Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 5, (May 2016), pp. 713–728
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

Rules for computing fixpoints of a fuzzy closure operator

Partha Ghosh, Krishna Kundu

Received 4 February 2015; Revised 3 June 2015; Accepted 8 November 2015

Abstract. Applications of fuzzy closure operators in several areas of fuzzy system are well known. Especially its applications in the field of formal concept analysis (FCA) are considered in this paper. It is known that in FCA, computation of all formal concepts from data table with graded attributes can be reduced to the problem of computing fix points of two fuzzy closure operators, $\uparrow\downarrow$ and $\downarrow\uparrow$. It is also true that as the size of datasets grows, the fuzzy concepts generated from fuzzy context become larger in number. Therefore for large and complex datasets, it is very hard to deal with such a large number of fuzzy concepts. In this paper, we focus on establishing some rules for computing fix points of $\uparrow\downarrow$, and $\downarrow\uparrow$ from a data table with graded attributes. The motivation is to reduce considerably the number of generated fuzzy concepts than the number of all fuzzy concepts. Considering Gödel operations over unit interval [0,1], we establish all the rules theoretically.

2010 AMS Classification: ????

Keywords: Fuzzy sets, Fix points, Formal Concept Analysis(FCA), Fuzzy concepts.

Corresponding Author: Partha Ghosh (partha_0377@rediffmail.com)

1. Introduction

Formal Concept Analysis (FCA) is a data analysis technique for discovering conceptual structures in a large amount of data. R. Wille[33] and B. Ganter[13] developed this idea in 1982. It is a method for data analysis, knowledge representation and information management. Data are repesented in the form of concept lattice after generating so-called formal concepts. Over last two decades the theoretical development of FCA has established the core theory to a stable state. In this paper we deal with FCA of data in fuzzy setting.

As far as our knowledge is concerned, Burusco and Fuentes-Ganzáles[7, 8] first introduced the theory of FCA in fuzzy setting. Later a feasible way has been emerged to develop FCA and related structure in fuzzy setting by the works of Pollandt[27] and, independently by Bělohlávek[2]. Generating fuzzy concepts from a given data with fuzzy attributes is one of the fundamental problems in the theory of fuzzy concept lattice. Since fuzzy concepts are the fix points of a particular fuzzy operator that is associated with input data[1], the problem of generating fuzzy concepts turn out to be the problem of computing all fix points of this operator. Two important works have been reported (by Bělohlávek et al.[3, 4] . They evaluated experimentally or theoretically all fix points of a fuzzy closure. Inspired by a well known observation that each fuzzy concept lattice can be viewed as a crisp concept lattice [see [5] and [27], Bělohlávek extended Ganter's NEXTCLOSURE algorithm in fuzzy setting to compute all fix points of a fuzzy closure. He first transformed a fuzzy context into a formal context. Then he used Ganter's NEXTCLOSURE [see [12, 13]] algorithm to generate all fix points of an ordinary closure. Finally he transformed all fix points of ordinary closure back to fix points of a fuzzy closure. Lindig's Next Neighbor algorithm [22], motivated Bělohlávek et al.[4] to consider one more approach for generating all fix points of a fuzzy closure together with their subconcept-superconcept hierarchy. In [4], they also showed that the computing time only for generating fuzzy concepts by using the algorithm in [3] is less than that by using the algorithm in [4]. But for the case of generating fuzzy concepts along with their concept hierarchy, the algorithm proposed in [4] is considerably faster than that in [3]. The computational aspects concerning algorithms for generating all fix points of a fuzzy closure operator, along with their partial ordering are discussed in the paper by Belohlavek et al.[6]. Recently some interesting approaches are proposed to the fuzzy concept lattice theory [9, 14, 15, 21, 24, 28, 29, 30]. An extensive overview of the papers published between 2003 and 2011 on FCA with fuzzy attributes and rough FCA in knowledge discovery etc. can be found in [26]. Very recently some articles [23, 31, 32] have been published on reduction of formal concepts and concept lattice in formal concept analysis with fuzzy setting. In [23], the the authors studied the reduction of the concept lattices based on rough set theory [11, 20, 25] and proposed two kinds of reduction methods for the concept lattices. In [31], the authors proposed a method for reducing the number of formal concepts in formal concept analysis with fuzzy setting using Shanon entropy. A method for reducing the size of fuzzy concept lattice are proposed in [32] using Shannon entropy and Huffman coding.

In this paper we propose some rules for generating the fix points of two fuzzy closure operators $\uparrow\downarrow$ and $\downarrow\uparrow$ which would help in deducing the fuzzy concepts in fuzzy FCA. We establish all of these rules theoretically. All of our proposed rules are based on Gödel algebra $\mathbf{L} = \langle L, \wedge, \vee, \bigotimes, \rightarrow, 0, 1\rangle$.

The paper is organized as follows. In Section 2, some preliminary notion on fuzzy logic, fuzzy sets and Fuzzy concept lattice are recalled. In Section 3, the rules for computing fix points of two fuzzy closures are introduced and established. Finally in Section 4, we discuss the proposed rules for computing fix points with an illustration.

2. Mathematical background – explanations on the fundamentals applied

2.1. **Basics of fuzzy logic, fuzzy sets.** In this section we recall the basics of fuzzy sets and fuzzy logic (for more extensive overviews see the references [10, 16, 17, 19])

as needed for this paper.
Since fuzzy logic are developed using general structure of truth degree. We use a complete residuated lattice [18] as a basic structure of truth degree. A complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that (1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; (2) $\langle L, \otimes, 1\rangle$ is a monoid; (3) $\otimes$ and $\rightarrow$ satisfy so called adjointness property, i.e., $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$, for each $a, b, c \in L$. Operations $\otimes$ and $\rightarrow$ are known as "fuzzy conjunction" and "fuzzy implication". All elements a of L are called truth degrees. Usually, the common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$, with $\vee$ and $\wedge$ being maximum and minimum, respectively, $\otimes$ being a left-continuous $t-$norm with the corresponding $\rightarrow$. One of the most important pairs of adjoint operation on $[0, 1]$ is due to Gödel: $a \otimes b = min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = b$ else. One may consider a finite set $\{a_0 = 0, a_1, ..., a_n = 1\}(a_0 < a_1 < ..... < a_n)$ as the set of truth values with $\otimes$ given by $a \otimes b = a_{min(k,l)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_n$ for $a_k \leq a_l$ and $a_k \rightarrow a_l = a_l$ otherwise. Such an $\mathbf{L}$ is called a finite Gödel chain.
Now based on the structure of complete residuated Lattice $\mathbf{L}$, we present the basic notions of $\mathbf{L}$-set and fuzzy relation. An $\mathbf{L}$-set [13] A in a universe set X is a mapping $A : X \rightarrow L$. $A(x)$ is called the truth value (or membership value) of x in A which maps X to the membership space L. Similarly, an $\mathbf{L}$-relation I is a mapping $I : X \times Y \rightarrow L$ assigning to any $x \in X$ and $y \in Y$ a truth value $I(x, y)$ to which x and y is related under I. The collection of all $\mathbf{L}$-sets in X is denoted by the set L^X. For every $t \in L$, $A^t = \{x \in X \mid A(x) \geq t\}$ are called level sets or t-cut of A. We let $supp(A) = \{x \in X \mid A(x) > 0\}$. We call $supp(A)$ the *support* of A. An $\mathbf{L}$-set A is nontrivial if $supp(A) \neq \varnothing$. We use the notation $\bigvee$ for supremum and $\bigwedge$ for infimum. Let A_1 and A_2 be any two $\mathbf{L}$-sets of X. Then $A_1 \subseteq A_2$ if $A_1(x) \leq A_2(x)$ for all $x \in X$. The union $A_1 \cup A_2$ of A_1, $A_2 \in L^X$ is a subset of X defined by $(A_1 \cup A_2)(x) = A_1(x) \vee A_2(x)$ for all $x \in X$ and intersection $A_1 \cap A_2$ of A_1, A_2 is also a subset of X defined by $(A_1 \cap A_2)(x) = A_1(x) \wedge A_2(x)$ for all $x \in X$.

2.2. **Fuzzy contexts and fuzzy concepts.** Formal Concept Analysis is a mathematical technique based on lattice theory. It aims to formulate the philosophical understanding of a concept as a unit of two parts: its extent (the set of the objects which fall under this concept) and its intent (the set of attributes covered by this concept). In addition, certain objects have certain attributes; in other words, objects are related to attributes. The sets of objects and attributes together with their relation to each other form a "formal context". Ganter-Wille's approach was based on bivalent logic, in which objects (attributes) either crisply belong or not to the extent (intent) of the concept. But many of the information people facing are usually fuzzy and imprecise, so can not be described by a concept in the formal setting, e.g., if we consider the concept "POLITICAL LEADER" then the attributes (say, contributions in the society, contributions in the economy, power of leadership etc.) of "POLITICAL LEADER" can not be delineated. Therefore, it would not be the proper way to analyze the intent by bivalent logic. By introducing Fuzzy sets into formal context, one can express the fuzzy characteristic between the objects and attributes. The theory of concept lattices has been generalized from the point of view of fuzzy logic in [1, 2, 7, 8, 27]. In this sub-section we recall the basics of fuzzy concept lattice.

We start with a set X of objects, a set Y of attributes, a complete residuated lattice $\mathbf{L}$ and a fuzzy relation I between X and Y. The key idea of a fuzzy context ($\mathbf{L}$-context) is as follows: it is a triplet $\langle X, Y, I\rangle$, where $I(x, y) \in L$ (the set of truth values of complete residuated lattice L) is interpreted as the truth value of the fact, "the object $x \in X$ has the attribute $y \in Y$ ". For fuzzy sets $A \in L^X$ and $B \in L^Y$, Bělohlávek [1] and, independently, Pollandt[27] defined the fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ by

$A^{\uparrow}(y) = \bigwedge_{x\in X}\{A(x) \to I(x, y)\}$,

$B^{\downarrow}(x) = \bigwedge_{y\in Y}\{B(y) \to I(x, y)\}$.

One can easily interpret the element $A^{\uparrow}(y) \in A^{\uparrow}$ as the truth degree of "y is shared by all objects from A"and $B^{\downarrow}(x) \in B^{\downarrow}$ as the truth degree of "x has all attributes from B".

A fuzzy concept $\langle A, B\rangle$ consists of a fuzzy set A of objects (the extent of the concept) and a fuzzy set B of attributes (the intent of the concept) such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$. If $B\langle X, Y, I\rangle = \{\langle A, B\rangle\,|A^{\uparrow} = B, B^{\downarrow} = A\}$ denotes the set of all fuzzy concepts of the fuzzy context $\langle X, Y, I\rangle$, then the set $B\langle X, Y, I\rangle$ with the order relation:

$\langle A_1, B_1\rangle \le \langle A_2, B_2\rangle$ if and only if $A_1 \subseteq A_2$ (or, equivalently $B_1 \supseteq B_2$) is a complete lattice. The lattice $(B\langle X, Y, I\rangle\,, \le)$ is called a fuzzy concept lattice.

Example 2.1. Consider the following fuzzy context $\langle X, Y, I\rangle$ given by Table 1, inwhich the rows represent objects, and columns represent attributes of these objects. The truth values of the fuzzy context have been chosen randomly from [0,1].

Table 1. Fuzzy context

	1	2	3	4	5
1	0.9	0.7	0.2	0.4	1
2	0.8	1	0.3	0.7	0.9
3	0.2	0.2	0.2	0.1	0.3
4	0.3	0.6	0.3	0.2	0.2
5	0.5	0.8	0.4	0.3	0.4

Considering Gödel fuzzy logical connectives we see that
$\langle\{0.2/1, 0.9/2, 0.1/3, 0.2/4, 0.3/5\}, \{0.8/1, 1/2, 0.3/3, 0.7/4, 1/5\}\rangle$
is a fuzzy concept. $\{0.2/1, 0.9/2, 0.1/3, 0.2/4, 0.3/5\}$ is the extent of this fuzzy concept, and $\{0.8/1, 1/2, 0.3/3, 0.7/4, 1/5\}$ is the intent.

3. Proposed rules for computing all fix points of the fuzzy closure operators $\uparrow\downarrow$ and $\downarrow\uparrow$ from L-context

A fuzzy concept is completely determined by its extent, or by its intent. It is also a well-known fact that A is an extent iff A is a fixed point of $\uparrow\downarrow$: $L^X \to L^X$, i.e.,

$A^{\uparrow\downarrow} = A$, and B is an intent iff B is a fixed point of $\downarrow\uparrow$: $L^Y \to L^Y$, i.e., $B^{\downarrow\uparrow} = B$ [see in 2]. Therefore, in order to compute a fuzzy concept, it is sufficient to compute its extent. Some rules are presented for computing fix points of $\uparrow\downarrow$, and $\downarrow\uparrow$ based on Gödel operations. The main purpose of consideration of Gödel operations is given below:

If we use Gödel algebra to generate all fix points of the closure operator $\uparrow\downarrow$ from L-context, then it is obvious from the definition of $\uparrow\downarrow$ and definition of Gödel implication that membership values of each object in the fix points must be either 1 or some values of L. It is also true that for any L-context, there always exists a fix point of $\uparrow\downarrow$ in which the membership value of each object is the smallest truth value of each row of the L-context. Considering this fix point as an initial fix point, we think about any other fix points in which the membership value of any object, say $x_k \in X$, is different from 1 as well as smallest truth value of the row corresponding to the object $x_k \in X$ in the L-context table. If such type of fix points exist, then membership values of $x_k \in X$ in those fix points should be second smallest truth value, third smallest truth value, fourth smallest truth value and so on of the row corresponding to the object $x_k \in X$ in the L-context table. Based on these observations, we establish a set of rules for generating fix points of $\uparrow\downarrow$ so that we can easily compute the membership values of the fix points directly from L-context. Analogously, if we use Gödel algebra for generating the fix points of the closure operator $\downarrow\uparrow$ from L-context, then membership value of each attribute can be computed directly from L-context by considering all the smallest truth values of each column of the L-context.

Let us first introduce the notations used for computing fix points of $\uparrow\downarrow$ and $\downarrow\uparrow$. Since, the input for our analysis is an **L**-context $\langle X, Y, I\rangle$. Without loss of generality, we take $X = \{1, 2, ..., m\}$ and $Y = \{1, 2, ..., n\}$ be the sets of objects and attributes, respectively. Then $I(i, j) \in L$ is interpreted as the truth value of the fact, "the object $i \in X$ has the property $j \in Y$ ". Now, we denote each smallest value of the row corresponding to the object $i \in X$ by $r[i1]$, where $i = 1, 2, ..., m$. Similarly, $r[i2], r[i3], ..., r[in]$ denotes the second smallest, third smallest,,...,n-th smallest value of the row corresponding to the object $i \in X$, respectively. Also for each $i \in X$, we denote $col[r[ip]]$ as appearing column of $r[ip]$ in the **L**-context table, where $p = 1, 2, ..., n$. There may be several equal truth values in the row corresponding to the object $i \in X$. But for those equal truth values we always select each $r[ip]$ at random. From context table 1, we can see that for $i = 1$, $r[11] = 0.2, r[12] = 0.4, r[13] = 0.7, r[14] = 0.9, r[15] = 1$, and $col[r[11]] = 3, col[r[12]] = 4, col[r[13]] = 2, col[r[14]] = 1, col[r[15]] = 5$. Obviously, the column, $col[r[ip]]$ must corresponds some attribute $j \in Y$. In the similar manner, for each $j \in Y = \{1, 2, ..., n\}$, we denote the smallest, second smallest,...,m-th smallest value of the column corresponding to the column j by $c[1j], c[2j], ..., [mj]$, respectively, and for each $j \in Y$, $row[c[qj]]$ denotes the appearing row of $c[qj]$ in the **L**-context table, where $q = 1, 2, ..., m$. Again, there may be several equal truth values in the column corresponding to the attribute $j \in Y$. But for those equal truth values we always select $c[qj]$ at random. In context table 1, for $j = 1$, $c[11] = 0.2, c[21] = 0.3, c[31] = 0.5, c[41] = 0.8, c[51] = 0.9$, and $row[c[11]] = 3, row[c[21]] = 4, row[c[31]] = 5, row[c[41]] = 2, row[c[51]] = 1$. With these notation and using Gödel operation on L we now state and prove the following all rules for computing fix points of the fuzzy closure operators $\uparrow\downarrow$ and $\downarrow\uparrow$ from a **L**-context.

Rule 1. *Let $\langle X, Y, I\rangle$ be an* **L**-*context and $j \in Y$ be an attribute. Then $A_j \in L^X$ defined by $A_j(i) = I(i, j)$ for each $i \in X$ is a fix point of $\uparrow\downarrow$.*
Analogously, let $\langle X, Y, I\rangle$ be an **L**-*context and $i \in X$ be an object. Then $B_i \in L^Y$ defined by $B_i(j) = I(i, j)$ for each $j \in Y$ is a fix point of $\downarrow\uparrow$.*

Proof. To show that A is a fix point of $\uparrow\downarrow$, we need to prove $A^{\uparrow\downarrow} = A$.
Let for any $l \in Y$, $A_l(i) = I(i, l)$ for all $i \in X$. Then $A_l^{\uparrow}(l) = 1$ and thus $A_l^{\uparrow}(l) \to I(i, l) = I(i, l)$. Now for $j \in Y - \{l\}$, $A_l^{\uparrow}(j) \to I(i, j) = 1$, or $I(i, j)$. This is also obvious that if $A_l^{\uparrow}(j) \to I(i, j) = 1$, then for each $l \in Y$, $A_l^{\uparrow\downarrow}(i) = I(i, l) = A_l(i)$, where $i \in X$. Hence A_l is a fix point of $\uparrow\downarrow$. Now, we let $A_l^{\uparrow}(j) \to I(i, j) = I(i, j)$ for some $j \in Y - \{l\}$. For this case, to prove A_l is a fix point of $\uparrow\downarrow$, we need to prove that $I(i, j) \not< I(i, l)$ for those j which gives $A_l^{\uparrow}(j) \to I(i, j) = I(i, j)$. This is true, because if $I(i, j) < I(i, l)$, then $A_l^{\uparrow}(j) \leq I(i, l)$ which is a contradiction, since from Gödel implication, $A_l^{\uparrow}(j) \to I(i, j) = I(i, j)$ implies $A_l^{\uparrow}(j) > I(i, l)$. Thus, if $A_l^{\uparrow}(j) \to I(i, j) = I(i, j)$ for some or all $j \in Y - \{l\}$, then $I(i, j) \geq I(i, l)$. Hence $A_l^{\uparrow\downarrow}(i) = I(i, l) = A_l(i)$, i.e., A_l is a fix point of $\uparrow\downarrow$.
Proof is similar for analogous part. □

In context table 1, we can easily check that for each $j \in Y$,
$A_1 = \{I(1,1)/1, I(2,1)/2, I(3,1)/3, I(4,1)/4, I(5,1)/5\}$
$=\{0.9/1, 0.8/2, 0.2/3, 0.3/4, 0.5/5\}$,
$A_2 = \{I(1,2)/1, I(2,2)/2, I(3,2)/3, I(4,2)/4, I(5,2)/5\}$
$=\{0.7/1, 1/2, 0.2/3, 0.6/4, 0.8/5\}$,
$A_3 = \{I(1,3)/1, I(2,3)/2, I(3,3)/3, I(4,3)/4, I(5,3)/5\}$
$=\{0.2/1, 0.3/2, 0.2/3, 0.3/4, 0.4/5\}$,
$A_4 = \{I(1,4)/1, I(2,4)/2, I(3,4)/3, I(4,4)/4, I(5,4)/5\}$
$=\{0.4/1, 0.7/2, 0.1/3, 0.2/4, 0.3/5\}$
$A_5 = \{I(1,5)/1, I(2,5)/2, I(3,5)/3, I(4,5)/4, I(5,5)/5\}$
$=\{1/1, 0.9/2, 0.3/3, 0.2/4, 0.4/5\}$
are fix points of $\uparrow\downarrow$.
Similarly, we can also check that for each $i \in X$
$B_1 = \{I(1,1)/1, I(1,2)/2, I(1,3)/3, I(1,4)/4, I(1,5)/5\}$
$=\{0.9/1, 0.7/2, 0.2/3, 0.4/4, 1/5\}$,
$B_2 = \{I(2,1)/1, I(2,2)/2, I(2,3)/3, I(2,4)/4, I(2,5)/5\}$
$=\{0.8/1, 1/2, 0.3/3, 0.7/4, 0.9/5\}$,
$B_3 = \{I(3,1)/1, I(3,2)/2, I(3,3)/3, I(3,4)/4, I(3,5)/5\}$
$=\{0.2/1, 0.2/2, 0.2/3, 0.1/4, 0.3/5\}$,
$B_4 = \{I(4,1)/1, I(4,2)/2, I(4,3)/3, I(4,4)/4, I(4,5)/5\}$
$=\{0.3/1, 0.6/2, 0.3/3, 0.2/4, 0.2/5\}$,
$B_5 = \{I(5,1)/1, I(5,2)/2, I(5,3)/3, I(5,4)/4, I(5,5)/5\}$

$=\{0.5/1, 0.8/2, 0.4/3, 0.3/4, 0.4/5\}$
are fix points of $\uparrow\downarrow$.

Rule 2. *Let $\langle X, Y, I\rangle$ be an **L**-context, $k \in X$ be an object, and $p < n$ be a positive integer. Also, let for each $p' \in \{1, 2, ..., p\}$, $A_{k,p'} \in L^X$ is a fix point of $\uparrow\downarrow$, where*

$$A_{k,p'}(i) = \begin{cases} r[i1] \; for \; i \in X - \{k\} \\ r[kp'] \; if \; i = k. \end{cases}$$

If for $i \in X - \{k\}$ none of $r[i1]$ appears in the column of $r[kp]$, then $A_{k,p+1} \in L^X$ defined by

$$A_{k,p+1}(i) = \begin{cases} r[i1] \; for \; i \in X - \{k\} \\ r[k(p+1)] \; if \; i = k \end{cases}$$

*is a fixpoint of $\uparrow\downarrow$. Analogously, let $\langle X, Y, I\rangle$ be an **L**-context, $l \in Y$ be an attribute, and $q < m$ be a positive integer. Also, let for each $q' \in \{1, 2, ..., q\}$, $B_{l,q'} \in L^Y$ are fixpoints of $\downarrow\uparrow$, where*

$$B_{l,q'}(j) = \begin{cases} c[1j] \; for \; j \in Y - \{l\} \\ c[q'l] \; if \; j = l. \end{cases}$$

If for $j \in Y - \{l\}$ none of $c[1j]$ appears in the row of $c[ql]$, then $B_{l,q+1} \in L^Y$ defined by

$$B_{l,q+1}(i) = \begin{cases} c[1j] \; for \; j \in Y - \{l\} \\ c[(q+1)l] \; if \; j = l \end{cases}$$

is a fixpoint of $\downarrow\uparrow$.

Proof. To show that $A_{k,p+1}$ is a fixed point of $\uparrow\downarrow$, we need to prove $A_{k,p+1}^{\uparrow\downarrow} = A_{k,p+1}$. Since for each $j \in Y$,

$$A_{k,p+1}^{\uparrow}(j) = \bigwedge\{r[11] \to I(1, j), ..., A_{k,p+1}(k) \to I(k, j), ..., r[m1] \to I(m, j)\}$$

and

$$r[i1] \leq I(i, j) \text{ for } i \in X - \{k\},$$

for all $j \in Y$, $r[i1] \to I(i, j) = 1$ for $i \in X - \{k\}$. Again $A_{k,p+1}(k) = r[k(p + 1)]$. If $r[k(p + 1)] = r[kp]$, then the rule follows. We assume that $r[k(p + 1)] \neq r[kp]$. Then $A_{k,p+1}(k) = r[k(p + 1) > r[kp']$ for each $p' \in \{1, 2, ..., p\}$. Now, for each $p' \in \{1, 2, ..., p\}$, $A_{k,p+1}^{\uparrow}(col[r[kp']]) = r[kp']$. Let $Y' = \{col[r[kp']] \in Y \mid p' \in \{1, 2, ..., p\}\}$. Then $A_{k,p+1}^{\uparrow}(j) = 1$ for $j \in Y - Y'$. Thus

$$
\begin{aligned}
A^{\uparrow\downarrow}_{k,p+1}(k) &= \bigwedge_{j\in Y=Y'\bigcup(Y-Y')} (A^{\uparrow}_{k,p+1}(j)\to I(k,j)) \\
&= \bigwedge_{j\in Y-Y'} (A^{\uparrow}_{k,p+1}(j)\to I(k,j)) \wedge \bigwedge_{j\in Y'} (A^{\uparrow}_{k,p+1}(j)\to I(k,j)) \\
&= \bigwedge_{j\in Y-Y'} (1\to I(k,j)) \wedge \bigwedge_{j\in Y'} (r[kj]\to I(k,col[r[kj]])) \\
&= \bigwedge_{j\in Y-Y'} (1\to I(k,j)) \wedge \bigwedge_{j\in Y'} (r[kj]\to r[kj]) \\
&= \bigwedge\{1\to r[k(p+1)], 1\to r[k(p+2)], ..., 1\to r[kn]\}\wedge 1 \\
&= r[k(p+1)] = A_{k,p+1}(k).
\end{aligned}
$$

Let $Y'' = Y - \{col[r[kp]]\}$. Since $A_{k,p}\in L^X$ is a fix point of $\uparrow\downarrow$, where

$$
A_{k,p}(i) = \begin{cases} r[i1]\ for\ i\in X-\{k\} \\ r[kp]\ if\ i=k, \end{cases}
$$

$A^{\uparrow\downarrow}_{k,p}(k) = r[kp]$ and $A^{\uparrow\downarrow}_{k,p}(i) = \bigwedge_{j\in Y=Y''\bigcup(Y-Y'')} (A^{\uparrow}_{k,p}(j)\to I(i,j)) = r[i1]$ for $i\in X-\{k\}$.

Now for $i\in X-\{k\}$,

$$
\begin{aligned}
&A^{\uparrow\downarrow}_{k,p+1}(i) \\
=& \bigwedge_{j\in Y=Y'\bigcup(Y-Y')} (A^{\uparrow}_{k,p+1}(j)\to I(i,j)) \\
=& \bigwedge_{j\in Y=Y''\bigcup(Y-Y'')} (A^{\uparrow}_{k,p+1}(j)\to I(i,j)) \wedge (A^{\uparrow}_{k,p+1}(col[r[kp]])\to I(i,col[r[kp]])) \\
=& \bigwedge_{j\in Y''\bigcup(Y-Y'')} (A^{\uparrow}_{k,p+1}(j)\to I(i,j)) \wedge (r[kp]\to I(i,col[r[kp]])).
\end{aligned}
$$

Since for $i\in X-\{k\}$ none of $r[i1]$ appears in $col[r[kp]]$ which gives $I(i,col[r[kp]])\neq r[i1]$. Also $A_{k,p}\in L^X$ is a fix point of $\uparrow\downarrow$ and $A_{k,p}(i) = A_{k,p+1}(i)$ for $i\in X-\{k\}$. Thus for all $i\in X-\{k\}$,

$$
\bigwedge_{j\in Y''\bigcup(Y-Y'')} (A^{\uparrow}_{k,p+1}(j)\to I(i,j)) = r[i1], \text{ i.e., } A^{\uparrow\downarrow}_{k,p+1}(i) = A_{k,p+1}(i).
$$

Hence $A_{k,p+1}$ is a fix point of $\uparrow\downarrow$.

Proof is similar for analogous part. □

In context table 1, if we consider the object $2\in X$, then for each $p'\in\{1,2,3\}$, we can easily check that each $A_{2,p'}\in L^X$ is a fix points of $\uparrow\downarrow$, where

$$
A_{2,p'}(i) = \begin{cases} r[i1]\ for\ i\in X-\{2\} \\ r[2p']\ if\ i=2 \end{cases}
$$

i.e., $A_{2,1} = \{0.2/1, 0.3/2, 0.1/3, 0.2/4, 0.3/5\}$, $A_{2,2} = \{0.2/1, 0.7/2, 0.1/3, 0.2/4, 0.3/5\}$, and $A_{2,3} = \{0.2/1, 0.8/2, 0.1/3, 0.2/4, 0.3/5\}$ are fix points of $\uparrow\downarrow$. Now we can see that for $i \in X - \{2\}$ none of $r[i1]$ appears in $col[r[23]] = 1$. Thus using Rule 2, $A_{2,4} \in L^X$ defined by

$$A_{2,4}(i) = \begin{cases} r[i1]\ for\ i \in X - \{2\} \\ r[24]\ if\ i = 2 \end{cases}$$

i.e., $A_{2,4} = \{0.2/1, 0.9/2, 0.1/3, 0.2/4, 0.3/5\}$ is a fix point of $\uparrow\downarrow$.

Rule 3. *Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context, $k \in X$ be an object, and $p < n$ be a positive integer. Also, let for each $p' \in \{1, 2, ..., p\}$, $A_{k,p'} \in L^X$ is a fix point of $\uparrow\downarrow$, where*

$$A_{k,p'}(i) = \begin{cases} r[i1]\ for\ i \in X - \{k\} \\ r[kp']\ if\ i = k. \end{cases}$$

If for $i \in X - \{k\}$ some $r[i1]$ appears in the column of $r[kp]$ but each of these r[i1] are less than $r[kp]$, then $A_{k,p+1} \in L^X$ defined by

$$A_{k,p+1}(i) = \begin{cases} r[i1]\ for\ i \in X - \{k\} \\ r[k(p+1)]\ if\ i = k \end{cases}$$

is a fixpoint of $\uparrow\downarrow$.

Analogously, let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context, $l \in Y$ be an attribute, and $q < m$ be a positive integer. Also, let for each $q' \in \{1, 2, ..., q\}$, $B_{l,q'} \in L^Y$ are fix points of $\downarrow\uparrow$, where

$$B_{l,q'}(j) = \begin{cases} c[1j]\ for\ j \in Y - \{l\} \\ c[q'l]\ if\ j = l. \end{cases}$$

If for $j \in Y - \{l\}$ some $c[1j]$ appears in the row of $c[ql]$ but each of these $c[1j]$ are less than $c[ql]$, then $B_{l,q+1} \in L^Y$ defined by

$$B_{l,q+1}(i) = \begin{cases} c[1j]\ for\ j \in Y - \{l\} \\ c[(q+1)l]\ if\ j = l \end{cases}$$

is a fixpoint of $\downarrow\uparrow$.

Proof. To show that $A_{k,p+1}$ is a fixed point of $\uparrow\downarrow$, we need to prove $A^{\uparrow\downarrow}_{k,p+1} = A_{k,p+1}$. Since for each $j \in Y$,

$$A^{\uparrow}_{k,p+1}(j) = \bigwedge\{r[11] \to I(1, j), ..., A_{k,p+1}(k) \to I(k, j), ..., r[m1] \to I(m, j)\}$$

and

$$r[i1] \leq I(i, j) \text{ for } i \in X - \{k\},$$

for all $j \in Y$, $r[i1] \to I(i, j) = 1$ for $i \in X - \{k\}$. Again $A_{k,p+1}(k) = r[k(p + 1)]$. If $r[k(p + 1)] = r[kp]$, then the rule follows. We assume that $r[k(p + 1)] \neq r[kp]$. Then $A_{k,p+1}(k) > r[kp']$ for each $p' \in \{1, 2, ..., p\}$. Now, for each $p' \in \{1, 2, ..., p\}$,

$A^{\uparrow}_{k,p+1}(col[r[kp']]) = r[kp']$. Let $Y' = \{col[r[kp']] \in Y \mid p' \in \{1, 2, ..., p\}\}$. Then $A^{\uparrow}_{k,p+1}(j) = 1$ for $j \in Y - Y'$. Thus

$$\begin{aligned} A^{\uparrow\downarrow}_{k,p+1}(k) &= \bigwedge_{j \in Y = Y' \bigcup \{Y - Y'\}} (A^{\uparrow}_{k,p+1}(j) \to I(k,j)) \\ &= \bigwedge_{j \in Y - Y'} (A^{\uparrow}_{k,p+1}(j) \to I(k,j)) \wedge \bigwedge_{j \in Y'} (A^{\uparrow}_{k,p+1}(j) \to I(k,j)) \\ &= \bigwedge_{j \in Y - Y'} (1 \to I(k,j)) \wedge \bigwedge_{j \in Y'} (r[kj] \to I(k, col[r[kj)]])) \\ &= \bigwedge_{j \in Y - Y'} (1 \to I(k,j)) \wedge \bigwedge_{j \in Y'} (r[kj] \to r[kj]) \\ &= \bigwedge \{1 \to r[k(p+1)], 1 \to r[k(p+2)], ..., 1 \to r[kn]\} \wedge 1 \\ &= r[k(p+1)] = A_{k,p+1}(k). \end{aligned}$$

Since corresponding to some or all $i' \in X - \{k\}$, $r[i'1] < I(k, col[r[i'1]]) = r[kp](\neq r[k(p+1)])$. Without loss of generality, we assume $col[r[i'1]] = j' \in Y$. This implies that $I(i', j') = r[i'1]$ and $I(k, col[r[i'1]]) = I(k, j') = r[kp]$. Also, $A^{\uparrow}_{k,p+1}(j') = r[ip]$ and $r[kp] \to I(i', j') = r[kp] \to r[i'1] = r[i'1]$. Thus $A^{\uparrow\downarrow}_{k,p+1}(i') = r[i'1] = A_{k,p+1}(i')$. Also, since $col[r[kp']] \neq col[r[i1]]$ for remaining $i \in X - \{k\}$, $i \neq i'$, $A^{\uparrow}_{k,p+1}(col[r[ip]]) = 1$ and $A^{\uparrow\downarrow}_{k,p+1}(i) = r[i1] = A_{k,p+1}(i)$ for $i \in X - \{k\}$, $i \neq i'$. So $A_{k,p+1}$ is a fixed point of $\uparrow\downarrow$.

Proof is similar for analogous part. □

In context table 1, if we consider the object $5 \in X$, then for $p' \in \{1\}$ we can easily check that $A_{5,1} \in L^X$, where

$$A_{5,1}(i) = \begin{cases} r[i1] \; for \; i \in X - \{5\} \\ r[51] \; if \; i = 5 \end{cases}$$

i.e., $A_{5,1} = \{0.2/1, 0.3/2, 0.1/3, 0.2/4, 0.3/5\}$ is a fixed point of $\uparrow\downarrow$.
Now we can see that $r[31]$ and $r[41]$ appears in $col[r[51]] = 4$ and both of them are less than $r[51]$. Therefore $A_{5,2} \in L^X$ defined by

$$A_{5,2}(i) = \begin{cases} r[i1] \; for \; i \in X - \{5\} \\ r[52] \; if \; i = 5 \end{cases}$$

i.e., $A_{5,2} = \{0.2/1, 0.3/2, 0.1/3, 0.2/4, 0.4/5\}$ is a fix point of $\uparrow\downarrow$. Since $r[52] = r[53] = 0.4$, we may select any of them as second smallest element of the fifth row irrespective of any choice.

Rule 4. *Let $\langle X, Y, I \rangle$ be an* **L**-*context, $k \in X$ be an object, and $p < n$ be a positive integer. Also, let for each $p' \in \{1, 2, ..., p\}$, $A_{k,p'} \in L^X$ are fix points of $\uparrow\downarrow$, where*

$$A_{k,p'}(i) = \begin{cases} r[i1] \; for \; i \in X - \{k\} \\ r[kp'] \; if \; i = k. \end{cases}$$

If for $i \in X - \{k\}$ some $r[i1]$ appears in the column of $r[kp]$ and none of these $r[i1]$

are greater than $r[kp]$ *but may be equal with* $r[kp]$*, then* $A_{k,p+1} \in L^X$ *defined by*

$$A_{k,p+1}(i) = \begin{cases} r[i2] \; for \; i \in X' = \{i \in X - \{k\} | col[r[i1] = col[r[kp]] \; and \; r[i1] = r[kp]\} \\ r[k(p+1)] \; if \; i = k \\ r[i1] \; for \; i \in X - (\{k\} \cup X') \end{cases}$$

is a fixpoint of $\uparrow\downarrow$*.*

Analogously, let $\langle X, Y, I \rangle$ *be an* **L***-context,* $l \in Y$ *be an attribute, and* $q < m$ *be a positive integer. Also, let for each* $q' \in \{1, 2, ..., q\}$*,* $B_{l,q'} \in L^Y$ *are fix points of* $\downarrow\uparrow$*, where*

$$B_{l,q'}(j) = \begin{cases} c[1j] \; for \; j \in Y - \{l\} \\ c[q'l] \; if \; j = l \end{cases}$$

If for $j \in Y - \{l\}$ *some or all* $c[1j]$ *appears in the row of* $c[ql]$ *and none of these* $c[1j]$ *greater than* $c[ql]$ *but may be equal with* $c[1j]$*, then* $B_{l,q+1} \in L^Y$ *defined by*

$$B_{l,q+1}(i) = \begin{cases} c[2j] \; for \; j \in Y' = \{j \in Y - \{l\} | row[c[1j]] = row[c[ql]] \; and \; c[1j] = c[ql]\} \\ c[(q+1)l] \; if \; j = l \\ c[1j] \; for \; j \in Y - (\{l\} \cup Y') \end{cases}$$

is a fixpoint of $\downarrow\uparrow$*.*

Proof. To show that $A_{k,p+1}$ is a fixed point of $\uparrow\downarrow$, we need to prove $A^{\uparrow\downarrow}_{k,p+1} = A_{k,p+1}$. Since for each $j \in Y$, $r[i1] \leq I(i,j)$ for $i \in X - X'$, where $X' = \{i \in X - \{k\} | col[r[i1] = col[r[kp]] \; and \; r[i1] = r[kp]\}$, for all $j \in Y$, $r[i1] \to I(i,j) = 1$, where $i \in X - X'$. Again, $A_{k,p+1}(k) = r[k(p+1)]$, and $A_{k,p+1}(i) = r[i2]$ for $i \in X' = \{i \in X - \{k\} | col[r[i1] = col[r[kp]] \; and \; r[i1] = r[kp]\}$. For $i \in X'$ if $r[kp] = r[k(p+1)] = r[i2]$, then obviously $A_{k,p+1}$ is a fix point of $\uparrow\downarrow$. Consider the case, when both of $r[k(p+1)]$ and $r[i2]$ are not equal to $r[kp](= r[i1])$, where $i \in X'$. In this case, $A^{\uparrow}_{k,p+1}(col[r[kp]]) = r[i1] = r[kp]$, $i \in X'$. Without loss of generality, we assume that $r[kp] \neq r[k(p+1)]$. Then $A^{\uparrow}_{k,p+1}(col[r[kp']]) = r[kp']$ for each $p' \in \{1, 2, ..., p-1\}$. Let $Y' = \{col[r[kp']] \in Y \,|\, p' \in \{1, 2, ..., p\}\}$. Then $A^{\uparrow}_{k,p+1}(j) = 1$ for $j \in Y - Y'$, $j \neq col[r[kp]]$. Again if $i \in X'$, then $r[i1] = I(k, col[r[i1]]) = r[kp]$. We put $col[r[i1]] = j'$. Thus

$$\begin{aligned} & A^{\uparrow\downarrow}_{k,p+1}(k) \\ &= \bigwedge_{j \in Y = Y' \bigcup (Y - Y') \bigcup \{col[r[kp]]\}} (A^{\uparrow}_{k,p+1}(j) \to I(k,j), r[kp] \to I(k, col[r[kp]])) \\ &= \bigwedge_{j \in Y - Y'} (A^{\uparrow}_{k,p+1}(j) \to I(k,j)) \wedge \bigwedge_{j \in Y'} (A^{\uparrow}_{k,p+1}(j) \to I(k,j)) \wedge 1 \\ &= \bigwedge_{j \in Y - Y'} (1 \to I(k,j)) \wedge \bigwedge_{j \in Y'} (r[kj] \to I(k, col[r[kj)]]) \\ &= r[k(p+1)] \wedge 1 \\ &= r[k(p+1)] = A_{k,p+1}(k). \end{aligned}$$

For $i \in X'$,

$$\begin{aligned}&A^{\uparrow\downarrow}_{k,p+1}(i)\\ &= \bigwedge\{1 \to I(i,1), ..., r[kp] \to I(i,j') = r[i1], ..., 1 \to I(i,n)\}\\ &= r[i2]\end{aligned}$$

and

$$A^{\uparrow\downarrow}_{k,p+1}(i) = \bigwedge_{j\in Y}(I(i,j)) = r[i1] = A_{k,p+1}(i) \text{ for remaining } i \in X - X'.$$

So $A_{k,p+1}$ is a fix point of $\uparrow\downarrow$.

Proof is similar for analogous part. □

In context table 1, if we want to compute fix points considering the attribute $2 \in Y$, then for $q' \in \{1\}$ we can easily check that $B_{1,1} \in L^Y$, where

$$B_{2,1}(j) = \begin{cases} c[1j] \; for \; j \in Y - \{2\} \\ c[12] \; if \; j = 2 \end{cases}$$

i.e., $B_{2,1} = \{0.2/1, 0.2/2, 0.2/3, 0.1/4, 0.2/5\}$ is a fix point of $\downarrow\uparrow$.

Now we can see that both $c[11]$ and $c[14]$ appears in $row[c[12]] = 3$, but $c[11] = c[12]$. Thus $B_{2,2} \in L^Y$ defined by

$$B_{2,2}(i) = \begin{cases} c[21] \; for \; j \in Y' = \{1\} \\ c[22] \; for \; j = 2 \\ c[1j] \; for \; j \in Y - \{2\} \cup Y'\} \end{cases}$$

i.e., $B_{2,2} = \{0.3/1, 0.6/2, 0.2/3, 0.1/4, 0.2/5\}$ is a fix point of $\downarrow\uparrow$.

Rule 5. *Let $\langle X, Y, I\rangle$ be an **L**-context, $k \in X$ be an object, and $p < n$ be a positive integer. Also, let for each $p' \in \{1, 2, ..., p\}$, $A_{k,p'} \in L^X$ are fix points of $\uparrow\downarrow$, where*

$$A_{k,p'}(i) = \begin{cases} r[i1] \; for \; i \in X - \{k\} \\ r[kp'] \; if \; i = k. \end{cases}$$

If for $i \in X - \{k\}$ some or all $r[i1]$ appears in the column of $r[kp]$ and at least one of these $r[i1]$, say $r[i'1](\neq r[i'2])$, $i' \in \{X\} - \{k\}$ is greater than $r[kp]$, then for any $p' \in \{p+1, ..., n\}$ the set $A_{k,p'} \in L^X$ defined by

$$A_{k,p'}(i) = \begin{cases} r[i1] \; for \; i \in X - \{k\} \\ r[kp'] \; if \; i = k \end{cases}$$

is not a fixpoint of $\uparrow\downarrow$

*Analogously, let $\langle X, Y, I\rangle$ be an **L**-context, $l \in Y$ be an attribute, and $q < m$ be a positive integer. Also, let for each $q' \in \{1, 2, ..., q\}$, $B_{l,q'} \in L^Y$ are fix points of $\downarrow\uparrow$, where*

$$B_{l,q'}(j) = \begin{cases} c[1j] \; for \; j \in Y - \{l\} \\ c[q'l] \; if \; j = l. \end{cases}$$

If for $j \in Y - \{l\}$ some or all $c[1j]$ appears in the row of $c[ql]$ and and at least one of these $c[1j]$, say $c[1j'](\neq c[2j'])$, $j' \in \{Y\} - \{l\}$ greater than $c[ql]$, then for any $q' \in \{q+1, ..., m\}$ the set $B_{l,q+1} \in L^Y$ defined by

$$B_{l,q'}(j) = \begin{cases} c[1j] \ for \ j \in Y - \{l\} \\ c[q'l] \ if \ j = l \end{cases}$$

is not a fixpoint of $\downarrow\uparrow$.

Proof. Since corresponding to some $i' \in X - \{k\}$, $r[i'1] > I(k, col[r[i'1]) = r[kp](\neq r[kp+1])$. Without loss of generality, we assume $col[r[i'1]] = j' \in Y$. This implies that $I(i', j') = r[i'1]$ and $I(k, col[r[i'1]]) = I(k, j') = r[kp]$. Since for all $j \in Y$, $r[i'1] \rightarrow I(i', j) = 1$ for $i' \in X - \{k\}$, for $A_{k,p+1}(k) = r[k(p+1)]$, $A^{\uparrow}_{k,p+1}(j') = r[i'p]$ and $r[i'p](= A^{\uparrow}_{k,p+1}(j')) \rightarrow r[i'1] = 1$. Thus $A^{\uparrow\downarrow}_{k,p+1}(i') \neq r[i'1] = A_{k,p+1}(i')$. So for any $p' \in \{p+1, ..., n\}$, the set $A_{k,p'} \in L^X$ defined by

$$A_{k,p'}(i) = \begin{cases} r[i1] \ for \ i \in X - \{k\} \\ r[kp'] \ if \ i = k \end{cases}$$

is not a fixpoint of $\uparrow\downarrow$

Proof is similar for analogous part. □

In context Table 1, if we consider the object $1 \in X$, then for $p' \in \{1\}$ we can easily check that $A_{1,1} \in L^X$, where

$$A_{1,1}(i) = \begin{cases} r[i1] \ for \ i \in X - \{1\} \\ r[11] \ if \ i = 1 \end{cases}$$

i.e., $A_{1,1} = \{0.2/1, 0.3/2, 0.1/3, 0.2/4, 0.3/5\}$ is a fixpoint of $\uparrow\downarrow$
Now we can see that $r[21](\neq r[22])$ appears in $col[r[11]] = 3$, and also $r[21] > r[11]$. Therefore $A_{1,2} \in L^X$ defined by

$$A_{1,2}(i) = \begin{cases} r[i1] \ for \ i \in X - \{1\} \\ r[12] \ if \ i = 1 \end{cases}$$

i.e., $A_{1,2} = \{0.4/1, 0.3/2, 0.1/3, 0.2/4, 0.3/5\}$ can not be a fix point of $\uparrow\downarrow$.

4. Discussion

Consider the fuzzy context given by Table 1, where truth values have been chosen randomly from [0,1]. Since for any L-context $\langle X, Y, I \rangle$, $A_{i,1} \in L^X$, where $A_{i,1}(i) = r[i1]$ for $i \in X$ is a fix point of $\uparrow\downarrow$, and $B_{j,1} \in L^Y$, where $B_{j,1}(j) = c[1j]$ for $j \in Y$ is a fix point of $\downarrow\uparrow$. Therefore we can generate fix points of $\uparrow\downarrow$ for each $i \in X$ starting with $A_{i,1} \in L^X$, and fix points of $\downarrow\uparrow$ for each $j \in Y$ starting with $B_{j,1} \in L^Y$. Now form the context table, we can see that because of rule 5 no fix point of $\uparrow\downarrow$ can be generated corresponding to the objects $1 \in X$, $3 \in X$, and $4 \in X$. Corresponding to $2 \in X$, the fix points $A_{2,2}$, $A_{2,3} \in L^X$ are generated using Rule 3, and the fix points $A_{2,4}$, $A_{2,5} \in L^X$ are generated using rule 2. Corresponding to $5 \in X$, the fix points $A_{5,2} = (A_{5,3}) \in L^X$ are generated using Rule 3, and the fix points $A_{5,4}$, $A_{5,5} \in L^X$

are generated using Rule 2. Now starting with $B_{j,1} \in L^Y$, we generate fix point $B_{1,2}$ of $\downarrow\uparrow$ corresponding to the attribute $1 \in Y$ using Rule 4. No more fix point can be generated corresponding to the attribute $1 \in Y$ using the above rules, since $col[c[12]] = col[c[21]]$ and $c[12] > c[21]$. Corresponding to the attribute $2 \in Y$, the fix point $B_{2,2}(= B_{1,2}) \in L^Y$ is generated using Rule 4, the fix points $B_{2,3}, B_{2,4} \in L^Y$ are generated using Rule 3, and $B_{2,5} \in L^Y$ is generated using Rule 2. Corresponding to the attribute $3 \in Y$, the fix point $B_{3,2}(= B_{1,2} = B_{2,2}) \in L^Y$ is generated using Rule 2, the fix points $B_{3,3}$ is generated using Rule 4, $B_{3,4}(= B_{3,3}) \in L^Y$ is generated using again the Rule 2. Corresponding to the attribute $4 \in Y$ no fix point can be generated because of Rule 5. Lastly, corresponding to the attribute $5 \in Y$, the fix point $B_{5,2} \in L^Y$ of $\downarrow\uparrow$ is generated using Rule 2, the fix points $B_{5,3}$ is generated using Rule 3, $B_{5,4}, B_{5,6} \in L^Y$ are generated using again the Rule 2. In previous section we have already discussed that the fix points A_1, A_2, A_3, A_4, A_5 of $\uparrow\downarrow$ corresponding to each attribute $j \in Y$, and the fix points B_1, B_2, B_3, B_4, B_5 of $\downarrow\uparrow$ corresponding to each object $i \in X$ can be obtained using Rule 1. All the fuzzy concepts corresponding to the above fix points are listed below, where a fuzzy concept corresponding to any fix point is denoted as $C(fixpoint)$.

1. $C(A_{i,1}) = \langle\{0.2/1, 0.3/2, 0.1/3, 0.2/4, 0.3/5\}, \{1/1, 1/2, 1/3, 1/4, 1/5\}\rangle$, $i \in X$
2. $C(A_{2,2}) = \langle\{0.2/1, 0.7/2, 0.1/3, 0.2/4, 0.3/5\}, \{1/1, 1/2, 0.3/3, 1/4, 1/5\}\rangle$
3. $C(A_{2,3}) = \langle\{0.2/1, 0.8/2, 0.1/3, 0.2/4, 0.3/5\}, \{1/1, 1/2, 0.3/3, 0.7/4, 1/5\}\rangle$
4. $C(A_{2,4}) = \langle\{0.2/1, 0.9/2, 0.1/3, 0.2/4, 0.3/5\}, \{0.8/1, 1/2, 0.3/3, 0.7/4, 1/5\}\rangle$
5. $C(A_{2,5}) = C(B_2)$
   $= \langle\{0.2/1, 1/2, 0.1/3, 0.2/4, 0.3/5\}, \{0.8/1, 1/2, 0.3/3, 0.7/4, 0.9/5\}\rangle$
6. $C(A_{5,2} = A_{5,3})$
   $= \langle\{0.2/1, 0.3/2, 0.1/3, 0.2/4, 0.4/5\}, \{1/1, 1/2, 1/3, 0.3/4, 1/5\}\rangle$
7. $C(A_{5,4}) = \langle\{0.2/1, 0.3/2, 0.1/3, 0.2/4, 0.5/5\}, \{1/1, 1/2, 0.4/3, 0.3/4, 0.4/5\}\rangle$
8. $C(A_{5,5}) = \langle\{0.2/1, 0.3/2, 0.1/3, 0.2/4, 0.8/5\}, \{0.5/1, 1/2, 0.4/3, 0.3/4, 0.4/5\}\rangle$
9. $C(A_1) = \langle\{0.9/1, 0.8/2, 0.2/3, 0.3/4, 0.5/5\}, \{1/1, 0.7/2, 0.2/3, 0.1/4, 0.2/5\}\rangle$
10. $C(A_2) = C(B_{2,5})$
    $= \langle\{0.7/1, 1/2, 0.2/3, 0.6/4, 0.8/5\}, \{0.3/1, 1/2, 0.2/3, 0.1/4, 0.2/5\}\rangle$
11. $C(A_3) = \langle\{0.2/1, 0.3/2, 0.2/3, 0.3/4, 0.4/5\}, \{1/1, 1/2, 1/3, 0.1/4, 0.2/5\}\rangle$
12. $C(A_4) = \langle\{0.4/1, 0.7/2, 0.1/3, 0.2/4, 0.3/5\}, \{1/1, 1/2, 0.2/3, 1/4, 1/5\}\rangle$
13. $C(A_5) = C(B_{5,5})$
    $= \langle\{1/1, 0.9/2, 0.3/3, 0.2/4, 0.4/5\}, \{0.2/1, 0.2/2, 0.2/3, 0.1/4, 1/5\}\rangle$
14. $C(B_{j,1}) = \langle\{1/1, 1/2, 1/3, 1/4, 1/5\}, \{0.2/1, 0.2/2, 0.2/3, 0.1/4, 0.2/5\}\rangle$, $j \in Y$
15. $C(B_{1,2} = B_{2,2} = B_{3,2})$
    $= \langle\{1/1, 1/2, 0.2/3, 1/4, 1/5\}, \{0.3/1, 0.6/2, 0.2/3, 0.1/4, 0.2/5\}\rangle$
16. $C(B_{2,3}) = \langle\{1/1, 1/2, 0.2/3, 0.6/4, 1/5\}, \{0.3/1, 0.7/2, 0.2/3, 0.1/4, 0.2/5\}\rangle$
17. $C(B_{2,4}) = \langle\{0.7/1, 1/2, 0.2/3, 0.6/4, 1/5\}, \{0.3/1, 0.8/2, 0.2/3, 0.1/4, 0.2/5\}\rangle$
18. $C(B_{3,3}) = \langle\{0.2/1, 1/2, 0.2/3, 1/4, 1/5\}, \{0.3/1, 0.6/2, 0.3/3, 0.1/4, 0.2/5\}\rangle$
19. $C(B_{5,2} = B_3)$
    $= \langle\{1/1, 1/2, 0.3/3, 0.2/4, 1/5\}, \{0.2/1, 0.2/2, 0.2/3, 0.1/4, 0.3/5\}\rangle$
20. $C(B_{5,3} = \langle\{1/1, 1/2, 1/3, 0.2/4, 1/5\}, \{0.2/1, 0.2/2, 0.2/3, 0.1/4, 0.4/5\}\rangle$
21. $C(B_{5,4}) = \langle\{1/1, 1/2, 0.3/3, 0.2/4, 0.4/5\}, \{0.2/1, 0.2/2, 0.2/3, 0.1/4, 0.9/5\}\rangle$
22. $C(B_1) = \langle\{1/1, 0.8/2, 0.1/3, 0.2/4, 0.3/5\}, \{0.9/1, 0.7/2, 0.2/3, 0.4/4, 1/5\}\rangle$
23. $C(B_4) = \langle\{0.2/1, 1/2, 0.1/3, 1/4, 1/5\}, \{0.3/1, 0.6/2, 0.3/3, 0.2/4, 0.2/5\}\rangle$

24. $C(B_5) = \langle\{0.2/1, 0.3/2, 0.1/3, 0.2/4, 1/5\}, \{0.5/1, 0.8/2, 0.4/3, 0.3/4, 0.4/5\}\rangle$

From above list we observe that there are several fix points of $\uparrow\downarrow$, which are equal, and also several fix points of $\downarrow\uparrow$, which are equal. It also may be possible that for a fix point of $\uparrow\downarrow$ there exists a fix point of $\downarrow\uparrow$ both of which produce same fuzzy concept.

5. Conclusion

This paper presents the rules, along with justification of their correctness for computing the fix points of the fuzzy closure operators $\uparrow\downarrow$ and $\downarrow\uparrow$ from an L-context. Based on Gödel operation on [0, 1], this is the first contribution on computing the fix points of the operators $\uparrow\downarrow$ and $\downarrow\uparrow$, directly from an L-context. As our rules based on Gödel operation on [0, 1], these rules may be used for generating all the one-sided fuzzy concepts and proto-fuzzy concepts. In our future work we will develop algorithms for generating the all one-sided fuzzy concepts and proto-fuzzy concepts by using the above rules. Our future research will also be directed towards the development of an algorithm for computing the fix points together with the lattice order on the set of the fix points of the closure operators $\uparrow\downarrow$ and $\downarrow\uparrow$.

References

[1] R. Bělohlávek, Lattices generated by binary fuzzy relations (extended abstract), in Abstract of the 4th international conference on Fuzzy sets Theory and its Applications, Liptovsky Jan' country-regionplaceSlovakia (1998) p.11.

[2] R. Bělohlávek, Fuzzy relational systems, Foundations and Principles, Kluwer Academic/ Plenum Publishers, Newyork 2002.

[3] R. Bělohlávek, Algorithms for fuzzy concept lattices, In Proc. Fourth Int. Conf. on Recent Advances in Soft Computing, CityplaceNottingham, country-regionUnited Kingdom (2002) 200–205.

[4] R. Bělohlávek, B. D. Baets, J. Outrata, and V. Vychodil, Lindig's Algorithm for Concept Lattices over Graded Attributes, In V. Torra, Y. Narukawa, and Y. Yoshida, Eds., MDAI 2007, LNAI 4617, CitySpringer-Verlag StateBerlin CityplaceHeidelberg (2007) 156–167.

[5] R. Bělohlávek, Reduction and a simple proof of characterization of fuzzy concept lattices, Fund. Inform. 46 (4) (2001) 277–285.

[6] R. Bělohlávek, B. D. Baets, J. Outrata and V. Vychodil, Computing the lattice of all fixpoints of a fuzzy closure operator, IEEE Transactions on Fuzzy Systems 18 (3) (2010) 546–557.

[7] A. Burusco, R. Fuentes-Ganzáles, The study of the L-fuzzy concept lattice, Mathware and Soft Computing 3 (1994) 209–218.

[8] A. Burusco, R. Fuentes-Ganzáles, Construction of the L-fuzzy concept lattice, Fuzzy Sets and Systems 97 (1998) 109-114.

[9] V. Cross, M. Kandasamy and W. Yi, Comparing Two Approaches to Creating Fuzzy Concept Lattices, In: Proc. of the North American Fuzzy Information Processing Society, El Paso, TX, March (2011)18-19.

[10] D. Dubois and H. Prade, Fuzzy Sets and Systems, Theory and Applications. Academic Press, StateplaceNew York 1980.

[11] A. Formica, Semantic Web search based on rough sets and Fuzzy Formal Concept Analysis, Knowledge-Based Systems 26 (2012) 40–47.

[12] B. Ganter, Two basics algorithms in concept analysis, Technische Hoschule Darmstadt, Darmstadt, Germany, Tech. Rep., FB4-Preprint no. 831, 1984.

[13] B. Ganter and R. Wille, Formal Concept Analysis, Mathematical Foundation, CityplaceSpringer-Verlag, StateBerlin 1999.

[14] P. Ghosh, K. Kundu and D. Sarkar, Fuzzy graph representation of a fuuzy concept lattice, Fuzzy Sets and Systems 161 (2010) 1669–1675.

[15] P. Ghosh and K. Kundu, An Algorithm for fuuzy concepts Using Graph, Mathware and Soft Computing 17 (2010) 27–38.

[16] A. J. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145-174.

[17] P. Hajek, Metamathematics of Fuzzy Logic, Kluwer, CityplaceDordrecht 1998.

[18] U. HöHle, On the fundamentals of fuzzy set theory, J. Math. Anal. Appl. 201 (1996) 786–826.

[19] J. G. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice-Hall, PTR, CityplaceUpper Saddle River, StateNJ 1995.

[20] A. M. Kozae, S. A. El-Sheikh, E. H. Aly and M. Hosny , Rough sets and its applications in a computer network, Ann. Fuzzy Math. Inform. 6 (3) (2013) 605–624.

[21] C. A. Kumar and S. Srinivas, Concept Lattice Reduction Using Fuzzy K-means Clustering, Expert Systems with Applications 37 (3) (2009) 2696–2704.

[22] C. Lindig, ”Fast concept analysis”, in Working with Conceptual Structures-Contributions to ICCS 2000, G. Stumme, Ed. Aachen, Germany: Shaker-Verlag (2000) 152–161.

[23] M. Liu, M. Shao, W. Zhang and C. Wu, Reduction method for concept lattices based on rough set theory and its application, Comput. Math. Appl. 53 (9) (2007) 1390–1410.

[24] Juraj Macko, User-Friendly Fuzzy FCA, In P. Cellier, F. Distel, and B. Ganter(Eds.): In ICFCA 2013, LNAI 7880, Springer-Verlag Berlin Heidelberg (2013) 156–171.

[25] Z. Pawlak, Rough Sets: Theoretical aspects of reasoning about data, Kluwer Academic Publishers, Boston 1991.

[26] J. Poelmans, D. I. Ignatov, S. O. Kuznetsov and G. Dedene, Fuzzy and rough formal concept analysis: a survey, Int. J. Gen. Syst. 43 (2) (2014) 105–134.

[27] S. Pollandt, Fuzzy Begriffe, CitySpringer-Verlag, StateBerlin/CityplaceHeidelberg 1997.

[28] M. W. Shao, M. Liu and W. X. Zhang, Set Approximations in Fuzzy Formal Concept Analysis, Fuzzy Sets and Systems 158 (2007) 2627–2640.

[29] P. K. Singh, C. A. Kumar, Bipolar fuzzy graph representation of concept lattice, Inform. Sci. 288 (2014) 437û448.

[30] P. K. Singh, C. A. Kumar and J. Li, Knowledge representation using interval-valued fuzzy formal concept lattice, Soft Comput. (2015) 1–18. doi:10.1007/s00500-015-1600-1.

[31] P. K. Singh, A. K. Cherukuri and J. Li, Concepts reduction in formal concept analysis with fuzzy setting using Shannon entropy, International Journal of Machine Learning and Cybernetics (2015) 1–11. doi:10.1007/s13042-014-0313-6.

[32] P. K. Singh and A. Gani, Fuzzy concept lattice reduction using Shannon entropy and Huffman coding, J. Appl. Non-Class. Log. 25 (2)(2015) 101–119.

[33] R. Wille, ”Restructuring lattice theory: An Approach Based on hierarchies of concepts”, in Ordered sets, Eds., I. Rival, Reidel, City Dordrecht, place City Boston (1982) 445–470.

Partha Ghosh (partha_0377@rediffmail.com)
Department of Mathematics, Goenka College of Commerce and Business Administration, 210, B. B. Ganguly Street, Kolkata-700012, India

Krishna Kundu (kundu.krishna@gmail.com)
Department of Applied Mathematics, University of Calcutta, 92, A.P.C. Road, Kolkata- 700009, India