Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 5, (May 2016), pp. 755–768
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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Fuzzy renewal processes on quasi-probability space

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Received 3 October 2015; Revised 25 October 2015; Accepted 13 December 2015

Abstract. Quasi-probability is a fuzzy measure. This paper will further discussed its properties. Bases on quasi-probability theory, modeling renewal precesses for quasi-random variables are investigated. First, a renewal process with independent fuzzy interarrival times is explored, some limit theorems on renewal variable and average renewal time are obtained, and a fuzzy elementary renewal theorem is proved. Second, a renewal reward process with fuzzy interarrival times and rewards is investigated, and the limit theorems on reward rate in quasi-probability measure are derived. All obtained results are natural extensions of the classical stochastic renewal process to the case where the measure tool is fuzzy.

2010 AMS Classification: 60F05, 28E10

Keywords: Quasi-probability measure, Fuzzy renewal process, Fuzzy renewal reward process, Convergence in quasi-probability measure, Convergence almost surely.

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1. Introduction

Probability theory is an efficient mathematical tool to handle the behaviors of random phenomena. Based on probability theory, stochastic renewal theory has been well developed in the past decades. In classical stochastic renewal process, the interarrival times and rewards are assumed to be independent and identically distributed random variables. In this condition, the authors of [15] and [23] have investigated elementary renewal theorem, delay renewal theorem and renewal reward theorem, and the papers [1, 3, 16, 24] have shown some applications of stochastic renewal process.

In this paper we study renewal process in fuzzy environments. In fuzzy systems, information and cost variables are usually vague or imprecise which is essentially different from the probabilistic variability [5, 6, 7], [10, 11, 12], [17, 18, 19, 20, 21] and [25, 26, 27, 28]. So it will be more reasonable to apply quasi-probability measure, which is an important extension of probability measure [2, 14, 22] in terms of their non-additive behavior and capacity of dealing with fuzziness, to investigate such renewal processes. Quasi-probability measure was introduced by Wang Zhenyuan [19], which offered an efficient tool to deal with fuzzy information fusion, subjective judgement, decision making, and so forth [4], [8], [9], [29].

Fuzzy renewal processes have been discussed in [6, 7], [12], [20, 21], [28]. Among them, a fuzzy renewal process was studied in [28]. Going deeper with the results in [28], a fuzzy alternating renewal process was explored in [12]; and the fuzzy renewal processes with T-independent variables and the corresponding fuzzy renewal theorems for the long-term expected renewal rate (reward rate) have been well developed in [20]. Up to now, the renewal processes for quasi-probability measure have not been investigated. To this end, this paper discusses the modeling of renewal processes on quasi-probability space. The work helps to build important theoretical foundations for the development of quasi-probability measure theory.

The paper is organized as follows: Section 1 is for introduction. In Section 2 some preliminaries are given. In Section 3 we define and study fuzzy renewal process. Fuzzy renewal reward process are shown in Section 4 and, ultimately, Section 5 is for conclusions.

2. Preliminaries

In this section, the definition and the properties of quasi- probability measure will be given. Then the quasi-random variables and its expected value will also be introduced. Some interested readers can refer to the book [19] for more details on quasi- probability measure theory.

2.1. The definition and properties of quasi-probability measure.

In this paper, let X be a nonempty set and $(X, \mathcal{F})$ be a measurable space. Here $\mathcal{F}$ is a $\sigma-$algebra of X. If $A \in \mathcal{F}$, then the complement of A is denoted by A^c.

Definition 2.1 ([19])**.** Let $\alpha \in (0, +\infty]$, an extended real function is called a T-function iff $\theta : [0, a] \to [0, +\infty]$ is continuous, strictly increasing, and such that $\theta(0) = 0,\ \theta^{-1}(\{\infty\}) = \varnothing$ or $\{\infty\}$, according to a being finite or not.

Suppose that $\alpha \in (0, +\infty]$, an extended real function $\theta : [0, a] \to [0, +\infty]$ is called a regular function, if θ is continuous, strictly increasing, and $\theta(0) = 0, \theta(1) = 1$ [8].

Obviously, if θ is a regular function, then θ^{-1} is also a regular function.

Definition 2.2 ([19])**.** μ is called quasi-additive iff there exists a T-function θ , whose domain of definition contains the range of μ , such that the set function $\theta \circ \mu$ defined on $\mathcal{F}$ by $(\theta \circ \mu)(E) = \theta\ [\mu(E)]\ \ (\forall E \in \mathcal{F})$, is additive; μ is called a quasi-measure iff there exists a T-function θ such that $\theta \circ \mu$ is a classical measure on $\mathcal{F}$. The T-function θ is called the proper T-function of μ.

Definition 2.3. Let μ be a quasi-measure on $\mathcal{F}$, if θ is a regular T-function of μ, and $\mu(X) = 1$, then μ is called a quasi-probability. The triplet $(X, \mathcal{F}, \mu)$ is called a quasi-probability space.

Example 2.4. Let μ be a probability measure. From Definition 2.3, we know that μ is a quasi-probability with $\theta(x) = x$ as its T-function.

Example 2.5 ([19])**.** Suppose that $X = \{1, 2, \cdots, n\}$, $\rho(X)$ is the power set of X. If

$$\mu(E) = (\frac{|E|}{n})^2,$$

where $|E|$ is the number of those points that belong to E, then μ is a quasi-probability with $\theta(x) = \sqrt{x}, \quad x \in [0, 1]$ as its T-function.

Theorem 2.6 ([19])**.** (1) *Any quasi-measure on a semiring is a quasi-additive fuzzy measure.*

(2) *Any quasi-additive fuzzy measure on a ring is a quasi-measure.*

Theorem 2.7 ([29])**.** *Let μ be a quasi-probability. Then $\mu(\varnothing) = 0$.*

Theorem 2.8. *Let μ be a quasi-probability on $\mathcal{F}$ and $A, B \in \mathcal{F}$. Then we have :*

(1) *If $A \subset B$, then $\mu(A) < \mu(B)$.*

(2) *If $\mu(A) = 0$, then $\mu(A^c) = 1$.*

(3) $\mu(A \bigcup B) \leq \theta^{-1}[(\theta \circ \mu)(A) + (\theta \circ \mu)(B)]$.

Definition 2.9. Let $(X, \mathcal{F}, \mu)$ be a quasi- probability space and $\xi = \xi(\omega), \omega \in \mathcal{F}$, be a real set function on $\mathcal{F}$. For any given real number x, if $\{\omega | \ \xi(\omega) \leq x\} \in \mathcal{F}$, then ξ is called a quasi-random variable, denoted by q-random variable.

Definition 2.10. The distribution function of q-random variable ξ is defined by

$$F_\mu(\mathrm{x}) = \mu\{\omega \in \mathcal{F} | \ \xi(\omega) \leq \mathrm{x}\}.$$

Let ξ and η be two q-random variables. $\forall x, y \in R$, if

$$\mu(\xi \leq x, \eta \leq y) = \theta^{-1}[(\theta \circ \mu)(\xi \leq x) \cdot (\theta \circ \mu)(\eta \leq y)],$$

then ξ and η are independent q-random variables.

The q-random variables $\xi_1, \ \xi_2, \ ..., \ \xi_n$ ... are said to be identically distribution iff

$$\mu\{\xi_i \in B\} = \mu\{\xi_j \in B\}, \ i, j = 1, 2, ...$$

for any Borel set $\mathcal{B}$ of $\mathcal{R}$ [8].

Definition 2.11 ([8])**.** Let ξ be a q-random variable whose distribution function is $F_\mu(x)$. If $\int_{-\infty}^{+\infty} |x| dF_\mu(x) < \infty$, then the expected value of ξ is defined by $E_\mu[\xi] = \int_{-\infty}^{+\infty} x dF_\mu(x)$.

Theorem 2.12 ([8])**.** *Let $\xi_1, \ \xi_2, \ ... \ \xi_n, \ ...$ be a sequence of q-random variables with finite expected values. Then*

(1) *For any constant λ, $E_\mu[\lambda\xi] = \lambda E_\mu[\xi]$.*

(2) *For any $m \geq 1$, $E_\mu[\Sigma_{i=1}^m \xi_i] = \Sigma_{i=1}^m E_\mu[\xi_i]$.*

(3) *If $\xi_1, \ \xi_2$ are independent and identically distributed q-random variables and $\xi_1 \leq \xi_2$, then $E_\mu[\xi_1] \leq E_\mu[\xi_2]$.*

Theorem 2.13 ([8])**.** *Suppose that $\xi_1, \ \xi_2, \ ... \ \xi_n$ are independent q-random variables. If n partial derivations of θ^{-1} are continuous, then*

$$E_\mu[\xi_1 \xi_2 ... \xi_n] = C_{n\alpha} E_\mu[\xi_1] E_\mu[\xi_2] ... E_\mu[\xi_n].$$

Here $C_{n\alpha}$ is a constant.

2.2. **Convergence concepts of q-random variables sequence.**

Definition 2.14. Suppose that $\xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ is a sequence of q-random variables. If there exists a q-random variable ξ, such that $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \mu\{|\ \xi_n - \xi\ |\geq \varepsilon\} = 0,$$

namely,

$$\lim_{n\to\infty} \mu\{|\ \xi_n - \xi\ |< \varepsilon\} = 1,$$

then we say that $\{\xi_n\}$ converges in quasi-probability to ξ. Denoted by

$$\xi_n \to \xi \qquad (\mu).$$

Definition 2.15. Suppose that $\xi, \xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ are q-random variables defined on the quasi-probability space $(X, \mathcal{F}, \mu)$. If

$$\mu\{\lim_{n\to\infty} \xi_n = \xi\} = 1,$$

then we say that $\{\xi_n\}$ converges almost surely to ξ. Denoted by

$$\lim_{n\to\infty} \xi_n = \xi \qquad (\mu - a.s.).$$

Theorem 2.16 ([29]). *Suppose that $\xi,\ \xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ are q-random variables defined on the quasi-probability space $(X, \mathcal{F}, \mu)$. If $\{\xi_n\}$ converges almost surely to ξ, then $\{\xi_n\}$ converges in quasi-probability to ξ.*

Theorem 2.17 ([8]). *(Law of large numbers) Let $\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\ \cdots$ be the independent and identical distributed q-random variables, ξ_n have the same finite expected value, that is $\forall n$, $E_\mu[\xi_n] = a$, then we have*

$$\frac{1}{n}\sum_{k=1}^{n} \xi_k - E\left(\frac{1}{n}\sum_{k=1}^{n} \xi_k\right) \to 0 \qquad (\mu).$$

3. Fuzzy renewal process

In the section, we will discuss renewal process for q-random variables. For each integer n, let q-random variable ξ_n, defined on the quasi- probability space (X, $\mathcal{F}$, μ), be the interarrival time between the $(n-1)$th and nth event.

Definition 3.1. Suppose that $\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\ \cdots$ are the positive independent and identical distributed q-random variables, defined $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then

$$N(t) = \max_{n\geq 0}\{n | S_n \leq t\}$$

is called a renewal process.

Example 3.2. There are many ships enter a port. Let S_n denote the time of the nth ship entering the port, $n = 1, 2, \cdots$, if $\xi_1 = S_1,\ \xi_2 = S_2 - S_1,\ \cdots$, then $\xi_1,\ \xi_2,\ \cdots$ are independent and identical distributed q-random variables. And

$$N(t) = \max_{n\geq 0}\{n | \xi_1 + \xi_2 + \cdots + \xi_n \leq t\}$$

is a renewal process.

If $\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\ \cdots$ denote the interarrival times of successive events. Then S_n, a stationary independent increment process with respect to n, can be regarded as the waiting time until the occurrence of the nth event, and the renewal process $N(t)$ is the number of renewal in $(0,\ t]$.

According to Definition 3.1, for any time t and integer n, we can obtain the following fundamental relationships [13]:

$$N(t) \geq n \Leftrightarrow S_n \leq t, \qquad N(t) \leq n \Leftrightarrow S_{n+1} > t.$$

It follows from the fundamental relationships that $N(t) \geq n$ is equivalent to $S_n \leq t$ and $N(t) \leq n$ is equivalent to $S_{n+1} > t$, thus we have

$$\mu\{N(t) \geq n\} = \mu\{S_n \leq t\}, \qquad \mu\{N(t) \leq n\} = \mu\{S_{n+1} > t\}.$$

Theorem 3.3. *Let $N(t)$ be a renewal process with interarrival times $\xi_1,\ \xi_2,\ \cdots,$ $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, and $E_\mu[\xi_k] = u$ exists, $k = 1,\ 2,\ \cdots$. Noting that*

$$A = \{\lim_{t\to\infty} N(t) = +\infty\},$$

then $\mu\{A\} = 1$. And if $N(t) \in A$, we have

$$\frac{S_{N(t)}}{N(t)} \to u \qquad (\mu).$$

Proof. Let $N(+\infty) = \lim_{t\to\infty} N(t)$ denote to the total numbers of renewal occurrence. Because

$$\{N(+\infty) < +\infty\} \Leftrightarrow \{\xi_n = +\infty, \ \text{ for some } \ n\} \Leftrightarrow \left\{\bigcup_{n=1}^{+\infty}(\xi_n = +\infty)\right\},$$

we have

$$\mu\{N(+\infty) < +\infty\} = \mu\{\xi_n = +\infty, \ \text{ for some } \ n\} = \mu\left\{\bigcup_{n=1}^{+\infty}(\xi_n = +\infty)\right\}$$
$$\leq \theta^- \left[\sum_{n=1}^{+\infty}(\theta\circ\mu)\{\xi_n = +\infty\}\right] = \theta^-(0) = 0.$$

By virtue of Theorem 2.8,

$$\mu\{A\} = \mu\{N(+\infty) = +\infty\} = 1.$$

On the other hand, $\frac{S_{N(t)}}{N(t)}$ is the average of the former $N(t)$ renewal interarrival times. According to the law of large numbers (Theorem 2.17), when $N(t) \to +\infty$, $\frac{S_{N(t)}}{N(t)} \to u$. And

$$\lim_{t\to\infty} N(t) = +\infty,$$

thus we have

$$\frac{S_{N(t)}}{N(t)} \to u \qquad (\mu).$$

The proof of the theorem is completed. □

Lemma 3.4. *Let $N(t)$ be a renewal process with interarrival times $\xi_1,\ \xi_2,\ \cdots$, then*

$$\frac{1}{N(t)} \to 0 \qquad (\mu).$$

Proof. Since

$$\mu\{N(+\infty) = +\infty\} = 1,$$

we have

$$\mu\left\{\lim_{t\to\infty} N(t) = +\infty\right\} = \mu\left\{\lim_{t\to\infty} \frac{1}{N(t)} = 0\right\} = 1.$$

It follows from Definition 2.15 that

$$\frac{1}{N(t)} \to 0 \qquad (\mu - a.s.).$$

According to Theorem 2.16,

$$\frac{1}{N(t)} \to 0 \qquad (\mu)$$

is valid. □

Theorem 3.5. *Let $N(t)$ be a renewal process with interarrival times $\xi_1,\ \xi_2,\ \cdots,$ and $E_\mu[\xi_k] = u$ exists, $k = 1,\ 2,\ \cdots$. Noting that*

$$A = \{\lim_{t\to\infty} N(t) = +\infty\}.$$

If $N(t) \in A$, then we have

$$\frac{N(t)}{t} \to \frac{1}{u} \qquad (\mu).$$

Proof. Since $S_{N(t)}$ is the time of the last renewal prior to or at time t, and $S_{N(t)+1}$ is the time of the first renewal after time t, we have

$$S_{N(t)} \le t < S_{N(t)+1},$$

which implies

$$\frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}.$$

It follows from Theorem 3.3 that for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \mu\left\{\left|\frac{S_{N(t)}}{N(t)} - u\right| \ge \varepsilon\right\} = 0.$$

Now we prove

$$\lim_{t\to\infty} \mu\left\{\left|\frac{S_{N(t)+1}}{N(t)} - u\right| \ge \varepsilon\right\} = 0.$$

According to [20], for any $\varepsilon \in (0,\ 1)$, we can have

$$\begin{aligned} &\left\{\left|\frac{S_{N(t)+1}}{N(t)} - u\right| \ge \varepsilon\right\} \\ \subseteq\ &\left\{\frac{1}{N(t)} \cdot \left|\frac{S_{N(t)+1}}{N(t)+1} - u\right| \ge \frac{\varepsilon}{3}\right\} \bigcup \left\{\left|\frac{S_{N(t)+1}}{N(t)+1} - u\right| \ge \frac{\varepsilon}{3}\right\} \bigcup \left\{u \cdot \frac{1}{N(t)} \ge \frac{\varepsilon}{3}\right\}. \end{aligned}$$

Which implies

$$\begin{aligned}
&\mu\left\{\left|\frac{S_{N(t)+1}}{N(t)}-u\right|\geq\varepsilon\right\}\\
&\leq\mu\left(\left\{\frac{1}{N(t)}\cdot\left|\frac{S_{N(t)+1}}{N(t)+1}-u\right|\geq\frac{\varepsilon}{3}\right\}\bigcup\left\{\left|\frac{S_{N(t)+1}}{N(t)+1}-u\right|\geq\frac{\varepsilon}{3}\right\}\bigcup\left\{u\cdot\frac{1}{N(t)}\geq\frac{\varepsilon}{3}\right\}\right)\\
&\leq\theta^{-1}\left[(\theta\circ\mu)\left\{\frac{1}{N(t)}\geq\frac{\varepsilon}{3}\right\}+2(\theta\circ\mu)\left\{\left|\frac{S_{N(t)+1}}{N(t)+1}-u\right|\geq\frac{\varepsilon}{3}\right\}+(\theta\circ\mu)\left\{\frac{1}{N(t)}\geq\frac{\varepsilon}{3u}\right\}\right].
\end{aligned}$$

By Lemma 3.4 and Theorem 3.3, the following equality

$$\lim_{t\to\infty}\mu\left\{\left|\frac{S_{N(t)+1}}{N(t)}-u\right|\geq\varepsilon\right\}=0$$

is true.

It follows from [20] that for all $0<\varepsilon<\frac{1}{u}$,

$$\left\{\left|\frac{N(t)}{S_{N(t)}}-\frac{1}{u}\right|\geq\varepsilon\right\}\subseteq\left\{\left|\frac{S_{N(t)}}{N(t)}-u\right|\geq\frac{u^2\varepsilon}{1+u\varepsilon}\right\}.$$

By virtue of Theorem 3.3, we have

$$\lim_{t\to\infty}\mu\left\{\left|\frac{N(t)}{S_{N(t)}}-\frac{1}{u}\right|\geq\varepsilon\right\}\leq\lim_{t\to\infty}\mu\left\{\left|\frac{S_{N(t)}}{N(t)}-u\right|\geq\frac{u^2\varepsilon}{1+u\varepsilon}\right\}=0.$$

In the similar way, we can obtain

$$\lim_{t\to\infty}\mu\left\{\left|\frac{N(t)}{S_{N(t)+1}}-\frac{1}{u}\right|\geq\varepsilon\right\}=0.$$

Since

$$\frac{N(t)}{S_{N(t)+1}}<\frac{N(t)}{t}<\frac{N(t)}{S_{N(t)}},$$

by the same way as [20], we can prove that for any $\varepsilon>0$,

$$\begin{aligned}
&\lim_{t\to\infty}\mu\left\{\left|\frac{N(t)}{t}-\frac{1}{u}\right|\geq\varepsilon\right\}\\
&=\lim_{t\to\infty}\mu\left(\left\{\frac{N(t)}{t}\geq\frac{1}{u}+\varepsilon\right\}\bigcup\left\{\frac{N(t)}{t}\leq\frac{1}{u}-\varepsilon\right\}\right)\\
&\leq\lim_{t\to\infty}\theta^{-1}\left[(\theta\circ\mu)\left\{\frac{N(t)}{t}\geq\frac{1}{u}+\varepsilon\right\}+(\theta\circ\mu)\left\{\frac{N(t)}{t}\leq\frac{1}{u}-\varepsilon\right\}\right]\\
&\leq\lim_{t\to\infty}\theta^{-1}\left[(\theta\circ\mu)\left\{\left|\frac{N(t)}{S_{N(t)}}-\frac{1}{u}\right|\geq\varepsilon\right\}+(\theta\circ\mu)\left\{\left|\frac{N(t)}{S_{N(t)+1}}-\frac{1}{u}\right|\geq\varepsilon\right\}\right]\\
&=\lim_{t\to\infty}\theta^{-1}\left[\theta\left(\mu\left\{\left|\frac{N(t)}{S_{N(t)}}-\frac{1}{u}\right|\geq\varepsilon\right\}\right)+\theta\left(\mu\left\{\left|\frac{N(t)}{S_{N(t)+1}}-\frac{1}{u}\right|\geq\varepsilon\right\}\right)\right]\\
&=\theta^{-1}\left[\theta\left(\lim_{t\to\infty}\mu\left\{\left|\frac{N(t)}{S_{N(t)}}-\frac{1}{u}\right|\geq\varepsilon\right\}\right)+\theta\left(\lim_{t\to\infty}\mu\left\{\left|\frac{N(t)}{S_{N(t)+1}}-\frac{1}{u}\right|\geq\varepsilon\right\}\right)\right]\\
&=\theta^{-1}[\theta(0)+\theta(0)]=\theta^{-1}(0)=0.
\end{aligned}$$

This implies that

$$\frac{N(t)}{t} \to \frac{1}{u} \qquad (\mu).$$

The proof of the theorem is complete. □

Definition 3.6. Suppose that $\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\ \cdots$ are the positive independent and identical distributed q-random variables, N is a q-random variable which only takes positive integers. If for any integer n, the event $\{N = n\}$ and $\xi_{n+1},\ \xi_{n+2},\ \cdots$ are independent of each other, then N is the stopping time of $\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\ \cdots$.

Theorem 3.7. *Let $\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\ \cdots$ be the positive independent and identical distributed q-random variables, $E_\mu[\xi_1] < +\infty$. If N is the stopping time of $\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\ \cdots$, and $E_\mu[N] < +\infty$, then we have*

$$E_\mu[\sum_{n=1}^{N} \xi_n] \le C_{2\alpha} E_\mu[N] \cdot E_\mu[\xi_1].$$

Here $C_{2\alpha}$ is a constant.

Proof. Assume that

$$I_n = \begin{cases} 1, & N \ge n, \\ 0, & N < n. \end{cases}$$

Then I_n is identified by $\xi_1,\ \xi_2,\ \cdots,\ \xi_{n-1}$, and I_n is independent of ξ_n. By Theorem 2.13, we have

$$\begin{aligned} & E_\mu[\sum_{n=1}^{N} \xi_n] = E_\mu[\sum_{n=1}^{\infty} \xi_n I_n] = \sum_{n=1}^{\infty} E_\mu[\xi_n I_n] \le \sum_{n=1}^{\infty} C_{2\alpha} E_\mu[\xi_n] \cdot E_\mu[I_n] \\ = \quad & C_{2\alpha} E_\mu[\xi_1] \sum_{n=1}^{\infty} E_\mu[I_n] = C_{2\alpha} E_\mu[\xi_1] \sum_{n=1}^{\infty} \mu\{N \ge n\} = C_{2\alpha} E_\mu[\xi_1] \cdot E_\mu[N]. \end{aligned}$$

□

Lemma 3.8. *Let $N(t)$ be a renewal process with interarrival times $\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\ \cdots$ and $u = E_\mu[\xi_1] < +\infty$, then*

$$E_\mu[\sum_{n=1}^{N(t)+1} \xi_n] \le C_{2\alpha} \cdot u \cdot (E_\mu[N(t)] + 1).$$

Here $C_{2\alpha}$ is a constant.

Proof. In fact,

$$\{N(t) + 1 = n\} = \{N(t) = n - 1\} = \{S_{n-1} \le t, S_n > t\}.$$

Thus event $\{N(t)+1 = n\}$ only depends on $\xi_1,\ \xi_2,\ \cdots,\ \xi_n$, that is to say, $\{N(t)+1 = n\}$ is independent of $\xi_{n+1}, \xi_{n+2}, \cdots$. This implies that $N(t)+1$ is the stopping time of $\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\ \cdots$, when $u = E_\mu[\xi_1] < +\infty$, according to Theorem 3.7, we have

$$E_\mu[\sum_{n=1}^{N(t)+1} \xi_n] \le C_{2\alpha} \cdot u \cdot (E_\mu[N(t)] + 1).$$

□

Theorem 3.9. *(Fuzzy elementary renewal theorem) Let $N(t)$ be a renewal process with interarrival times $\xi_1,\ \xi_2,\ \cdots$, and $E_\mu[\xi_k] = u$ exists, $k = 1,\ 2,\ \cdots$. Then we have*

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{C_{2\alpha}\cdot u}.$$

Here $C_{2\alpha}$ is a constant.

Proof. Let $u = E_\mu[\xi_1] < +\infty$. Since

$$\sum_{n=1}^{N(t)+1} \xi_n > t,$$

it follows from Lemma 3.8 that

$$C_{2\alpha}\cdot u\cdot (E_\mu[N(t)] + 1) > t,$$

therefore, we have

$$\underline{\lim_{t\to\infty}} \frac{E[N(t)]}{t} \geq \frac{1}{C_{2\alpha}\cdot u}.$$

On the other hand, for any $M > 0$, we can define a new renewal process $\{\bar{N}(t),\ t \geq 0\}$ with interarrival times $\bar{\xi}_1,\ \bar{\xi}_2,\ \cdots,\ \bar{\xi}_n,\ \cdots$, here $\bar{\xi}_n$ $(n = 1,\ 2\ ,\ \cdots)$ are defined as follows:

$$\bar{\xi}_n = \begin{cases} \xi_n, & \xi_n \leq M, \\ M, & \xi_n > M. \end{cases}$$

Noting that

$$\bar{S}_n = \bar{\xi}_1 + \bar{\xi}_2 + \cdots + \bar{\xi}_n,$$

and

$$\bar{N}(t) = \sup\{n \mid \bar{S}_n \leq t\}.$$

Since $\bar{\xi}_n \leq M$, $n = 1,\ 2,\ \cdots$, one has

$$\bar{S}_{\bar{N}(t)+1} \leq t + M.$$

According to Lemma 3.8 ,

$$E_\mu\left[\sum_{n=1}^{\bar{N}(t)+1} \xi_n\right] \leq C_{2\alpha}\cdot u_M\cdot (E_\mu[\bar{N}(t)] + 1).$$

Here $u_M = E_\mu[\bar{\xi}_1]$.
When

$$C_{2\alpha}\cdot u_M\cdot (E_\mu[\bar{N}(t)] + 1) \geq t + M,$$

we have

$$\underline{\lim_{t\to\infty}} \frac{E[\bar{N}(t)]}{t} \geq \frac{1}{C_{2\alpha}\cdot u_M}.$$

Let $M \to \infty$,

$$\underline{\lim_{t\to\infty}} \frac{E[\bar{N}(t)]}{t} \geq \frac{1}{C_{2\alpha}\cdot u}.$$

Furthermore, when $M \to \infty$,

$$E_\mu[\bar{N}(t)] = E_\mu[N(t)].$$

It implies that

$$\underline{\lim}_{t\to\infty} \frac{E[N(t)]}{t} \geq \frac{1}{C_{2\alpha} \cdot u}.$$

When

$$C_{2\alpha} \cdot u_M \cdot (E_\mu[\bar{N}(t)] + 1) \leq t + M,$$

we have

$$\overline{\lim_{t\to\infty}} \frac{E[\bar{N}(t)]}{t} \leq \frac{1}{C_{2\alpha} \cdot u_M}.$$

Since $\bar{S}_n \leq S_n$, $\bar{N}(t) \geq N(t)$, we know

$$E_\mu[\bar{N}(t)] \geq E_\mu[N(t)].$$

It implies that

$$\overline{\lim_{t\to\infty}} \frac{E[N(t)]}{t} \leq \frac{1}{C_{2\alpha} \cdot u_M}.$$

Let $M \to \infty$, we can have

$$\overline{\lim_{t\to\infty}} \frac{E[N(t)]}{t} \leq \frac{1}{C_{2\alpha} \cdot u}.$$

From above mention, one can conclude that

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{C_{2\alpha} \cdot u}.$$

The proof of the theorem is complete. □

4. Fuzzy renewal reward process

Let $\{N(t), t > 0\}$ be a renewal process with interarrival times $\{\xi_k\}$, and assume that each time a renewal occurs we receive a positive reward. We shall interpret η_k as the reward earned at each time of the kth renewal, and suppose that $\{\eta_k\}$ is a sequence of positive independent and identical distributed q-random variables.

Definition 4.1. Let $\xi_1,\ \xi_2,\ \cdots$ be positive independent and identical distributed interarrival times, and let $\eta_1,\ \eta_2,\ \cdots$ be positive independent and identical distributed rewards. Then

$$R(t) = \sum_{k=1}^{N(t)} \eta_k$$

is called a renewal reward process, where $N(t)$ is the renewal process with interarrival times $\xi_1,\ \xi_2,\ \cdots$.

Usually, a renewal reward process $R(t)$ denotes the total reward earned by time t [13].

Theorem 4.2. *Assume that $R(t)$ is a renewal reward process with interarrival times $\xi_1,\ \xi_2,\ \cdots$ and rewards $\eta_1,\ \eta_2,\ \cdots$, noting that*

$$A = \{\lim_{t\to\infty} N(t) = +\infty\},$$

if $N(t) \in A$, $E\eta_1 < +\infty, E\xi_1 < +\infty$, then the reward rate

$$\frac{R(t)}{t} \to \frac{E\eta_1}{E\xi_1} \qquad (\mu).$$

Proof. It follows from [20] that for all $\varepsilon \in (0,\ 1)$,

$$\begin{aligned}
& \left\{\left|\frac{R(t)}{t} - \frac{E\eta_1}{E\xi_1}\right| \geq \varepsilon\right\} \\
\subseteq \quad & \{|\frac{N(t)}{t} - \frac{1}{E\xi_1}| \geq \frac{\varepsilon}{3}\} \cup \{|\frac{R(t)}{N(t)} - E\eta_1| \geq \frac{\varepsilon}{3}\} \cup \{|\frac{1}{E\xi_1}(\frac{R(t)}{N(t)} - E\eta_1)| \geq \frac{\varepsilon}{3}\} \\
& \cup\{|\frac{N(t)}{t} - \frac{1}{E\xi_1}| \geq \frac{\varepsilon}{3E\eta_1}\}.
\end{aligned}$$

Since θ, θ^{-1} are continuous and strictly increasing, we have

$$\begin{aligned}
& \lim_{t\to\infty} \theta^{-1}(\theta \circ \mu) \left\{\left|\frac{R(t)}{t} - \frac{E\eta_1}{E\xi_1}\right| \geq \varepsilon\right\} \\
\leq \quad & \lim_{t\to\infty} \theta^{-1}(\theta \circ \mu)(\{|\frac{N(t)}{t} - \frac{1}{E\xi_1}| \geq \frac{\varepsilon}{3}\} \cup \{|\frac{R(t)}{N(t)} - E\eta_1| \geq \frac{\varepsilon}{3}\} \\
& \cup\{|\frac{1}{E\xi_1}(\frac{R(t)}{N(t)} - E\eta_1)| \geq \frac{\varepsilon}{3}\} \cup \{|\frac{N(t)}{t} - \frac{1}{E\xi_1}| \geq \frac{\varepsilon}{3E\eta_1}\}) \\
\leq \quad & \lim_{t\to\infty} \theta^{-1}[(\theta \circ \mu)\{|\frac{N(t)}{t} - \frac{1}{E\xi_1}| \geq \frac{\varepsilon}{3}\} + (\theta \circ \mu)\{|\frac{R(t)}{N(t)} - E\eta_1| \geq \frac{\varepsilon}{3}\} \\
& +(\theta \circ \mu)\{|\frac{1}{E\xi_1}(\frac{R(t)}{N(t)} - E\eta_1)| \geq \frac{\varepsilon}{3}\} + (\theta \circ \mu)\{|\frac{N(t)}{t} - \frac{1}{E\xi_1}| \geq \frac{\varepsilon}{3E\eta_1}\}] \\
= \quad & \theta^{-1}[\theta(\lim_{t\to\infty} \mu\{|\frac{N(t)}{t} - \frac{1}{E\xi_1}| \geq \frac{\varepsilon}{3}\}) + \theta(\lim_{t\to\infty} \mu\{|\frac{R(t)}{N(t)} - E\eta_1| \geq \frac{\varepsilon}{3}\} \\
& +\theta(\lim_{t\to\infty} \mu\{|\frac{1}{E\xi_1}(\frac{R(t)}{N(t)} - E\eta_1)| \geq \frac{\varepsilon}{3}\} + \theta(\lim_{t\to\infty} \mu\{|\frac{N(t)}{t} - \frac{1}{E\xi_1}| \geq \frac{\varepsilon}{3E\eta_1}\}].
\end{aligned}$$

It follows from Theorem 3.3 that

$$\lim_{t\to\infty} \mu\{|\frac{R(t)}{N(t)} - E\eta_1| \geq \frac{\varepsilon}{3}\} = 0$$

and

$$\lim_{t\to\infty} \mu\{|\frac{1}{E\xi_1}(\frac{R(t)}{N(t)} - E\eta_1)| \geq \frac{\varepsilon}{3}\} = \lim_{t\to\infty} \mu\{|\frac{R(t)}{N(t)} - E\eta_1| \geq \frac{\varepsilon E\xi_1}{3}\} = 0.$$

Furthermore, by Theorem 3.5 and $\theta(0) = 0$, we have

$$\lim_{t\to\infty} \theta^{-1}(\theta \circ \mu)\{|\frac{R(t)}{t} - \frac{E\eta_1}{E\xi_1}| \geq \varepsilon\} = \theta^{-1}(0) = 0.$$

Finally, one can conclude that

$$\lim_{t\to\infty} \mu\{|\frac{R(t)}{t} - \frac{E\eta_1}{E\xi_1}| \geq \varepsilon\} = 0,$$

which implies that

$$\frac{R(t)}{t} \to \frac{E\eta_1}{E\xi_1} \qquad (\mu).$$

Now the theorem is proved. □

Example 4.3. Suppose that $\{N(t), t \geq 0\}$ is a renewal process, $\delta(t)$ is the age at t. Now we compute the limitation:

$$\lim_{t\to\infty} \frac{\int_0^t \delta(s)ds}{t}.$$

In fact, we can denote that

$$R(t) = \int_0^t \delta(s)ds.$$

And ξ is the time of a renewal process, so the reward of ξ is

$$\eta = \int_0^\xi \delta(s)ds = \frac{\xi^2}{2},$$

according to theorem 4.2,

$$\lim_{t\to\infty} \frac{\int_0^t \delta(s)ds}{t} = \frac{E[\xi^2]}{2E[\xi]}.$$

5. Conclusions

It is known that, quasi-probability theory and probability theory are inherent different. The former based on the quasi-probability measure which is a nonadditive measure, and the latter based on the probability measure which is an additive measure. In the fuzzy renewal process, all the renewal theorems are derived with convergence in measure, while in the stochastic renewal process, all the renewal theorem are given with almost sure convergence. Interestingly, from the result comparisons in renewal process and renewal reward process, we can clearly see that the results in fuzzy renewal process we obtained share a high consistency with the corresponding results of stochastic renewal process.

In this paper, based on quasi-probability measure theory, we investigated the fuzzy renewal process and the fuzzy reward process for q-random variables. One key fuzzy renewal theorem was obtained, that is, a fuzzy elementary renewal theorem. One can see that, Some important items derived in fuzzy renewal theory such as average renewal time (Theorem 3.3), renewal rate (Theorem 3.5), expected renewal rate (Theorem 3.9) and reward rate (Theorem 4.2) own a high homology in convergence mode to the stochastic corresponding. Our future work is focused on further research the limitation of expected reward rate. All investigations helped to lay important theoretical foundations for the systematic and comprehensive development of quasi-probability measure theory.

Acknowledgements. This work was supported by National Nature Science Foundation of China (61375075); Natural Science Foundation of Hebei Province, China (A2014201166); Hebei Education Department , China (QN20131055).

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