\mu}\,\overline{\nu}$ for every i-v fuzzy left ideal $\overline{\nu}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

(6) *$\overline{\lambda} \cap \overline{\mu} \cap \overline{\nu} \leq \overline{\lambda}\,\overline{\mu}\,\overline{\nu}$ for every fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy left ideal $\overline{\nu}$ and every i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

(7) *$\overline{\lambda} \cap \overline{\mu} \cap \overline{\nu} \leq \overline{\lambda}\,\overline{\mu}\,\overline{\nu}$ for every fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy left ideal $\overline{\nu}$ and every i-v fuzzy quasi-ideal $\overline{\mu}$ of S.*

(8) *$\overline{\lambda} \cap \overline{\mu} \cap \overline{\nu} \leq \overline{\lambda}\,\overline{\mu}\,\overline{\nu}$ for every fuzzy right ideal $\overline{\lambda}$ and every i-v fuzzy left ideal $\overline{\nu}$ and every generalized i-v fuzzy bi-ideal $\overline{\mu}$ of S.*

Proof. (1) $\Rightarrow$ (8). Let $\overline{\lambda}$, $\overline{\mu}$ and $\overline{\nu}$ be any i-v fuzzy right ideal, any generalized i-v fuzzy bi-ideal and any i-v fuzzy left ideal of S, respectively. Let $a \in S$. Then since S is regular, there exists $x \in S$ such that $a = axa$. Hence we have

$$
\begin{aligned}
\left(\overline{\lambda}\,\overline{\mu}\,\overline{\nu}\right)(a) &= \underset{a=yz}{sup^i}\left\{min^i\left\{\overline{\lambda}(y), \left(\overline{\mu}\,\overline{\nu}\right)(z)\right\}\right\} \\
&\geq min^i\left\{\overline{\lambda}(ax), \left(\overline{\mu}\,\overline{\nu}\right)(a)\right\} \\
&= min^i\left\{\overline{\lambda}(a), \underset{a=pq}{sup^i}\left\{min^i\left\{\overline{\mu}(p), \overline{\nu}(q)\right\}\right\}\right\} \\
&\geq min^i\left\{\overline{\lambda}(a), min^i\left\{\overline{\mu}(a), \overline{\nu}(xa)\right\}\right\} \\
&\geq min^i\left\{\overline{\lambda}(a), min^i\left\{\overline{\mu}(a), \overline{\nu}(a)\right\}\right\} \\
&= min^i\left\{\overline{\lambda}(a), \overline{\mu}(a), \overline{\nu}(a)\right\} \\
&= \left(\overline{\lambda}\cap\overline{\mu}\cap\overline{\nu}\right)(a),
\end{aligned}
$$

and so $\overline{\lambda}\,\overline{\mu}\,\overline{\nu} \geq \overline{\lambda}\cap\overline{\mu}\cap\overline{\nu}$. Hence (1) $\Rightarrow$ (8).

It is clear that (8) $\Rightarrow$ (6) $\Rightarrow$ (7). It can be seen in a similar way that (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4). Clearly (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5).

(3) $\Rightarrow$ (1). Let $\overline{\lambda}$ be any i-v fuzzy right ideal and $\overline{\mu}$ be any i-v fuzzy left ideal of S. Since every i-v fuzzy left ideal is an i-v fuzzy quasi-ideal of S, by (3), we have $\overline{\lambda}\cap\overline{\mu} \leq \overline{\lambda}\,\overline{\mu}$. Again since $\overline{\lambda}$ is an i-v fuzzy right ideal and $\overline{\mu}$ is any i-v fuzzy left ideal of S, we have $\overline{\lambda}\overline{\mu} \leq \overline{\lambda}\cap\overline{\mu}$. Thus $\overline{\lambda}\overline{\mu} = \overline{\lambda}\cap\overline{\mu}$. By Theorem 3.3 S is regular and hence we obtain (3) $\Rightarrow$ (1). Similarly (5) $\Rightarrow$ (1). Finally, assume that (7) holds. Let $\overline{\lambda}$ and $\overline{\nu}$ respectively be an i-v fuzzy right ideal and an i-v fuzzy left ideal of S. Since $\overline{S}$ is itself an i-v fuzzy quasi-ideal of S and by Proposition 2.18, we have $\overline{\lambda}\cap\overline{\nu} = \overline{\lambda}\cap\overline{S}\cap\overline{\nu} \leq \overline{\lambda}\;\overline{S}\;\overline{\nu} \leq \overline{\lambda}\;\;\overline{\nu}$. Clearly $\overline{\lambda}\,\overline{\nu} \leq \overline{\lambda}\cap\overline{\nu}$. Hence $\overline{\lambda}\,\overline{\nu} = \overline{\lambda}\cap\overline{\nu}$. It follows from Theorem 3.3, that S is regular and (7) $\Rightarrow$ (1). □

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